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On the definition of a probabilistic inner product space

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RIASSUNTO: *In questo articolo presentiamo una definizione di spazio probabilistico con prodotto interno basata sulla recente definizione di spazio normato probabilistico. Tale definizione consente di comprendere le piú importanti classi di spazî probabilistici con prodotto interno.*

ABSTRACT: *In this paper we give a definition of a probabilistic inner product space which is based on the new and recently formulated definition of a probabilistic normed space. This definition is sufficiently general to encompass the most important classes of probabilistic inner product spaces.*

1 – Introduction

The purpose of this paper is to present a definition of a probabilistic inner product space which is based on the new definition of a probabilisitic normed space that was recently introduced and studied in [1], [2], [8]. It employs the probabilistic generalization of the triangle inequality rather than a probabilistic generalization of the Cauchy-Schwarz inequality, and thereby overcomes many of the obstacles encountered in the earlier approaches of M.L. SENECHAL [13], and J.M. FORTUNY [4]. Thus it leads naturally to the definition of the most important classes of probabilistic

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inner product spaces, namely the Menger spaces and the Šerstnev spaces; and it also includes probabilistic inner product spaces generated by families of mappings from a probability space into a real inner product space. However, a number of important issues, e.g., questions of continuity and a reasonable definition of orthogonality, remain to be settled.

In the sequel we generally follow the notation and terminology of [12].

2 – Preliminaries

A *distribution function* (briefly, a *d.f.*) is a function F from the extended real line $\overline{\mathbf{R}} = [-\infty, +\infty]$ into the unit interval $I = [0, 1]$ that is nondecreasing and satisfies $F(-\infty) = 0$, $F(+\infty) = 1$. We normalize all d.f.'s to be left–continuous on the unextended real line $\mathbf{R} =]-\infty, +\infty[$. In particular then, for every a in $[-\infty, +\infty[$, the functions ϵ_a defined by

$$\epsilon_a = \begin{cases} 0, & x \in [-\infty, a]\,, \\ 1, & x \in]a, +\infty]\,, \end{cases} \tag{1}$$

are d.f.'s, as is the function ϵ_∞ defined by $\epsilon_\infty(x) = 0$ for x in $[-\infty, +\infty[$ and $\epsilon_\infty(+\infty) = 1$. The set of all d.f.'s will be denoted by Δ and the subset of all F's in Δ satisfying $F(0) = 0$ will be denoted by Δ^+. The sets Δ and Δ^+ are partially ordered by the usual pointwise partial ordering of functions: ϵ_∞ is the minimal element of both Δ and Δ^+; $\epsilon_{-\infty}$ is the maximal element of Δ, and ϵ_0 the maximal element of Δ^+ .

A *triangle function* is a binary operation on Δ^+ that is commutative, associative, nondecreasing in each place, and has ϵ_0 as identity. Continuity of a triangle function means continuity with respect to the topology of weak convergence in Δ^+.

Typical (continuous) triangle functions are convolution and the operations τ_T and τ_S, which are, respectively, given by

$$\tau_T(F, G)(x) = \sup_{u+v=x} T\,(F(u), G(v))\,, \tag{2}$$

and

$$\tau_S(F, G)(x) = \inf_{u+v=x} S\,(F(u), G(v))\,, \tag{3}$$

for all F, G in Δ^+ and all x in $\mathbf{R}$ [12; Secs. 7.2 and 7.3]. Here T is a continuous *t-norm* and S is a continuous *t-conorm*, i.e., both are continuous binary operations on I that are commutative, associative and nondecreasing in each place; T has 1 as identity and S has 0 as identity. If T is a t-norm and T^* is defined on $I \times I$ via

$$T^*(x, y) = 1 - T\,(1 - x, 1 - y)\,,$$

then T^* is a *t-conorm*, specifically the t-conorm of T.

It follows without difficulty from (1)-(3) that, for every continuous t-norm T and every continuous t-conorm S, the triangle functions τ_T and τ_S satisfy the condition

$$\tau\,(\epsilon_a, \epsilon_b) = \epsilon_{a+b}, \quad \text{for all } a, b \geq 0. \tag{4}$$

The most important t-norms are the functions W, Prod, and M which are defined, respectively, by

$$\begin{aligned} W(a, b) &= \max(a + b - 1, 0), \\ \mathrm{Prod}(a, b) &= a\,b, \\ M(a, b) &= \min(a, b). \end{aligned}$$

Their corresponding t-conorms are given, respectively, by

$$\begin{aligned} W^*(a, b) &= \min(a + b, 1), \\ \mathrm{Prod}^*(a, b) &= a + b - a\,b, \\ M^*(a, b) &= \max(a, b). \end{aligned}$$

The set of all t-norms is partially ordered by the usual pointwise partial ordering of functions. Thus we have $W \leq \mathrm{Prod} \leq M$. Moreover, M is the maximal t-norm and, since $\min(a, b) \leq \max(a, b)$ for all a, b in I, it follows at once that

$$T(a, b) \leq S(a, b), \tag{5}$$

for every t-norm T and every t-conorm S. Correspondingly, the set of all triangle functions is partially ordered via

$$\tau_1 \leq \tau_2 \quad \text{iff} \quad \tau_1\,(F, G) \leq \tau_2\,(F, G)\,, \text{for all } F, G \text{ in } \Delta^+.$$

It follows from (2), (3) and (5) that $\tau_T \leq \tau_S$ for every continuous t-norm T and every continuous t-conorm S.

Definition 2.1. *A probabilistic metric (briefly,* PM*) space is a triple $(S, \mathcal{F}, \tau)$, where S is a nonempty set, τ is a triangle function, and $\mathcal{F}$ is a mapping from $S \times S$ into Δ^+ such that, if F_{pq} denotes the value of $\mathcal{F}$ at the pair (p, q), the following conditions hold for all p, q, r in S:*

(M1a) $F_{pp} = \epsilon_0$;

(M1b) $F_{pq} \neq \epsilon_0$ *if* $p \neq q$;

(M2) $F_{pq} = F_{qp}$;

(M3) $F_{pr} \geq \tau\,(F_{pq}, F_{qr})$.

If (M1a), (M2) *and* (M3) *are satisfied, then $(S, \mathcal{F}, \tau)$ is a probabilistic pseudometric space* [12].

Every metric space can be regarded as a special kind of PM space. For if (S, d) is a metric space, if $\mathcal{F} : S \times S \to \Delta^+$ is defined via $F_{pq} = \epsilon_{d(p,q)}$, and if τ is a triangle function such that $\tau\,(\epsilon_a, \epsilon_b) \geq \epsilon_{a+b}$ for all $a, b \geq 0$ — e.g., if τ is given by (2) or (3) (see (4)) — then $(S, \mathcal{F}, \tau)$ is a PM space from which the original metric space can be immediately recovered.

Definition 2.2. *A probabilistic normed* (*briefly,* PN) *space is a quadruple $(S, \mathcal{N}, \tau, \tau^*)$, where S is a real linear space, τ and τ^* are continuous triangle functions such that $\tau \leq \tau^*$, and $\mathcal{N}$ is a mapping from S into Δ^+ such that, if N_p denotes the value of $\mathcal{N}$ at the point p, the following conditions hold for all p, q in S:*

(N1a) $N_\theta = \epsilon_0$, *where θ is the null vector in S;*

(N1b) $N_p \neq \epsilon_0$ *if* $p \neq \theta$;

(N2) $N_{-p} = N_p$;

(N3) $N_{p+q} \geq \tau\,(N_p, N_q)$;

(N4) $N_p \leq \tau^*\left(N_{\alpha\, p}, N_{(1-\alpha)\, p}\right)$ *for all α in I.*

If (N1a), (N2), (N3) *and* (N4) *are satisfied, then $(S, \mathcal{N}, \tau, \tau^*)$ is a probabilistic pseudonormed space* [1].

If $(S, \|\cdot\|)$ is a real normed space, if τ is a triangle function such that $\tau\,(\epsilon_a, \epsilon_b) \geq \epsilon_{a+b}$ for all $a, b \geq 0$, and if $\mathcal{N} : S \to \Delta^+$ is defined via $N_p = \epsilon_{\|p\|}$, then $(S, \mathcal{N}, \tau, \tau^*)$ is a PN space.

If $(S, \mathcal{N}, \tau, \tau^*)$ is a PN space and if $\mathcal{F} : S \times S \to \Delta^+$ is defined via $F_{pq} = N_{p-q}$, then $(S, \mathcal{F}, \tau)$ is a PM space. If $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$ for some continuous t-norm T and its associated t-conorm T^*, then $(S, \mathcal{N}, \tau_T, \tau_{T^*})$ is a *Menger* PN *space*, which we denote briefly by $(S, \mathcal{N}, T)$. If $\tau^* = \tau_M$ and equality holds in (N4), then $(S, \mathcal{N}, \tau, \tau_M)$ is a *Šerstnev* PN *space*. In this case, as shown in [1], the conditions

$$N_p = \tau_M \left(N_{\alpha\, p}, N_{(1-\alpha)\, p}\right), \quad \text{for all } p \text{ in } S \text{ and all } \alpha \text{ in } I, \tag{6}$$

and (N2), taken together, are equivalent to Šerstnev's condition

$$N_{\lambda\, p}(x) = N_p\left(\frac{x}{|\lambda|}\right), \quad \text{for all } \lambda \text{ and } x \text{ in } \mathbf{R}, \tag{7}$$

where, by convention, $N_p(x/0) = \epsilon_0(x)$ [14], [15].

3 – Probabilistic inner product spaces

In going from real to probabilistic inner products, the first thing to notice is that, since real inner products can assume negative values, we will need to deal with d.f.'s in Δ rather than with d.f.'s confined to the subspace Δ^+. Accordingly, we begin with the following:

DEFINITION 3.1. *A multiplication on Δ is a binary operation τ on Δ that is commutative, associative, nondecreasing in each place, and whose restriction to Δ^+ is a triangle function.*

Multiplications of particular interest to us are the extensions of the functions τ_T and τ_S defined on $\Delta \times \Delta$ by

$$\tau_T(F, G)(x) = \sup_{u+v=x} T\left(F(u), G(v)\right), \tag{8}$$

and

$$\tau_S(F, G)(x) = l^- \left[\inf_{u+v=x} T^*\left(F(u), G(v)\right)\right], \tag{9}$$

respectively. Here T is a continuous t-norm, S a continuous t-conorm and, for any F in Δ, $l^- F$ is the left-continuous normalization of F, i.e., $l^- F(x) = F(x-)$ for every x in $\mathbf{R}$.

The fact that τ_T is indeed a multiplication on Δ was established in [10; Section 9]. There it was also shown that, in Δ, τ_T is *not* continuous with respect to the topology of weak convergence.

Minor modifications of the arguments given in [10; Section 5] suffice to show that the extended τ_S is also a multiplication on Δ. But, like τ_T, the multiplication τ_S is *not* continuous on Δ. To see this, for every positive integer n, let $F_n = \epsilon_{-n}$ and $G_n = G$ where $G(x) = 1/2$ for all x in $\mathbf{R}$. Then the sequence $\{F_n\}$ converges weakly to $\epsilon_{-\infty}$, while $\{G_n\}$ converges weakly to G; but for every x in $\mathbf{R}$, we have

$$\lim_{n\to\infty} \tau_S(\epsilon_{-n}, G_n)(x) = \frac{1}{2} \neq 1 = \tau_S(\epsilon_{-\infty}, G)(x).$$

For any F in Δ, we let $\overline{F}$ denote the d.f. in Δ defined via

$$\overline{F}(x) = l^-\left(1 - F(-z)\right), \quad \text{for all } x \text{ in } \mathbf{R}. \tag{10}$$

Note that $\overline{\overline{F}} = F$ for every F in Δ and that $F = \overline{F}$ if and only if F is symmetric.

Definition 3.2. *A probabilistic inner product (briefly,* PIP*) space is a quadruple $(S, \mathcal{G}, \tau, \tau^*)$, where S is a real linear space, τ and τ^* are multiplications on Δ such that $\tau \leq \tau^*$ and $\mathcal{G}$ is a mapping from $S \times S$ into Δ such that, if $G_{p,q}$ denotes the value of $\mathcal{G}$ at the pair (p, q) and if the function $\mathcal{N} : S \to \Delta^+$ is defined via*

$$N_p(x) = \begin{cases} G_{p,p}(x^2), & x > 0, \\ 0, & x \leq 0, \end{cases} \tag{11}$$

the following conditions hold for all p, q, r in S:

(P1a) *$G_{p,p} \in \Delta^+$ and $G_{\theta,\theta} = \epsilon_0$, where θ is the null vector in S;*
(P1b) *$G_{p,p} \neq \epsilon_0$ if $p \neq \theta$;*
(P2) *$G_{\theta,p} = \epsilon_0$;*
(P3) *$G_{p,q} = G_{q,p}$;*
(P4) *$G_{-p,q} = \overline{G}_{p,q}$;*
(P5) *$N_{p+q} \geq \tau\left(N_p, N_q\right)$;*
(P6) *$N_p \leq \tau^*\left(N_{\alpha p}, N_{(1-\alpha)p}\right)$ for every α in I;*
(P7) *$\tau\left(G_{p,r}, G_{q,r}\right) \leq G_{p+q,r} \leq \tau^*\left(G_{p,r}, G_{q,r}\right)$.*

If $\tau = \tau_T$ *and* $\tau^* = \tau_{T^*}$ *for some continuous t-norm* T *and its associated t-conorm* T^**, then* $(S, \mathcal{G}, \tau_T, \tau_{T^*})$ *is a* Menger PIP space*, which we denote by* $(S, \mathcal{G}, T)$*. If* $\tau^* = \tau_M$ *and equality holds in* (P6)*, then* $(S, \mathcal{G}, \tau, \tau_M)$ *is a Šerstnev* PIP *space. If* (P1a) *and* (P2)-(P7) *are satisfied, then* $(S, \mathcal{G}, \tau, \tau^*)$ *is a probabilistic pseudo-inner product space.*

It is immediate that $(S, \mathcal{N}, \tau, \tau^*)$ is a PN space and we shall refer to $\mathcal{N}$ as the probabilistic norm derived from the probabilistic inner product $\mathcal{G}$. Note again that if $(S, \mathcal{N}, \tau, \tau_M)$ is a Šerstnev PIP space, then, in view of the fact that $N_{-p} = N_p$, (P6) may be replaced by (7).

If, for any p, q in S and any x in $\mathbf{R}$, we interpret the number $G_{p,q}(x)$ as "the probability that the inner product of p and q is less than x", then (P1)-(P4) are natural probabilistic versions of the corresponding properties of real inner products; (P5) is the triangle inequality for the associated probabilistic norm, (P6) is a probabilistic version of the homogeneity property of a norm and is also needed to ensure that $\mathcal{N}$ is indeed a probabilistic norm, and (P7) is a weak distributivity property which generalizes the usual bilinearity property of an inner product.

4 – Examples

If $(S, \langle\cdot,\cdot\rangle)$ is a real inner product space, if τ is a multiplication on Δ such that $\tau(\epsilon_a, \epsilon_b) = \epsilon_{a+b}$ for all a, b in $\mathbf{R}$, and if $\mathcal{G} : S \times S \to \Delta$ is defined via $G_{p,q} = \epsilon_{\langle p,q\rangle}$, then $(S, \mathcal{G}, \tau, \tau)$ is a PIP space. Thus, just as ordinary metric and normed spaces may, respectively, be viewed as special cases of PM and PN spaces, a real inner product space may be viewed as a special instance of a PIP space.

Definition 4.1. *Let* $(\Omega, \mathcal{A}, P)$ *be a probability space,* $(V, \langle\cdot,\cdot\rangle)$ *a real inner product space, and* S *a set of functions from* Ω *into* V*. Then* $(S, \mathcal{G})$ *is an* EP*-space with base* $(\Omega, \mathcal{A}, P)$ *and target* $(V, \langle\cdot,\cdot\rangle)$ *if the following conditions hold:*

(i) *S, under pointwise addition and scalar multiplication, is a real linear space. The zero element in S is the constant function θ given by $\theta(\omega) = n$ for all ω in Ω, where n is the null vector in V.*

(ii) *For all p, q in S and all x in $\mathbf{R}$, the set $\{\omega \text{ in } \Omega : \langle p(\omega), q(\omega)\rangle < x\}$ belongs to $\mathcal{A}$, i.e., the composite function $\langle p, q\rangle$ from Ω into $\mathbf{R}$*

defined by $\langle p,q\rangle(\omega) = \langle p(\omega), q(\omega)\rangle$ is P-measurable, or, in other words, is a real random variable.

(iii) *For all p, q in S, $\mathcal{G}(p,q)$ is the distribution function of $\langle p,q\rangle$, i.e., for all x in* $\mathbf{R}$,

$$G_{p,q}(x) = P\{\omega \text{ in } \Omega : \langle p(\omega), q(\omega)\rangle < x\}, \tag{12}$$

If, for any p in S, $\langle p,q\rangle = 0$ a.s. only if $p = \theta$, then $(S,\mathcal{G})$ is a canonical EP*-space.*

THEOREM 4.2. *If $(S,\mathcal{G})$ is an* EP*-space, then $(S,\mathcal{G},\tau_W,\tau_{W^*})$ is a pseudo-*PIP *space. If $(S,\mathcal{G})$ is a canonical* EP*-space, then $(S,\mathcal{G},\tau_W,\tau_{W^*})$ is a* PIP *space, i.e. $(S,\mathcal{G},W)$ is a Menger* PIP *space.*

PROOF. The properties (P1a), (P2), (P3) and (P4) are immediate, as is (P1b) when $(S,\mathcal{G})$ is canonical.

Next, it follows from Definition 4.1 and (11) that $(S,\mathcal{N})$ is an E-normed space. As shown in [12; Theorem 15.1.7], such a space is a pseudo-PN space in the sense of Šerstnev in which $\tau = \tau_W$. Condition (P5) is just the triangle inequality for this space; and since (7) holds, (6) yields (P6) with $\tau = \tau_M$ and, since $\tau_M < \tau_{W^*}$, *a fortiori*, with $\tau = \tau_{W^*}$.

It remains to establish (P7). Using (12), for any x in $\mathbf{R}$, we have

$$\begin{aligned} G_{p+q,r}(x) &= P\{\omega \text{ in } \Omega : \langle p(\omega)+q(\omega), r(\omega)\rangle < x\} \\ &= P\{\omega \text{ in } \Omega : \langle p(\omega), r(\omega)\rangle + \langle q(\omega), r(\omega)\rangle < x\}. \end{aligned}$$

Thus $G_{p+q,r}$ is the d.f. of the sum of the random variables $\langle p,r\rangle$ and $\langle q,r\rangle$. Let $C_{\langle p,r\rangle,\langle q,r\rangle}$ be the copula of these random variables, so that $C_{\langle p,r\rangle,\langle q,r\rangle}(G_{p,r},G_{q,r})$ is their joint d.f. Then (see [5], [9] or [11]), we have

$$G_{p+q,r} = \sigma_{C_{\langle p,r\rangle,\langle q,r\rangle}}(G_{p,r},G_{q,r}),$$

where, for any pair of d.f.'s F and G and any copula C,

$$\sigma_C(F,G) = \iint\limits_{u+v<x} dC(F(u),G(v)).$$

Next (see [7], [9] or [11]), for any copula C and for any pair of d.f.'s, F and G,

$$\tau_W(F,G) \le \sigma_C(F,G) \le \tau_{W^*}(F,G).$$

This yields (P7), with $\tau = \tau_W$ and $\tau^* = \tau_{W^*}$, and completes the proof.

In particular Theorem 4.1 applies to the product of random variables or of random vectors on a probability space $(\Omega, \mathcal{A}, P)$. In this case, S is the set of random variables or vectors on Ω, while the target is $\mathbf{R}^k$ $(k \ge 1)$ endowed with the usual inner product

$$\langle x, y\rangle = \sum_{j=1}^{k} x_j\, y_j \qquad (x, y \text{ in } \mathbf{R}^k).$$

Let $(S, \mathcal{G})$ be an EP-space with base $(\Omega, \mathcal{A}, P)$ and target $(V, \langle\cdot,\cdot\rangle)$. Then, for every ω in Ω, the function i_ω from $S \times S$ into $\mathbf{R}$ defined by $i_\omega(p,q) = \langle p(\omega), q(\omega)\rangle$ is a pseudo-inner product on S. Since distinct functions p and q may agree at a particular point ω in Ω, so that $p(\omega) = q(\omega)$ while $p \ne q$, i_ω need not be an inner product on S. Now, noting that

$$G_{p,q}(x) = P\{\omega \text{ in } \Omega : i_\omega(p,q) < x\}$$

is the P-measure of the set of all pseudo-inner products i_ω for which the inner product of p and q is less than x, we have that the EP-space $(S, \mathcal{G})$ is a *pseudo-inner product generated space*, which is generated by the collection $\{i_\omega : \omega \text{ in } \Omega\}$. H. Sherwood has shown that a pseudo PM space is an E-metric space if and only if it is a pseudo-metrically generated space [16]; and similarly (but with more difficulty) that a pseudo PN space is an E-normed space if and only if it is a pseudo-norm generated space [17]. Whether this equivalence still holds for PIP spaces, i.e., whether every pseudo-inner product generated space is an EP space, is at present an open question.

5 – Final Remarks

We begin with a simple but, nevertheless, somewhat surprising result. Recall that a t-norm T is positive if $T(a,b) > 0$ whenever $a > 0$ and $b > 0$.

THEOREM 5.1. *Let* $(S, \mathcal{G}, T)$ *be a Menger* PIP *space and suppose that there is a pair of points* p *and* q *in* S *such that* $G_{p,q}$ *is strictly positive on* $\mathbf{R}$ *and* $G_{p,p}$*,* $G_{q,q}$ *are both strictly positive on* $]0, +\infty[$*. Then* T *cannot be a positive t-norm.*

PROOF. By (P7) and (P3), we have

$$\begin{aligned} G_{p+q,p+q} &\geq \tau_T\left(G_{p+q,p}, G_{p+q,q}\right) \\ &\geq \tau_T\left(\tau_T\left(G_{p,p}, G_{p,q}\right), \tau_T\left(G_{p,q}, G_{q,q}\right)\right). \end{aligned}$$

Consequently, for all t, u, v, w in $\mathbf{R}$ such that $t + u + v + w = 0$,

$$0 = G_{p+q,p+q}(0) \geq T\left(T\left(G_{p,p}(t), G_{p,q}(u)\right), T\left(G_{p,q}(v), G_{q,q}(w)\right)\right).$$

Choosing $t > 0$ and $w > 0$ yields $G_{p,p}(t) > 0$ and $G_{q,q}(w) > 0$. Since $G_{p,q}(x) > 0$ for all real x, it follows that T is not positive.

LEMMA 5.2. *If* F *in* Δ *is such that*

$$\tau_T\left(F, \overline{F}\right) = \epsilon_0, \tag{13}$$

where $\overline{F}$ *is given by* (10)*, then* F *is a proper d.f., i.e.,* $\lim_{x\to+\infty} F(x) = 1$ *and* $\lim_{x\to-\infty} F(x) = 0$*.*

PROOF. By (8) and (1) we have

$$\sup_{u+v=x} T\left(F(u), \overline{F}(v)\right) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0. \end{cases}$$

Suppose that $\lim_{t\to+\infty} F(t) = a < 1$. Then, since $T(a, 1) = a$, we have $T\left(F(u), \overline{F}(v)\right) \leq a$ for all u, v in $\mathbf{R}$, whence (13) cannot hold.

Next, if $\lim_{t\to-\infty} F(t) = b > 0$, then

$$\lim_{t\to+\infty} \overline{F}(t) = \lim_{t\to+\infty} (1 - F(-t)) = 1 - \lim_{t\to-\infty} F(t) = 1 - b < 1,$$

and, again, (13) cannot hold.

In what follows, we need the notion of the (left-continuous) *quasi-inverse* of a distribution function F. This is the function $F^\wedge$ defined on the half-open interval $]0,1]$ by

$$F^\wedge(t) = \sup\{x : F(x) < t\}\,. \tag{14}$$

LEMMA 5.3. *Suppose F is in Δ. Then $\tau_M\left(F, \overline{F}\right) = \epsilon_0$, if and only if $F = \epsilon_c$ for some c in $\mathbf{R}$.*

PROOF. By Lemma 5.2, F is a proper d.f., whence $\overline{F}$ is also proper. Thus the duality theorem of [6] can be applied. This gives

$$F^\wedge(t) + \overline{F}^\wedge(t) = 0, \quad \text{for any } t \text{ in }]0,1[, \tag{15}$$

where $F^\wedge$ is the quasi-inverse of F. But $\overline{F}^\wedge(t) = -F^\wedge(1-t)$, whence, by (15), $F^\wedge(t) = F^\wedge(1-t)$. Since $F^\wedge$ is nondecreasing, this last equality can hold if and only if $F^\wedge$ is constant, say equal to c. Hence $F = \epsilon_c$ and the lemma is proved.

THEOREM 5.4. *If $(S, \mathcal{G}, M)$ is a Menger* PIP *space, then it is a real inner product space, i.e., there exists a real inner product $\langle\cdot,\cdot\rangle : S\times S \to \mathbf{R}$ such that $G_{p,q} = \epsilon_{\langle p,q\rangle}$ for all p, q in S.*

PROOF. Since $\tau_M = \tau_{M^*}$ [12; Theorem 7.5.6], it follows at once from (P7) that, for all p, q, r in S,

$$G_{p+q,r} = \tau_M\left(G_{p,r}, G_{q,r}\right).$$

Letting $q = -p$ and using (P2) and (P4) yields

$$\epsilon_0 = \tau_M\left(G_{p,r}, \overline{G}_{p,r}\right).$$

Thus, by the above lemmas, $G^\wedge_{p,r} = c$ for some c in $\mathbf{R}$. Now,let $\langle\cdot,\cdot\rangle$ be the mapping from $S \times S$ into $\mathbf{R}$ defined by $\langle p, q\rangle =$ the value of $G^\wedge_{p,q}$, for any p, q in S. Then a few calculations (employing the characterization of a normed space $(V, \|\cdot\|)$ given in [1], i.e., the fact that the condition $\|\lambda\, p\| = |\lambda|\,\|p\|$ for every p in V and every real λ may be replaced by the conditions $\|-p\| = \|p\|$ and $\|p\| = \|\alpha\, p\| + \|(1-\alpha)\, p\|$ for every p in V and all α in $[0,1]$) yield that $(S, \langle\cdot,\cdot\rangle)$ is a real inner product space.

The above result merits a comment. It is been known for a long time (see, e.g., Corollary 8.2.2 and Theorem 8.2.3 in [12]) that if $(S, \mathcal{F}, M)$ is a Menger PM space, then

$$F^{\wedge}_{p,r}(c) \leq F^{\wedge}_{p,q}(c) + F^{\wedge}_{q,r}(c),$$

for all p, q, r in S and all c in $[0,1]$. Thus, each of the functions d_c defined on $S \times S$ via $d_c(p,q) = F^{\wedge}_{p,q}(c)$ is a pseudo-metric on S. Similarly, if $(S, \mathcal{N}, M)$ is a Menger PN space, then each of the functions ν_c defined on S via $\nu_c(p) = N^{\wedge}_{p}(c)$ is a pseudo-norm on S. It therefore is no surprise that many results – mainly in the area of fixed-point theory – that hold in ordinary metric or normed spaces extend at once to Menger spaces under M. Nevertheless, in spite of this simple observation, there have been many papers with the words "probabilistic metric" or "probabilistic norm" in their title that belong to what C. Fenske [3] has so aptly called "nonsense literature ... devoted to absurd generalizations of the contraction principle". Anyone tempted to "enrich" the theory of probabilistic inner product spaces with contributions of this nature is strongly advised to take careful note of Theorem 5.4.

Questions of continuity, questions concerning the Cauchy-Schwarz inequality — a version of which was used in the earlier definitions of a PIP space proposed by M.L. Senechal [13] and J.M. Fortuny [4] — and questions concerning definitions of orthogonality will be considered in subsequent papers.

REFERENCES

[1] C. Alsina – B. Schweizer – A. Sklar: *On the definition of a probabilistic normed space*, Aequationes Math., **46** (1993), 91-98.

[2] C. Alsina – B. Schweizer – A. Sklar: *Continuity properties of probabilistic norms*, J. Math. Anal. Appl. (to appear).

[3] C. Fenske: Math. Rev., **83j**:54038

[4] J.M. Fortuny: *Espacios con producto interior probabilístico*, Stochastica **8** (1984), 229-248.

[5] M.J. Frank: *Associativity in a class of operations on a space of distribution functions*, Aequationes Math., **12** (1975), 121-144.

[6] M.J. Frank – B. Schweizer: *On the duality of generalized infimal and supremal convolutions*, Rend. Mat., (6) **12** (1979), 1-23.

[7] M.J. Frank – R.B. Nelsen – B. Schweizer: *Best possible bounds for the distribution of a sum – a problem of Kolmogorov*, Probab. Th. Rel. Fields, **74** (1987), 199-211.

[8] B. Lafuerza Guillén – J.A. Rodríguez Lallena – C. Sempi: *Some classes of probabilistic normed spaces*, (to appear).

[9] R. Moynihan – B. Schweizer – A. Sklar: *Inequalities among operations on probability distribution functions*, General Inequalities, edited by E. F. Beckenbach, (Birkhäuser, Basel) (1978), 133-149.

[10] B. Schweizer: *Multiplications on the space of probability distribution functions*, Aequationes Math., **12** (1975), 156-183.

[11] B. Schweizer: *Thirty years of copula*, Advances in probability distributions with given marginals: beyond the copulas, edited by G. Dall'Aglio, S. Kotz, G. Salinetti (Kluwer, Dordrecht) 1991, 13-50.

[12] B. Schweizer – A. Sklar: *Probabilistic metric spaces*, (New York, Elsevier North-Holland) 1983.

[13] M.L. Senechal: *Approximate functional equations and probabilistic inner product spaces*, Illinois Institute of Technology, Ph.D. Thesis, 1965.

[14] A.N. Šerstnev: *Random normed spaces: problems of completeness*, Kazan' Gos. Univ. Učen. Zap., **122** (1962), 3-20.

[15] A.N. Šerstnev: *On the notion of a random normed space*, Dokl. Akad. Nauk. SSSR, **149** (1963), 280-283.

[16] H. Sherwood: *On E-spaces and their relation to other classes of probabilisitic metric spaces*, J. London Math. Soc., **44** (1969), 441-448.

[17] H. Sherwood: *Isomorphically isometric probabilistic normed linear spaces*, Stochastica, **3** (1979), 71-77.

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