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# Left-invariant Lorentzian metrics on 3-dimensional Lie groups

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Riassunto: *Si determina i tensori di curvatura Riemanniana di tutte le metriche di Lorentz invarianti a sinistra nel gruppo di Lie tridimensionale.*

Abstract: *We find the Riemann curvature tensors of all left-invariant Lorentzian metrics on 3-dimensional Lie groups.*

## 1 – Introduction

The geometry of left-invariant Riemannian metrics on 3-dimensional Lie groups is so well understood that the theory can be regarded as essentially complete [9, §4]. For the other possible signature of metric tensors in dimension three, however, very little was known [2], [10]. In this paper we complete the study of this Lorentzian case, presenting a classification of all left-invariant Lorentzian metric tensors on Lie groups of dimension 3 by determining their curvatures and the symmetries of their sectional curvature functions.

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The motivation and methodology of our study derives from our classification of the symmetries of sectional curvature in dimension three, which we now recall. Let $M$ be a smooth 3-manifold and $g$ a pseudoriemannian metric tensor on $M$. Let $G_2(M)$ denote the Grassmannian bundle with fibers $G_2(T_xM)$, the space of (2-dimensional) planes in the tangent space $T_xM$ at a point $x \in M$. Observe that each $G_2(T_xM)$ may be regarded as a (real) algebraic variety, diffeomorphic to the (real) projective plane $\mathbb{P}^2$. As in [3], [4], [5], we shall regard the sectional curvature $K_x$ at each point $x \in M$ as a rational mapping of algebraic varieties $G_2(T_xM) \to \mathbb{R}$, or a rational function for short. The group of all automorphisms of $G_2(T_xM)$ is isomorphic to $PGL_3 \equiv PGL_3(\mathbb{R})$, the group of projective automorphisms of $\mathbb{P}^2$.

In [4] we determined the possible symmetry groups of $K$ at $x$; *i.e.*, the largest subgroup of $PGL_3$ which leaves $K_x$ invariant as a rational function. We shall refer to any one of these as a sectional curvature symmetry, or SCS for short. We also showed the existence of naturally reductive homogeneous spaces with constant SCS, and gave general descriptions of some examples of them. In [5], we exhibited explicit forms of the metric tensors on some of these examples. We also gave some inhomogeneous examples utilizing warped products, and began the study of how the SCS and CF-type can vary on a connected space.

In continuation of these two papers, we present here a classification of all left-invariant Lorentzian metric tensors on Lie groups of dimension 3 by determining the SCS of their curvatures. As in [4], [5], we defer the exhibition of more explicit forms of these metric tensors to a later article. As in [5], we present much of the information in the form of figures for greater efficiency (and, we hope, clarity). At present, there are only two subcases (out of almost 200) for which we have not yet determined the exact SCS; we hope to see these completed soon. We have given the full Riemann tensor in each case, however, so no traditional geometric information is lacking. (We have left it to the reader to work out the Ricci and scalar curvatures as exercises.)

Our Lorentzian metric tensors will have signature $+--$. When necessary, we distinguish among the possible orderings $+--$, $-+-$, $--+$. (To convert to the other signature convention $++-$, see [11, p. 92].) Thus a vector $v$ is *timelike* if $g(v,v) > 0$, *lightlike* or *null* if $g(v,v) = 0$, *spacelike* if $g(v,v) < 0$, and *causal* if $g(v,v) \geq 0$.

When convenient, we regard the Riemann tensor $R_{ijkl}$ as a quadratic form on $\bigwedge^2 TM$; cf. [3], [4]. In local coordinates,

$$R = \begin{bmatrix} R_{1212} & R_{1213} & R_{1223} \\ R_{1213} & R_{1313} & R_{1323} \\ R_{1223} & R_{1323} & R_{2323} \end{bmatrix} .$$

Then the sectional curvature appears as a rational function on $G_2(M)$ in the form of a quotient of two quadratic functions:

$$K = \frac{R}{\bigwedge^2 g} .$$

Here, we recall that if $A$ is a matrix regarded as a linear transformation $\mathbb{R}^n \to \mathbb{R}^n$, then the induced mapping $\bigwedge^2 A : \bigwedge^2 \mathbb{R}^n \to \bigwedge^2 \mathbb{R}^n$ is given by the matrix classically called the *second compound* of $A$, the matrix whose entries are the determinants of $2 \times 2$ submatrices of $A$ in an appropriate ordering [8, Sec. 7.2]. For example, $\bigwedge^2 \operatorname{diag}[1, -1, -1] = \operatorname{diag}[-1, -1, 1]$.

Also recall that the associated tensor $R^{ij}_{kl}$ represents the curvature operator $\bar{R}\colon \bigwedge^2 TM \to \bigwedge^2 TM$ in local coordinates. Note that if $R$ and $\bar{R}$ are written as matrices with respect to the same local coordinates, then $R = (\bigwedge^2 g)\bar{R}$.

We denote the Lorentz group in $(n = p + q)$ dimensions of signature $(p, q)$ by $O^q_p = O^q_p(\mathbb{R})$, thus the (usual) orthogonal group by $O_n = O_n(\mathbb{R})$. Projectivization of any group of linear transformations is indicated by a prefixed $P$; for example $PGL_3 = GL_3/\{aI\,;\, 0 \neq a \in \mathbb{R}\} \cong SL_3$.

Again, Parker thanks Cordero and the Departamento at Santiago for their extraordinary hospitality during his visits. He also apologizes for the delay of this paper due to his disability.

## 2 – Preliminary Recollections

For the convenience of the reader, we state some of the main results of [4] and [5]. This is Theorem 2.2 of [4]:

Theorem 2.1. *At each point $x$ of a Lorentzian 3-manifold $(M, g)$, there exists a choice of $g$-orthonormal coordinates with respect to which* *the Riemann tensor $R_x$ on $\bigwedge^2 T_x M$ takes on exactly one of these canonical forms:*

$$\begin{aligned}
&\text{CF1} \quad \operatorname{diag}[B, C, A];\\
&\text{CF2} \quad \begin{bmatrix} B & 0 & 0 \\ 0 & -A & F \\ 0 & F & A \end{bmatrix}, \quad F \neq 0;\\
&\text{CF3} \quad \begin{bmatrix} B & 0 & 0 \\ 0 & -\lambda \pm \frac{1}{2} & \pm\frac{1}{2} \\ 0 & \pm\frac{1}{2} & \lambda \pm \frac{1}{2} \end{bmatrix},\\
&\text{CF4} \quad \begin{bmatrix} -\lambda & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\lambda & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \lambda \end{bmatrix}.
\end{aligned}$$

We note that these forms can also be characterized in terms of eigenvectors of $\bar{R}$: timelike, spacelike, double null, and triple null, respectively; compare [7, § 4.3].

We also give Table 1 of [4].

**Table 1 – Lorentzian SCS**

| Canonical form of $R_x$ | Symmetry group of $K_x = R_x / \bigwedge^2 g_x$ |
|---|---|
| CF1 $\operatorname{diag}[B, C, A]$ | $A = -B = -C$   $PGL_3$<br>$B = C \neq -A$   $PO_2$<br>$A = -B \neq -C$, $A = -C \neq -B$   $PO_1^1$<br>generic   $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ |
| CF2 | $\mathbb{Z}_2$ |
| CF3 | $B \neq -\lambda$   $\mathbb{Z}_2$<br>$B = -\lambda$   $PHT$ |
| CF4 | 1 |

Recall the group $HT$ of horocyclic translations (called "null rotations" in relativity because there is a fixed null direction). The identity component of this group consists of the matrices

$$\exp\begin{bmatrix} 0 & -t & t \\ t & 0 & 0 \\ t & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -t & t \\ t & 1-\dfrac{t^2}{2} & \dfrac{t^2}{2} \\ t & -\dfrac{t^2}{2} & 1+\dfrac{t^2}{2} \end{bmatrix}, \ t \in \mathbb{R}.$$

Each component of $O_1^2(--+)$ contains one component of $HT$. See also [1], where these are called parabolic matrices, and [6].

In the Riemannian case, only CF1 and only the groups $PGL_3$ , $PO_2$ , and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ occur. The SCS is $PO_2$ whenever two of $A$, $B$, $C$ are equal but all three are not equal.

Next, we summarize Theorem 3.1 of [4].

THEOREM 2.2. *If $M = G$ is an irreducible, naturally reductive, Lorentzian homogeneous space of dimension 3, then either $M$ is flat or of constant negative curvature. In the former case, $M$ is Minkowskian 3-space or one of its quotients by a discrete group of translations. In the latter, $M$ is $SO_1^{2+}(\mathbb{R})$ or one of its coverings or quotients by a discrete subgroup.*

In [5] we began the study of how the SCS and CF-type can vary on a connected space. For later use and comparison, we briefly recall some of our results.

As in [5], we identify the space of possible curvature tensors at a point $p$ with $Sym_3$, the symmetric $3 \times 3$ matrices regarded as quadratic forms on $\bigwedge^2\mathbb{R}^3$. We coordinatize $Sym_3 \cong \mathbb{R}^6$ via (see [4], [5])

$$\begin{bmatrix} B & D & E \\ D & C & F \\ E & F & A \end{bmatrix} \longmapsto (B, C, A, D, E, F)\,.$$

If the metric tensor $g_p = \mathrm{diag}[1, -1, -1]$, then $\bigwedge^2 g_p = \mathrm{diag}[-1, -1, 1]$. A change of normal coordinates at $p$ acts on $T_pM \cong \mathbb{R}^3$ by an element of $O_1^2(+--)$, the Lorentz group for $g_p$. It is an easy exercise in linear algebra to verify that $\bigwedge^2 O_1^2(+--) = SO_1^2(--+)$. Thus we consider the action of $SO_1^2(--+)$ on $Sym_3$ where $A\cdot R = A^tRA$ for $A\in SO_1^2(--+)$ and $R\in Sym_3$.

Figure 1 from [5] shows how the orbits of CF1-types of $R$ are related to each other. It represents a plane perpendicular to the line of constant curvature in $BCA$-space. The view is from the third octant, toward the origin along this line. In the figure, "$\dim d$" means the full orbit in $\mathbb{R}^6$ has dimension $d$.

Figure 2 from [5] represents a plane parallel to $\{C = A\}$ in $CAF$-space. The view is from the fourth quadrant of the $CA$-plane toward the origin along the line $\{C = -A\}$. The lines have slopes of $\pm 1/\sqrt{2}$, and the two bullets for CF3 have coordinates $\pm(1/\sqrt{2}, 1/2)$. Considering the $B$-axis as perpendicular to the page, we have a representation of a translate of the 3-space with axes $B$, $C = A$, and $F$ along the $B$-axis in $BC\ A\ F$-space. The part of the $B$-axis with $B > C$ is $PO_1^1$ I and the part with $B < C$ is $PO_1^1$ II. The $SO_1^2$-orbits of $PO_2$ from CF1 are the portions of the elliptic cone $\{(2B - C + A)^2 + 4F^2 = (C + A)^2\}$ above and below the CF1 regions, $PO_2$ I to the right and $PO_2$ II to the left.

Figure 3 from [5] shows the three coordinate planes of $DEF$-space. Parts (a) and (b) have $B = C = -A$. The orbits of the other canonical forms are contained in the axes; for example, rotations of CF1 are in the $D$-axis, and those CF2 with $B = -A$ and certain boosts of CF3 are in the $F$-axis. Part (c) still has $B + C + 2A = 0$, but now we allow $B \neq C$. Note that the four CF3 points lie at $D = \pm 1$, not at $D = 0$; their $E$- and $F$-coordinates are $\pm 1/2\sqrt{2}$. The CF4 lines can be either CF4$^+$ or CF4$^-$, depending on the values of $B$ and $C$. The open regions are parts of the orbit of CF2, the others all being inside the axes. While rotations act naturally in $AEF$-space, their action in $DEF$-space is more complicated. Rotations acting here on CF4 sweep out the two branches of the quartic surface $D^2(E^2 + F^2) = (E^2 - F^2)^2$. We recall that a rotation acting through an angle $\theta$ clockwise about the $A$-axis in $AEF$-space, acts through an angle $2\theta$ counterclockwise about the line $\{B = C,\ D = 0\}$ in $BCD$-space.

For comparison with these, and for later use, we include contiguity relations for the Riemannian case in Figure 4. Here, one uses $\bigwedge^2 O_3 = SO_3$. The view is from the first octant back toward the origin along the line of constant curvature.

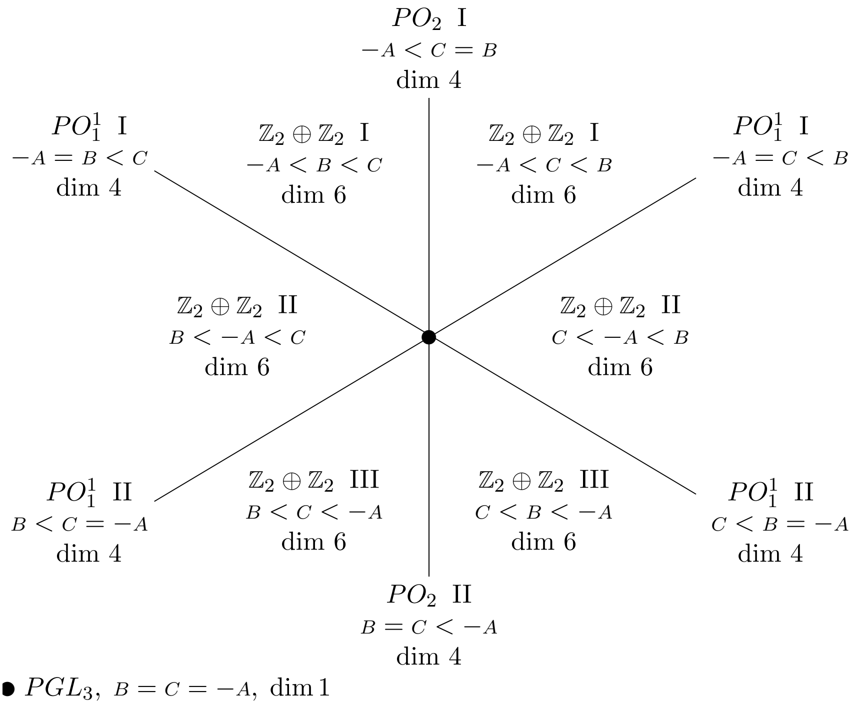


Fig. 1: Contiguity relations for CF1.

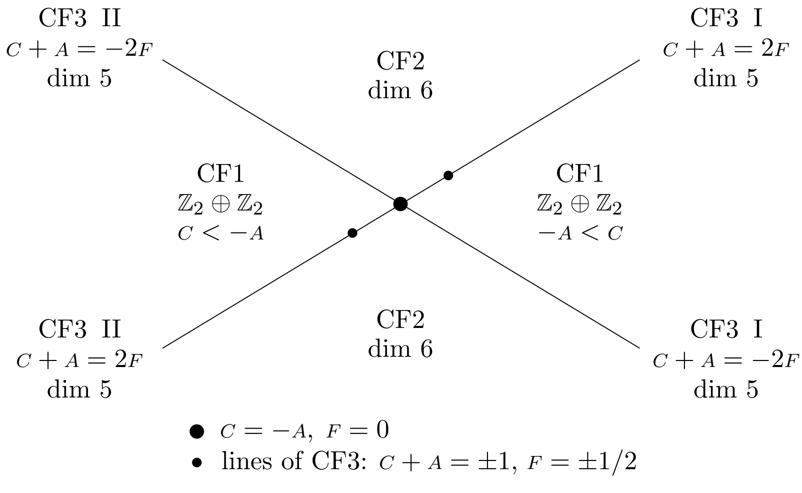


Fig. 2: Contiguity relations for CF2 and CF3.

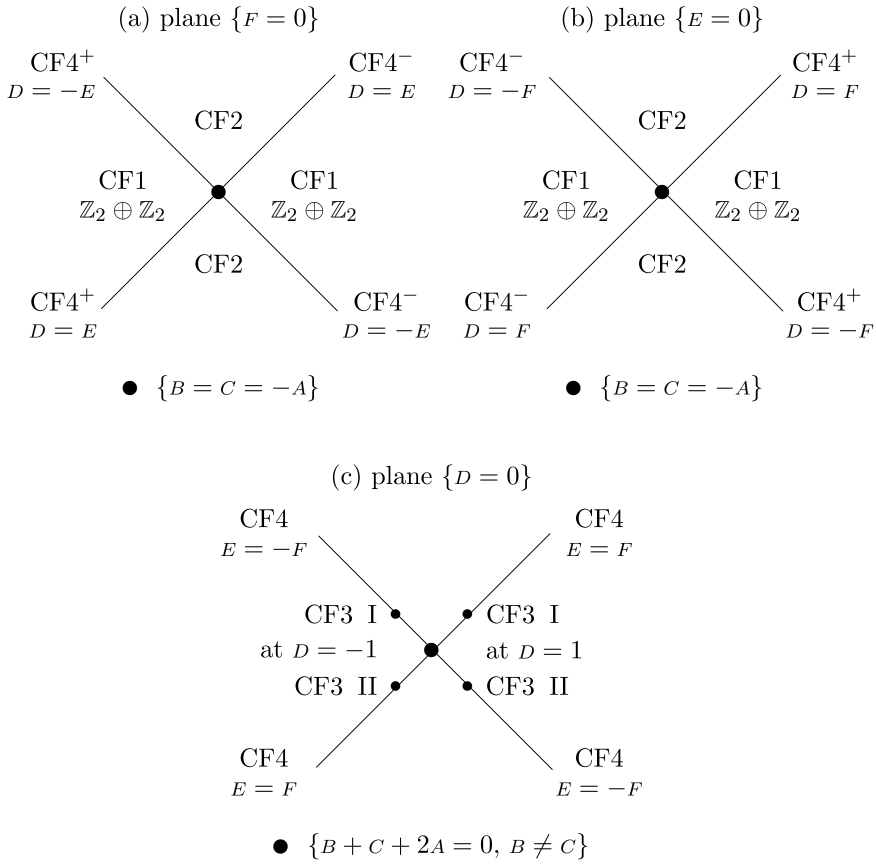


Fig. 3: Contiguity Relations for CF4.

## 3 – Unimodular Groups

We begin by considering $\mathbb{R}^3$ with the Lorentzian scalar product given by $\eta = \mathrm{diag}[\,1, -1, -1\,]$. Let $e_1, e_2, e_3$ denote the usual basis vectors of $\mathbb{R}^3$. The *Euclidean cross product* on $\mathbb{R}^3$ is then determined by $e_1 \times e_2 = e_3$, $e_2 \times e_3 = e_1$, and $e_3 \times e_1 = e_2$. We shall use the *Lorentzian cross product* determined in contrast by $e_1 \times e_2 = e_3$, $e_2 \times e_3 = -e_1$, and $e_3 \times e_1 = e_2$, extended to all of $\mathbb{R}^3$ as a bilinear and skewsymmetric operation; see [11, p. 262, ex. 22, and p. 124, ex. 7].

Now let $\mathfrak{g}$ be a 3-dimensional Lie algebra with a Lorentzian scalar product $\beta$. We choose a $\beta$-orthonormal basis $e_1, e_2, e_3$ and identify $\mathfrak{g}$ with $\mathbb{R}^3$ and $\beta$ with $\eta$.

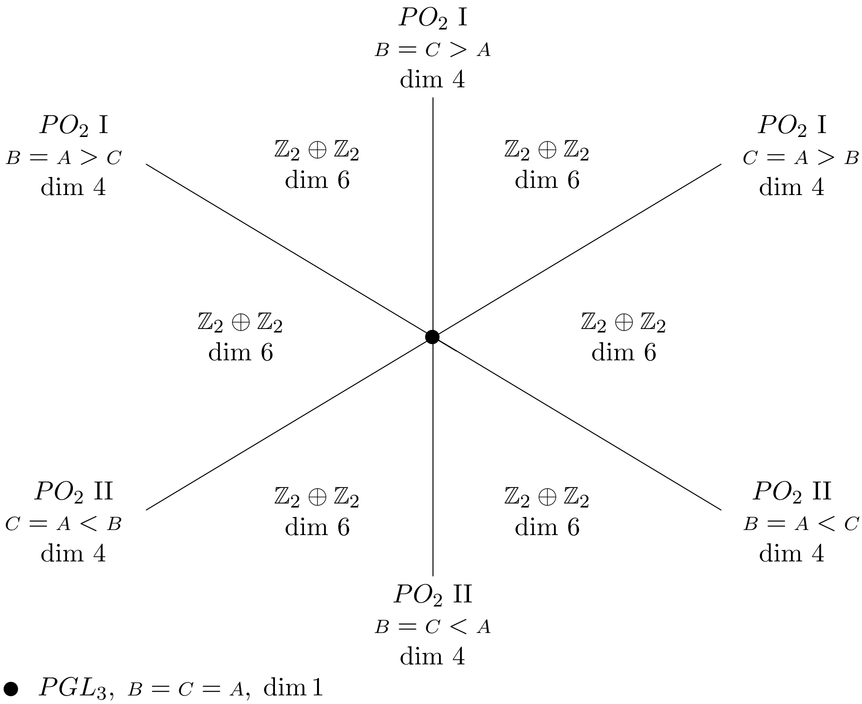


Fig. 4: Riemannian contiguity relations.

LEMMA 3.1. *The bracket operation on* $\mathfrak{g}$ *is related to the Lorentzian cross product by* $[u, v] = L(u \times v)$ *for a unique linear* $L : \mathfrak{g} \longrightarrow \mathfrak{g}$. *Further,* $\mathfrak{g}$ *is unimodular if and only if* $L$ *is* $\beta$*-selfadjoint.*

The proof is a direct parallel of the proof of Lemma 4.1 in [9] and we omit the details.

**Table 2 – Unimodular Lie algebras.**

| rk $E$ | def/indef | Lie alg | representative group and description |
|---|---|---|---|
| 3 | d | $\mathfrak{so}_3$ | $SO_3$ rigid motions of elliptic plane |
| 3 | i | $\mathfrak{so}_1^2$ | $SO_1^{2+}$ rigid motions of hyperbolic plane |
| 2 | d | $\mathfrak{e}_2$ | $E_2$ rigid motions of Euclidean plane |
| 2 | i | $\mathfrak{e}_1^1$ | $E_1^1$ rigid motions of Minkowski plane |
| 1 | – | $\mathfrak{h}_3$ | $H_3$ Heisenberg |
| 0 | – | $\mathbb{R}^3$ | $\mathbb{R}^3$ Abelian |

We call $L$ the *structure operator* since giving it is equivalent to giving the structure equations of $\mathfrak{g}$. We shall also use the analogous Euclidean selfadjoint operator $E$ determined *via* the Euclidean cross product defined with respect to the *same* basis of $\mathfrak{g}$, now regarded as a Euclidean orthonormal basis. (Note, this is the $L$ of [9].) Letting $\iota = \mathrm{diag}[-1, 1, 1]$, we find that $E = L\iota$. Observe that $E$ diagonalizes, since it is a Euclidean selfadjoint operator. To identify a unimodular Lie algebra with structure operator $L$, we form $E$, diagonalize it, and consult Table 2 (adapted from [9]).

From O'Neill [11, pp. 261–262, ex. 19], we find the following canonical forms for Lorentzian selfadjoint operators.

Lemma 3.2. *A Lorentzian selfadjoint operator on* $\mathbb{R}^3$ *appears, with respect to some* $\eta$*-orthonormal basis, in exactly one of these canonical forms:*

$$\text{SA1} \quad \mathrm{diag}[a, b, c]\,;$$

$$\text{SA2} \quad \begin{bmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix}, \quad b \neq 0\,;$$

$$\text{SA3} \quad \begin{bmatrix} a \pm \frac{1}{2} & \pm\frac{1}{2} & 0 \\ \mp\frac{1}{2} & a \mp \frac{1}{2} & 0 \\ 0 & 0 & b \end{bmatrix};$$

$$\text{SA4} \quad \begin{bmatrix} a & 0 & -\frac{1}{\sqrt{2}} \\ 0 & a & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & a \end{bmatrix}.$$

Thus, in order to determine all left-invariant Lorentzian metric tensors on unimodular 3-dimensional Lie groups, it suffices to study the four cases of $L$ given by the preceding lemma.

First, we determine which Lie algebras occur in each case. If the canonical form of $L$ is SA1, then $L = \mathrm{diag}[\,a, b, c\,]$ and $E = \mathrm{diag}[-a, b, c\,]$. The structure equations of $\mathfrak{g}$ are

$$
\begin{aligned}
[e_3, e_2] &= a\, e_1\,, \\
[e_3, e_1] &= b\, e_2\,, \\
[e_1, e_2] &= c\, e_3\,.
\end{aligned}
$$

Clearly, all the Lie algebras listed in Table 2 occur here. We shall see that $\mathfrak{so}_3$ and $\mathbb{R}^3$ occur only in this case.

When $L$ is SA2, the structure equations of $\mathfrak{g}$ are

$$
\begin{aligned}
[e_3, e_2] &= a\, e_1 + b\, e_2\,, \\
[e_3, e_1] &= -b\, e_1 + a\, e_2\,, \\
[e_1, e_2] &= c\, e_3\,.
\end{aligned}
$$

We find $\det(E - \lambda I) = (\lambda^2 - a^2 - b^2)(c - \lambda)$ and eigenvalues $c$, $\pm\sqrt{a^2 + b^2}$. Recalling that $b \neq 0$, we obtain $\mathfrak{so}_1^2$ when $c \neq 0$ and $\mathfrak{e}_1^1$ when $c = 0$.

When $L$ is SA3, the structure equations of $\mathfrak{g}$ are

$$
\begin{aligned}
[e_3, e_2] &= \Big(a \pm \frac{1}{2}\Big)\, e_1 \mp \frac{1}{2}\, e_2\,, \\
[e_3, e_1] &= \pm\frac{1}{2}\, e_1 + \Big(a \mp \frac{1}{2}\Big)\, e_2\,, \\
[e_1, e_2] &= b\, e_3\,.
\end{aligned}
$$

We find $\det(E - \lambda I) = (\lambda^2 \pm \lambda - a^2)(b - \lambda)$ and eigenvalues $b$, $\frac{1}{2}\left(\varepsilon \pm \sqrt{1 + 4a^2}\right)$ where $\varepsilon = \mp 1$ according to the sign choice in $L$. We obtain Lie algebras as follows:

$$
\begin{array}{lll}
 & b, a \neq 0 & \mathfrak{so}_1^2\,; \\[2ex]
(3.1) & b \neq 0,\ a = 0 & \begin{array}{c|cc} & \varepsilon = 1 & \varepsilon = -1 \\ \hline b < 0 & \mathfrak{e}_1^1 & \mathfrak{e}_2 \\ b > 0 & \mathfrak{e}_2 & \mathfrak{e}_1^1 \end{array}\ ; \\[2ex]
 & b = 0,\ a \neq 0 & \mathfrak{e}_1^1\,; \\[2ex]
 & b = a = 0 & \mathfrak{h}_3\,.
\end{array}
$$

Finally, when $L$ is SA4 the structure equations of $\mathfrak{g}$ are

$$[e_3, e_2] = a\, e_1 + \frac{1}{\sqrt{2}}\, e_3\,,$$

$$[e_3, e_1] = a\, e_2 + \frac{1}{\sqrt{2}}\, e_3\,,$$

$$[e_1, e_2] = -\frac{1}{\sqrt{2}}\, e_1 + \frac{1}{\sqrt{2}}\, e_2 + a\, e_3\,.$$

We find $\det(E-\lambda I) = -\lambda^3 + a\,\lambda^2 + (a^2+1)\,\lambda - a^3$. It is easy to verify, by using elementary calculus for example, that: when $a<0$ there are one positive and two negative eigenvalues; when $a=0$ there is one negative, one zero, and one positive eigenvalue; and when $a>0$ there are one negative and two positive eigenvalues. Thus, we obtain $\mathfrak{e}_1^1$ when $a=0$ and $\mathfrak{so}_1^2$ when $a\neq 0$.

Next, we determine the CF-types and SCS in each case. The Riemannian case may be somewhat familiar, however, so first we review in our setting the results of [9, §4] concerning sectional curvature on unimodular groups.

### 3.1 – The Riemannian case

Only SA1 occurs in the Riemannian case, so we have $E=\mathrm{diag}[a,b,c]$. We plot the information in *cab*-space, the ordering of coordinates being chosen for convenience in presenting the figures. $R$ is always diagonal here, so we need only the following formulas:

$$R_{1212} = \frac{c}{2}(a+b-c) - \frac{1}{4}(c-a+b)(c+a-b)\,,$$
$$R_{1313} = \frac{b}{2}(c+a-b) - \frac{1}{4}(c-a+b)(a+b-c)\,,$$
$$R_{2323} = \frac{a}{2}(c-a+b) - \frac{1}{4}(c+a-b)(a+b-c)\,.$$

Figure 5 shows the three coordinate planes. In each, $E_2$ is in the open quadrants I and III, $E_1^1$ is in II and IV, and $H_3$ is on the axes except at the origin (large bullet), which is the Abelian $\mathbb{R}^3$. The diagonal lines

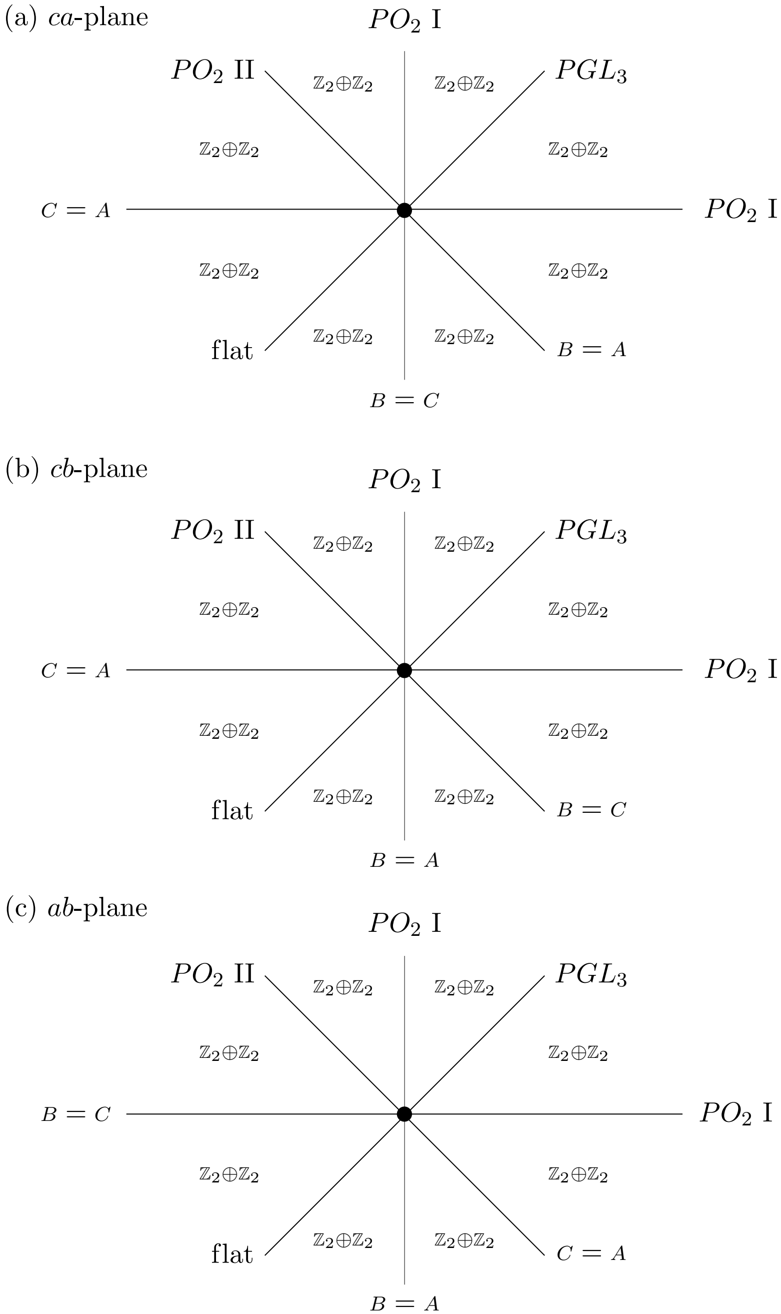


Fig. 5: $(+++)$ coordinate planes: Euclidean, Minkowskian, Heisenberg, and Abelian groups.

have slopes of $\pm 1$ and indicate metrics of particular SCS on $E_2$ and $E_1^1$. Each line (including the coordinate axes) is labeled with an SCS and (if appropriate) an equation in the coordinates of the 6-dimensional space of possible curvatures; cf. Section 2 and Figure 4 there. All the open regions have the same discrete SCS as indicated.

Of the other two representative Lie groups, $SO_3$ occurs in the open octants I and VII. In Figure 6, the view is of octant I looking back toward the origin along the line $c = a = b$ (large bullet) which indicates metrics of constant positive curvature. The lines in the figure now represent planes in $cab$-space, the open circles are intersections with coordinate planes, and the small bullets are lines of intersection in octant I. Each plane is labeled with the SCS, its equation in $cab$-space, and the equation of the curvature type in $BCA$-space; cf. Section 2. Note that the curvature type changes on passing through the line of constant curvature. To see octant VII, turn the figure upside down.

Finally, $SO_1^{2+}$ occurs in the remaining open octants. Figure 7 shows octants IV and VIII, somewhat more stylized than the other figures. Again, the lines represent planes which are labeled as before and the open circles are intersections with the coordinate planes. To see octants II and VI, turn the figure upside down.

As a simple application, we recover

THEOREM 3.3. *Among the unimodular Lie groups of dimension 3, only* $\mathbb{R}^3$, $E_2$, *and* $SO_3$ (*and their coverings and quotients*) *admit left-invariant Riemannian metrics of constant curvature. Those on* $\mathbb{R}^3$ *and* $E_2$ *are flat, and those on* $SO_3$ *have constant* (*strictly*) *positive curvature. No unimodular Lie group of dimension 3 admits a left-invariant Riemannian metric of* (*strictly*) *negative curvature.*

With only slightly more effort, one obtains the results on Ricci and scalar curvatures; we leave this to the reader.

### 3.2 – The Lorentzian case

We now turn to Lorentzian metrics and proceed by considering separately each of the canonical forms of the structure operator $L$.

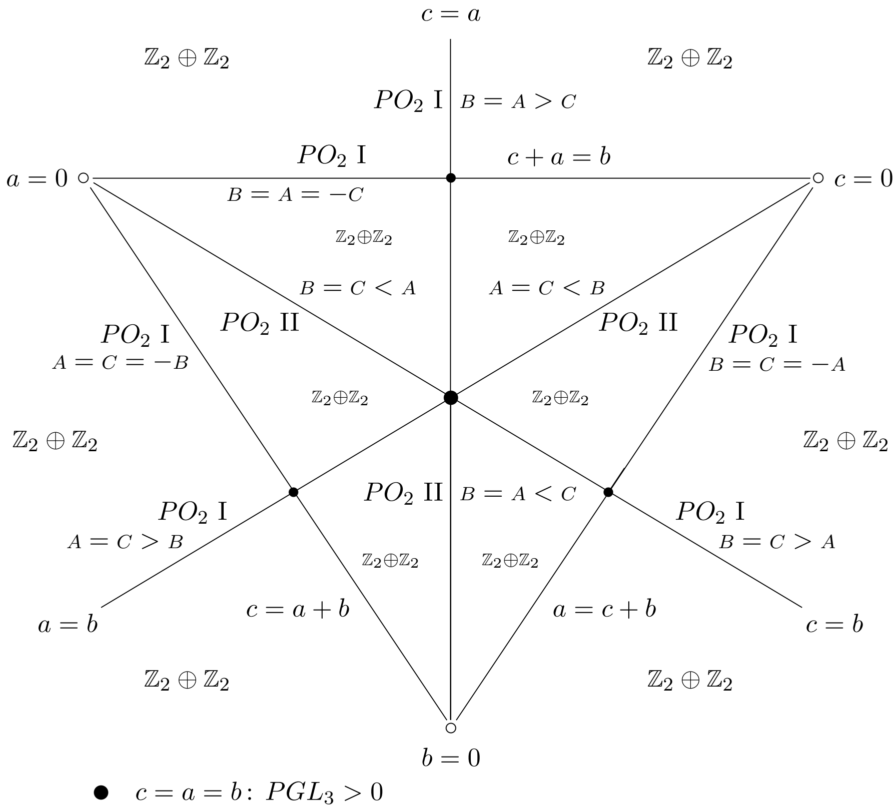


Fig. 6: $(+++)$ $SO_3$ in octant I.

*Canonical form SA1*

In this case $L = \mathrm{diag}[\,a, b, c\,]$ and $E = \mathrm{diag}[-a, b, c\,]$. To facilitate comparison with Riemannian metrics, we shall use the coordinate $\bar{a} = -a$ and plot the results in $c\bar{a}b$-space. We give results only for $(+--)$, those for the other two signatures $(-+-)$ and $(--+)$ being obtained by cyclic permutations of $a, b, c$. As in the Riemannian case, $R$ diagonalizes in this Lorentzian case (form CF1). We find

$$
\begin{aligned}
R_{1212} &= \frac{c}{2}(c+a-b) - \frac{1}{4}(c+a+b)(a+b-c)\,,\\
R_{1313} &= \frac{b}{2}(a+b-c) - \frac{1}{4}(c+a+b)(c+a-b)\,,\\
R_{2323} &= -\frac{a}{2}(c+a+b) - \frac{1}{4}(a+b-c)(c+a-b)\,.
\end{aligned}
$$

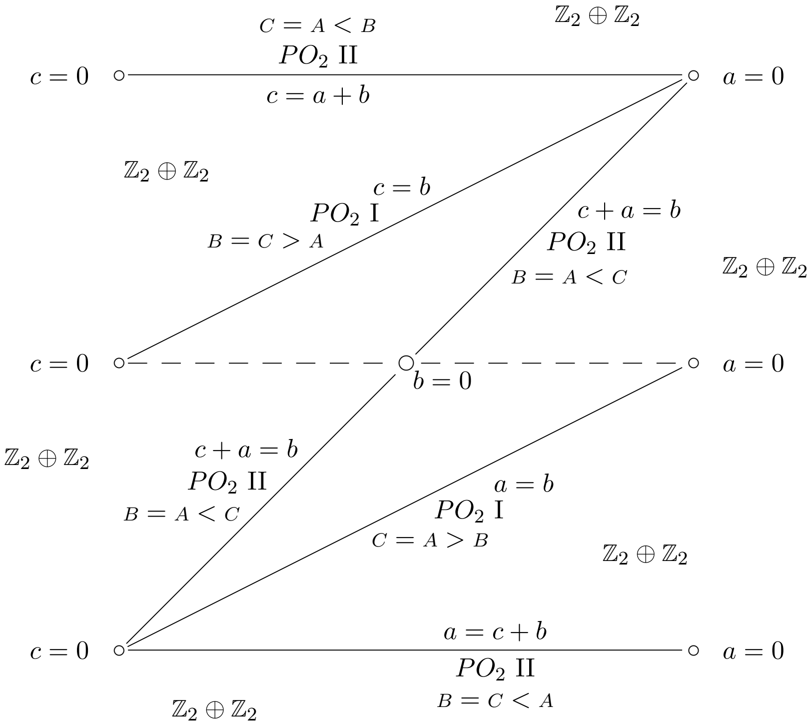


Fig. 7: $(+++)$ $SO_1^2$ in octants IV and VIII.

Figures 8, 9, and 10 are the Lorentzian versions of Figures 5, 6, and 7, with the same location of groups in Figures 5 and 8, and with the same labeling conventions in all. We observe now that in addition to $\mathbb{R}^3$ and $E_2$, $E_1^1$ also admits flat left-invariant Lorentzian metrics. This recovers parts (1) and (2) of Theorem 2 in [10]. We will recover part (3) from SA3.

Also, $SO_3$ now admits no left-invariant Lorentzian metric of constant curvature, but $SO_1^{2+}$ admits ones of constant positive curvature. This includes the results in Remarks 1 and 2 of [10], taking into account the differing signatures here $(+--)$ and there $(-++)$.

Unlike in the Riemannian case, $E_1^1$ and $SO_1^{2+}$ (or $SL_2$) are easy to distinguish here: the former admits a flat left-invariant Lorentzian metric and the latter does not; compare Corollary 4.7 in [9].

We may not yet formulate the Lorentzian version of Theorem 3.3, because there are still three cases of $L$ to consider.

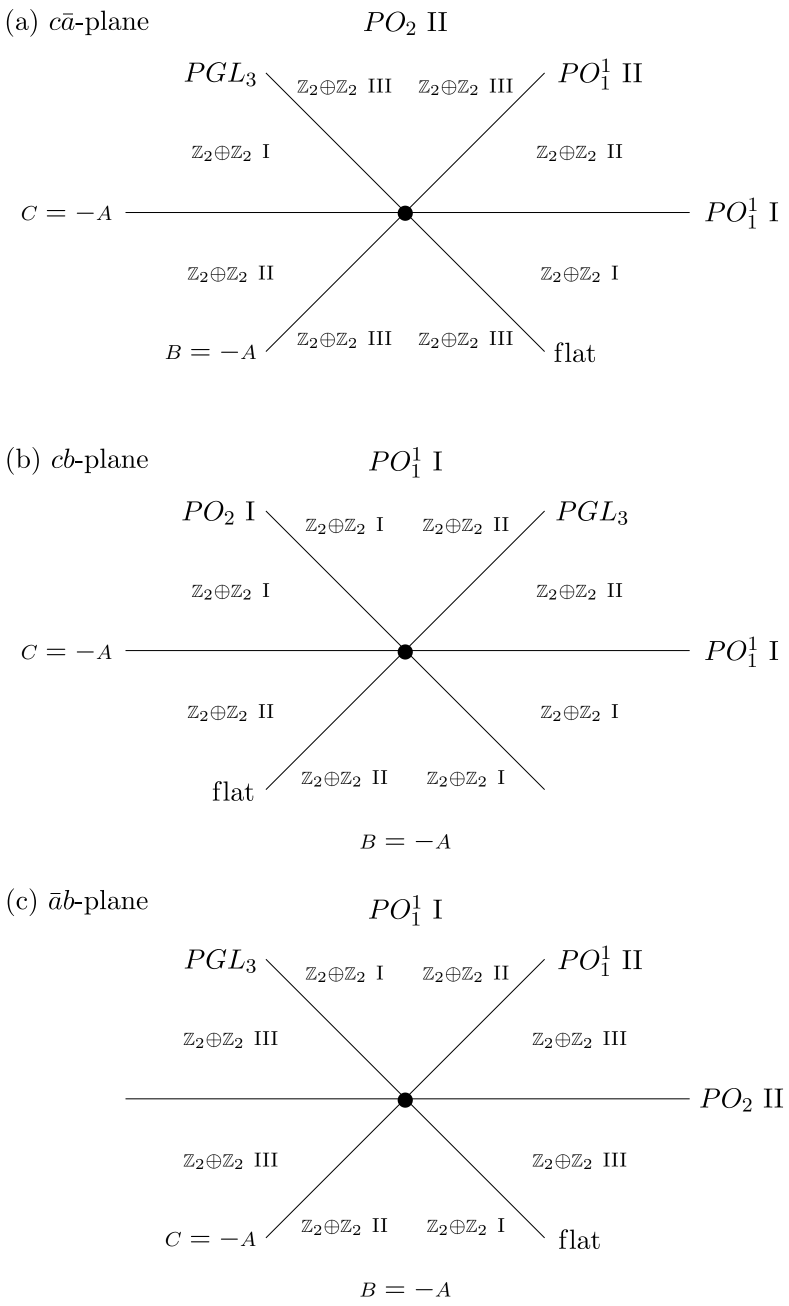


Fig. 8:SA1(+−−) coordinate planes: Euclidean, Minkowskian, Heisenberg, and Abelian groups.

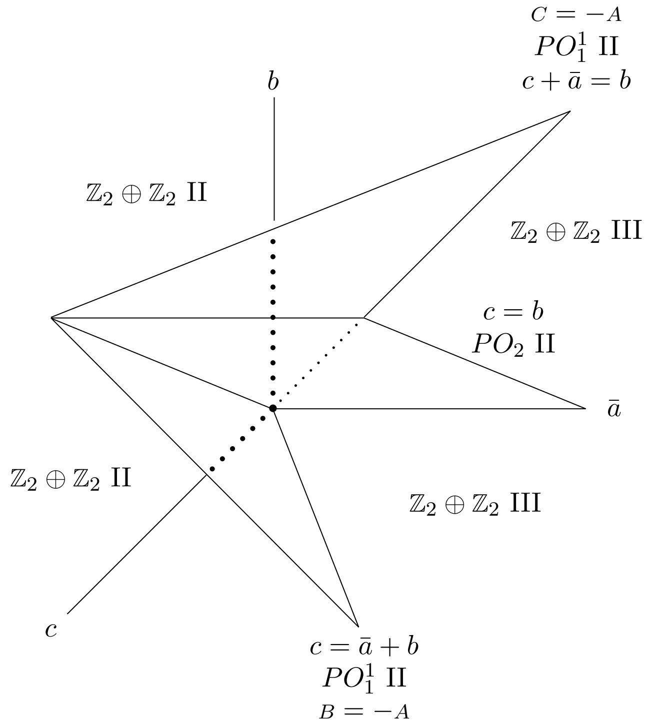


Fig. 9: SA1$(+--)$: $SO_3$ in octant I.

*Canonical form SA2*

We continue with $L$ of the form SA2. Calculating, we find the curvature tensor

$$R = \begin{bmatrix} b^2 + \dfrac{c^2}{4} - \dfrac{ac}{2} & 0 & 0 \\ 0 & -\left(b^2 + \dfrac{c^2}{4}\right) & b\,(2a - c) \\ 0 & b\,(2a - c) & b^2 + \dfrac{c^2}{4} \end{bmatrix}.$$

When $c = 2a$, we obtain $R = \operatorname{diag}[\, b^2, -(b^2 + a^2), b^2 + a^2 \,]$ of the type CF1 with SCS of $PO_1^1$ I and $-A = C < B$. For $a = 0$ this comes from a left-invariant Lorentzian metric on $E_1^1$, and for $a \neq 0$ from one on $SO_1^{2+}$. When $c \neq 2a$, $R$ is of the type CF2 with SCS of $\mathbb{Z}_2$. Again, the underlying group is $E_1^1$ or $SO_1^{2+}$ according as $c = 0$ or $c \neq 0$, respectively. Observe

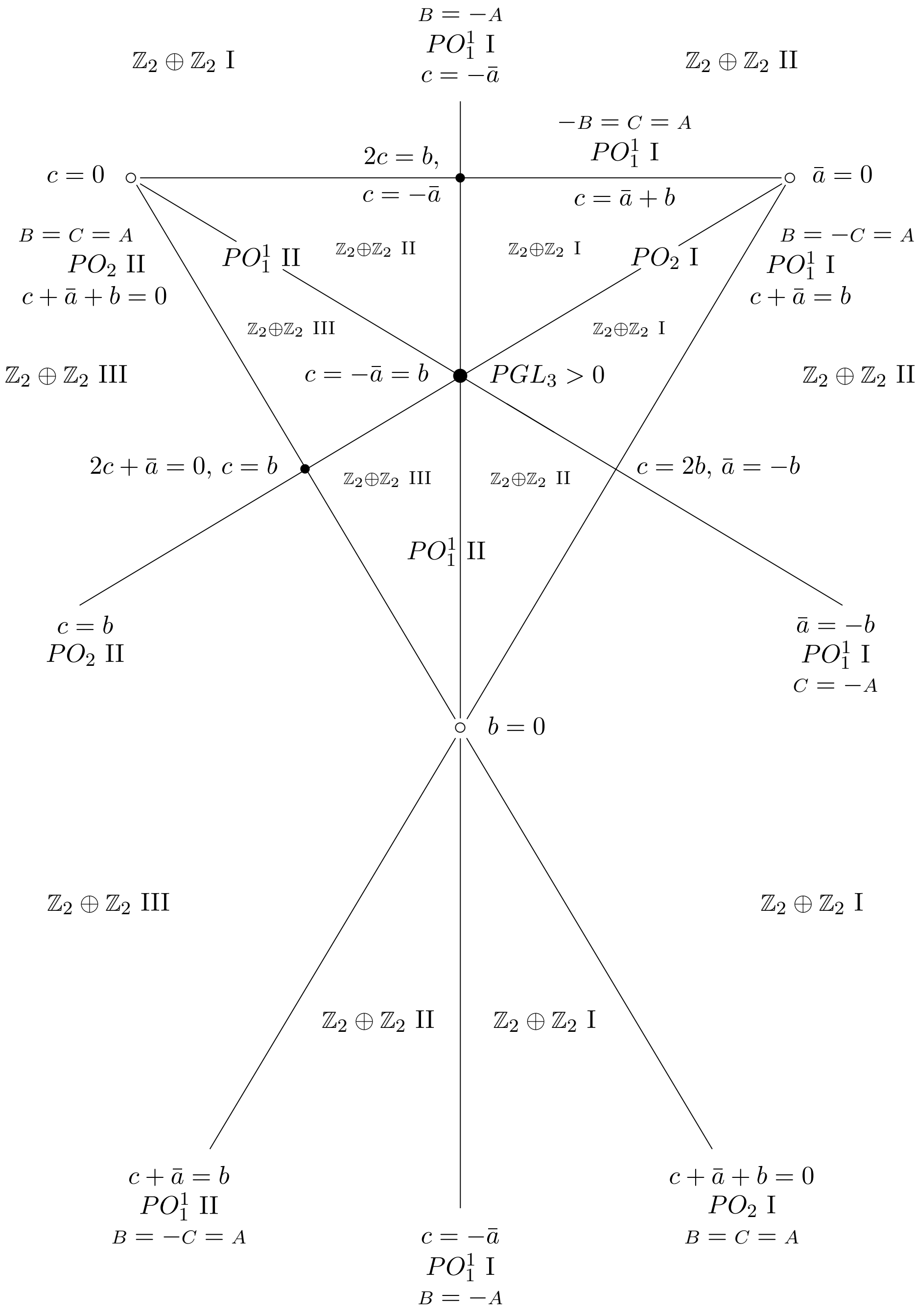


Fig. 10: SA1$(+--)$: $SO_1^2$ in octants IV and VIII

that we obtain only a special 3-parameter family of the full CF2. It is easy to verify that none of these occurred in SA1.

*Canonical form SA3*

Now we come to SA3. The curvature tensor is

$$R = \begin{bmatrix} \frac{3}{4}\,b^2 - ab & 0 & 0 \\ 0 & -\frac{b^2}{4} \pm \frac{1}{2}\,(b-2a) & \pm\frac{1}{2}\,(b-2a) \\ 0 & \pm\frac{1}{2}\,(b-2a) & \frac{b^2}{4} \pm \frac{1}{2}\,(b-2a) \end{bmatrix}.$$

When $b = 2a$, we obtain $R = \operatorname{diag}[\,a^2, -a^2, a^2\,]$ of the type CF1. For $a = 0$ this comes from a flat left-invariant Lorentzian metric on $H_3$, which recovers part (3) of Theorem 2 in [10]. For $a \neq 0$, it has SCS of $PO^1_1$ I with $B = -C = A$ and comes from a metric on $SO^{2+}_1$.

When $b \neq 2a$, there are three cases. For $b = 0$ it is a boosted type CF3 with SCS of $PHT$, and comes from a left-invariant Lorentzian metric on $E^1_1$. For $a = 0$ it is a boosted type CF3 with SCS of $\mathbb{Z}_2$ and comes from a metric on $E^1_1$ or $E_2$, depending on the sign of $b$ and the sign choice in $L$; cf. (3.1). For both $b, a \neq 0$ it is a boosted type CF3 and comes from a metric on $SO^{2+}_1$. When $b = a$ the SCS is $PHT$ and when $b \neq a$ the SCS is $\mathbb{Z}_2$.

*Canonical form SA4*

Finally, we consider SA4. The curvature tensor is

$$R = \begin{bmatrix} -\frac{a^2}{4} & -\frac{a}{\sqrt{2}} & -\frac{a}{\sqrt{2}} \\ -\frac{a}{\sqrt{2}} & 1 - \frac{a^2}{4} & 1 \\ -\frac{a}{\sqrt{2}} & 1 & 1 + \frac{a^2}{4} \end{bmatrix}.$$

When $a = 0$, this is a boosted type CF3 with SCS of $PHT$ and comes from a left-invariant Lorentzian metric on $E^1_1$. When $a \neq 0$, it comes from a metric on $SO^{2+}_1$. We have not been able to determine the exact SCS, but it is not $PGL_3$.

As promised earlier, we may now observe that indeed $\mathfrak{so}_3$ and $\mathbb{R}^3$ do occur only when $L$ is of type SA1. Finally, we give the Lorentzian version of Theorem 3.3.

THEOREM 3.4. *Among the unimodular Lie groups of dimension 3, only $SO_3$ (and its coverings and quotients) does* not *admit left-invariant Lorentzian metrics of constant curvature. Only those on $SO_1^2$ have constant (strictly) positive curvature; the rest are flat. No unimodular Lie group of dimension 3 admits a left-invariant Lorentzian metric of (strictly) negative curvature.*

With only slightly more effort, one can obtain results on Ricci and scalar curvatures; we leave this to the reader.

## 4 – Nonunimodular Groups

Now we consider the rest of the 3-dimensional Lie algebras $\mathfrak{g}$. We note in passing that these are in fact all solvable, and that $\mathfrak{e}_1^1$ and $\mathfrak{e}_2$ are the only other solvable algebras which are not nilpotent (or Abelian).

We shall exclude from our study the class $\mathfrak{S}$ of [9], [10]; its curvatures have already been completely determined [10]. Briefly, all left-invariant Lorentzian metrics on groups of this class have constant curvature, and this constant can be any real number.

Given a nonunimodular $\mathfrak{g}$ not in $\mathfrak{S}$, let $\mathfrak{u}$ denote its unimodular kernel,

$$\mathfrak{u} = \ker(\operatorname{tr}\operatorname{ad} : \mathfrak{g} \longrightarrow \mathbb{R}),$$

and let $\beta$ be a Lorentzian metric tensor on $\mathfrak{g}$. There are two cases.

If $\mathfrak{u}$ is not a null plane (*i.e.*, $\mathfrak{u}$ is not tangent to the lightcone of $\beta$ in $\mathfrak{g}$), then we can choose a $\beta$-orthonormal basis $e_1, e_2, e_3$ with $e_3 \perp \mathfrak{u}$ and $[e_1, e_3] \perp [e_2, e_3]$ in $\mathfrak{u}$, where $\perp$ denotes $\beta$-orthogonal. If $\mathfrak{u}$ is a spacelike plane, then $e_3$ is a timelike vector and we have the signature $--+$. If $\mathfrak{u}$ is a spacetime plane (also called a timelike plane in relativity), then $e_3$ is spacelike and we have the signature either $+--$ or $-+-$. Taking into account the semidirect product structure of our $\mathfrak{g}$, it easily follows that these last two signatures produce equivalent geometries. We shall consider only $+--$ explicitly and leave the transfer to $-+-$ to the reader.

If $\mathfrak{u}$ is a null plane, then we can choose a null basis $e_1, e_2, e_3$ with $e_3 \notin \mathfrak{u}$ a null vector, $e_1, e_2 \in \mathfrak{u}$, and with

$$\begin{aligned}\langle e_3, e_3\rangle = \langle e_2, e_2\rangle = \langle e_1, e_3\rangle = \langle e_1, e_2\rangle = 0\,,&\\ \langle e_3, e_2\rangle = -\langle e_1, e_1\rangle = 1\,,&\\ [e_1, e_3] \perp [e_2, e_3]\,.&\end{aligned}$$

In both cases the structure equations of $\mathfrak{g}$ are of the form

$$\begin{aligned}&[e_1, e_2] = 0\,,\\ &[e_1, e_3] = a\,e_1 + b\,e_2\,,\\ &[e_2, e_3] = c\,e_1 + d\,e_2\,, \quad a + d \neq 0\,,\end{aligned} \tag{4.1}$$

with at least two of $a, b, c, d$ different from zero (otherwise we are reduced to $H_3$ or $\mathbb{R}^3$). It follows from [9, p. 321] that nonunimodular $\mathfrak{g}$ with structure equations of this form are classified up to isomorphism by the invariant $D = (ad - bc)/(a + d)^2$.
Thus, we need consider only $\mathfrak{g}$ of the form (4.1) with $\beta = \mathrm{diag}[\,1, -1, -1]$, $\beta = \mathrm{diag}[-1, -1, 1\,]$, or

$$\beta = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

In the first case, the condition $[e_1, e_3] \perp [e_2, e_3]$ becomes $ac - bd = 0$. We find the curvature

$$R = \mathrm{diag}\Big[-\frac{1}{4}\,(b-c)^2 + ad,\ \frac{1}{4}\,(b-c)(3b+c) - a^2,\ \frac{1}{4}\,(b-c)(b+3c) + d^2\Big]$$

of type CF1. The SCS other than $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ are as follows:

$$bc + ad = b^2 - a^2 = c^2 - d^2 \qquad PGL_3 > 0\,;$$

$$\begin{aligned}&bc + ad = b^2 - a^2 \ \text{ and}\\ &\qquad b^2 - c^2 + d^2 - a^2 > 0 \qquad PO_2\ \mathrm{I}\,,\\ &\qquad b^2 - c^2 + d^2 - a^2 < 0 \qquad PO_2\ \mathrm{II}\,;\end{aligned}$$

$$
\begin{array}{lll}
bc + ad = c^2 - d^2 \ \text{ and} & & \\
\quad b^2 - c^2 + d^2 - a^2 > 0 & PO_1^1 \text{ I} & -A = B < C\,, \\
\quad b^2 - c^2 + d^2 - a^2 < 0 & PO_1^1 \text{ II} & C < B = -A\,; \\
b^2 - a^2 = c^2 - d^2 \ \text{ and} & & \\
\quad a^2 - b^2 + ad + bc > 0 & PO_1^1 \text{ I} & -A = C < B\,, \\
\quad a^2 - b^2 + ad + bc < 0 & PO_1^1 \text{ II} & B < C = -A\,.
\end{array}
$$

Figure 11 shows a slice $a = \text{const.} > 0$ in $abd$-space. There, $c > 0$ in quadrants I and III and $c < 0$ in quadrants II and IV. The $PO_2$ curve has the equation $d = a(b^2 - a^2)/(b^2 + a^2)$ and is asymptotic to $d = a$. The three pieces of $PO_1^1$ curve have the equation $d = a(b^2 + a^2)/(b^2 - a^2)$ and are asymptotic to $b = \pm a$ and $d = a$. To see $a < 0$, turn the figure upside down. Every nonunimodular Lie algebra not in $\mathfrak{S}$ is isomorphic to one with $a > 0$ and $-a < d \leq a$ [9]. Thus, this illustrates the result of [2] that Lie groups which admit left-invariant Riemannian metrics of constant negative curvature, admit left-invariant Lorentzian metrics of constant positive curvature.

Figure 12 shows the $bc$-plane regarded as the plane $a = d = 0$. Finally, we describe the remaining set where $a = b = 0$. When $d = \pm c \neq 0$ the SCS is $PGL_3$ with constant positive curvature. Otherwise the SCS is $PO_2$ I for $c^2 - d^2 < 0$ and $PO_2$ II for $c^2 - d^2 > 0$.

In the second case, the condition $[e_1, e_3] \perp [e_2, e_3]$ becomes $-ac - bd = 0$. We find the curvature

$$
R = \operatorname{diag}\left[\frac{1}{4}\,(b+c)^2 - ad,\ \frac{1}{4}\,(b+c)(3b-c) + a^2,\ -\frac{1}{4}\,(b+c)(b-3c) + d^2\right]
$$

of type CF1. Note carefully that $\bigwedge^2 \operatorname{diag}[-1,-1,1\,] = \operatorname{diag}[\,1,-1,-1\,]$, so the correct labeling is $R = \operatorname{diag}[A, B, C]$. When $a = b = 0$ the SCS is $PO_1^1$ I with $-A = B < C$. Figure 13 shows the $bc$-plane as the set $a = d = 0$. Finally, Figure 14 shows a slice $a > 0$ in $abd$-space. Now $c < 0$ in quadrants I and III and $c > 0$ in quadrants II and IV. To see $a < 0$, turn the figure upside down. This case contains the only instance of constant negative curvature we have observed so far, and illustrates the result in [2] that there are odd-dimensional Lie groups which admit

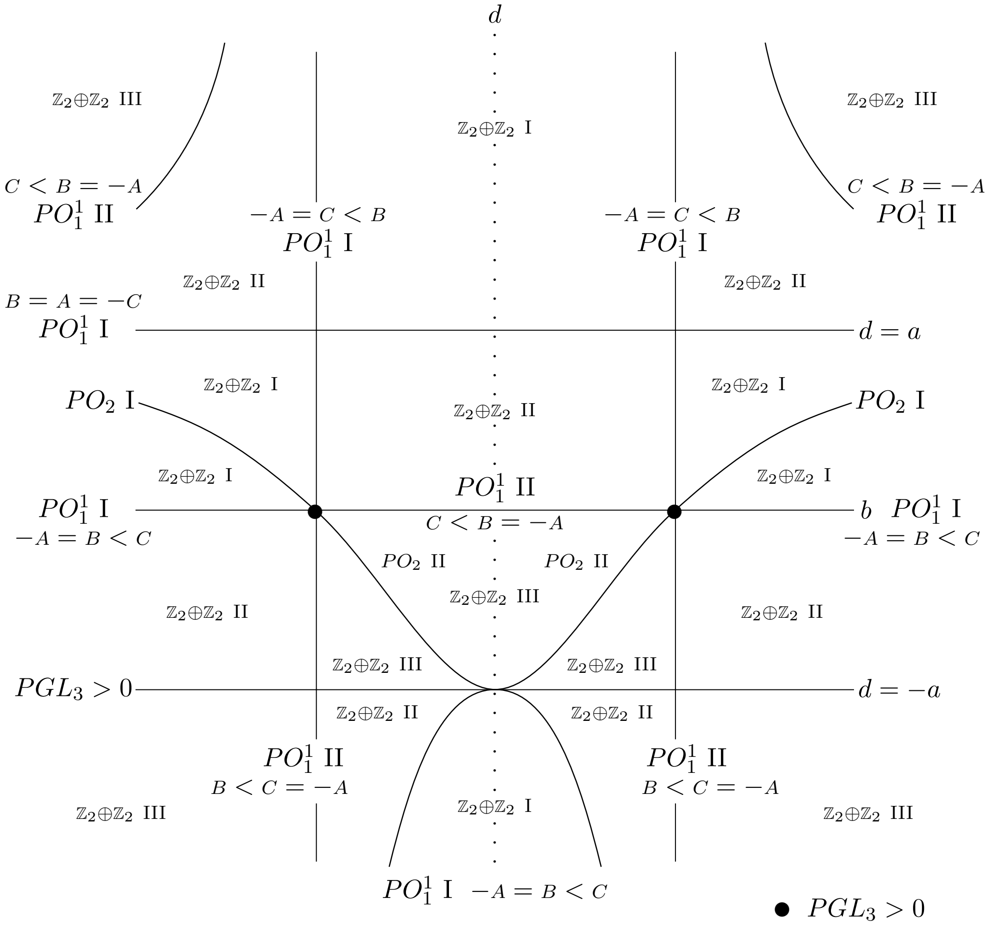


Fig. 11: Nonunimodular groups $+--$, $a \neq 0$.

left-invariant Riemannian metrics of constant negative curvature but do not admit left-invariant Lorentzian metrics of constant negative or zero curvature.

In the last case, the condition $[e_1, e_3] \perp [e_2, e_3]$ becomes $ac = 0$. We find the curvature

$$R = \begin{bmatrix} 0 & -\dfrac{c^2}{4} & 0 \\ -\dfrac{c^2}{4} & a(a-d) + bc & 0 \\ 0 & 0 & \dfrac{3c^2}{4} \end{bmatrix}.$$

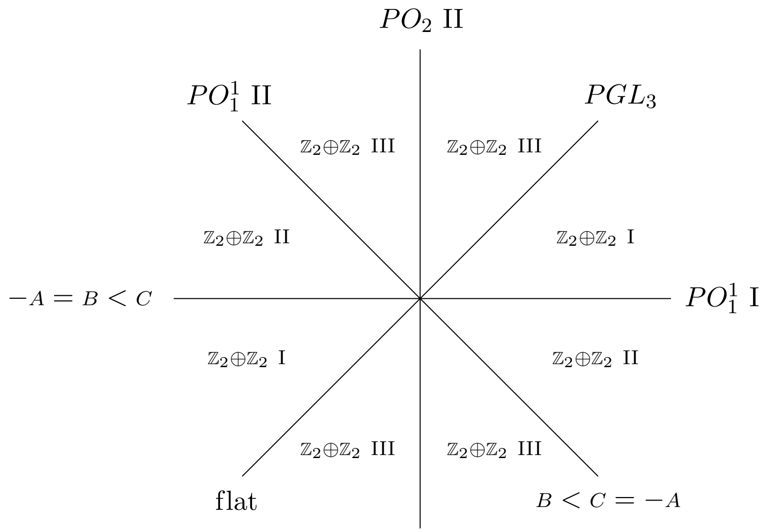


Fig. 12: Nonunimodular groups $(+--)$, $a = d = 0$.

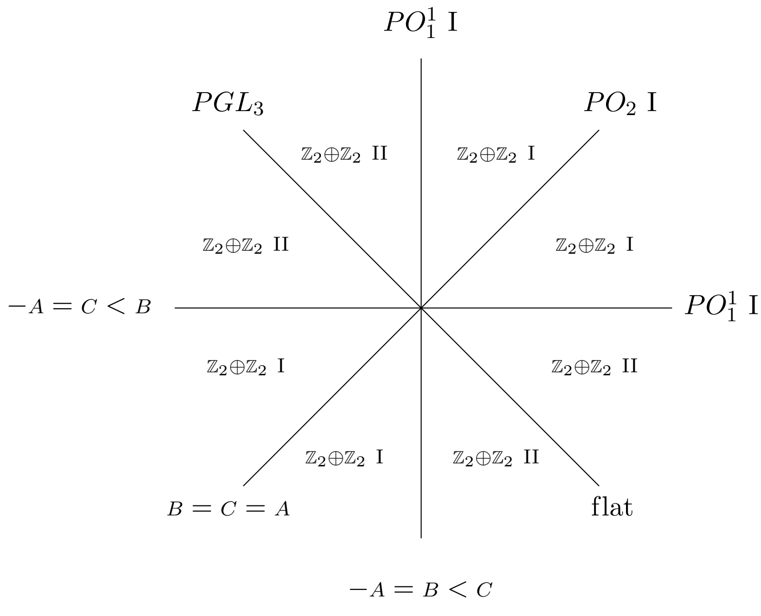


Fig. 13: Nonunimodular groups $(--+)$, $a = d = 0$.

When $a = c = 0$ the SCS is $PGL_3$ flat. When $a \neq 0 = c$, after a rotation and a boost we find the type CF3 with SCS of $PHT$. Finally, when $a = 0 \neq c$, we have not yet been able to determine the exact SCS.

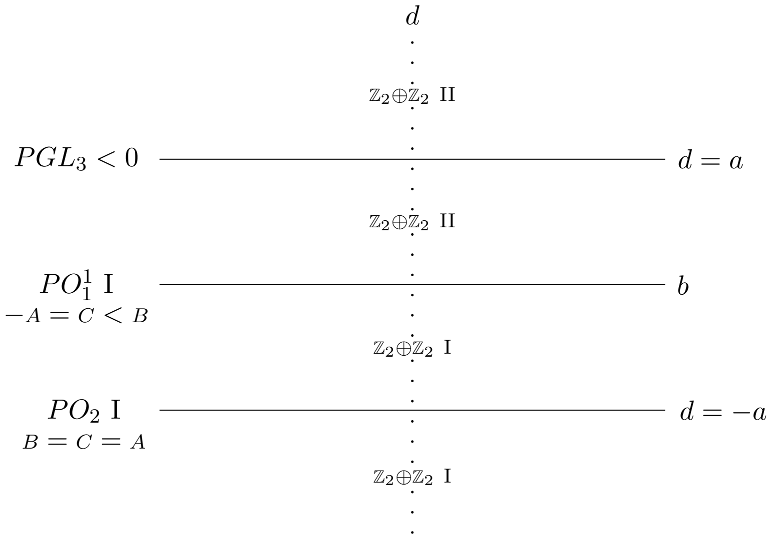


Fig. 14: Nonunimodular groups $--+$, $a \neq 0$.

## REFERENCES

[1] L. Auslander – L. Markus: *Flat Lorentz 3-manifolds*, Memoirs of the AMS 30. Providence: AMS, 1959.

[2] F. Barnet: *On Lie groups that admit left-invariant Lorentz metrics of constant sectional curvature*, Ill. J. Math., **33** (1989), 631-642.

[3] J. K. Beem – P. E. Parker: *Values of pseudoriemannian sectional curvature*, Comment. Math. Helv., **59** (1984), 319-331.

[4] L. A. Cordero – P. E. Parker: *Symmetries of sectional curvature on 3-manifolds*, Demonstratio Math., **28** (1995), 635-650.

[5] L. A. Cordero – P. E. Parker: *Examples of sectional curvature with prescribed symmetry on 3-manifolds*, Czech. Math. J., **45** (1995), 7-20.

[6] L. Greenberg: *Discrete subgroups of the Lorentz group*, Math. Scand., **10** (1962), 85-107.

[7] S. W. Hawking – G. F. R. Ellis: *The Large Scale Structure of Space-time*, Cambridge: U. P., 1973.

[8] N. Jacobson: *Basic Algebra I*, San Francisco: W. H. Freeman, 1974.

[9] J. Milnor: *Curvatures of left-invariant metrics on Lie groups*, Adv. in Math., **21** (1976), 293-329.

[10] K. Nomizu: *Left-invariant Lorentz metrics on Lie groups*, Osaka J. Math., **16** (1979), 143-150.

[11] B. O'Neill: *Semi-riemannian Geometry*, New York: Academic Press, 1983.

[12] J. A. Wolf: *Spaces of Constant Curvature*, 5-th ed. Boston: Publish or Perish, 1984.

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