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A unilateral problem for a nonlinear hyperbolic operator

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RIASSUNTO: *Si dimostra l'esistenza di soluzioni deboli per un problema con ostacolo associato ad un operatore iperbolico non lineare con smorzamento; i vincoli unilateri riguardano la funzione incognita.*

ABSTRACT: *We prove the existence of weak solutions for an obstacle problem associated to a nonlinear hyperbolic operator with damping; the unilateral constraints concern the unknown function.*

1 – Introduction

In [16] MEDEIROS and MILLA MIRANDA proved the global existence of solutions to the following hyperbolic mixed problem:

$$
(1.1)\quad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - M\left(\displaystyle\int_\Omega |\nabla u(x,t)|^2\,dx\right)\Delta u + (-\Delta)^\alpha \dfrac{\partial u}{\partial t} = f \text{ in } \Omega\times(0,\infty)\,,\\ u(x,0)=u_0(x),\ \dfrac{\partial u}{\partial t}(x,0)=u_1(x) \text{ in } \Omega\,,\\ u=0 \qquad \text{on } \Gamma\times(0,\infty)\,, \end{cases}
$$

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where: Ω is a bounded open subset of $\mathbb{R}^N$, with a smooth boundary $\partial\Omega = \Gamma$, $\Delta = \sum_{i=1}^N \partial^2/\partial x_i^2$, $|\nabla u(x,t)|^2 = \sum_{i=1}^N |\partial u/\partial x_i|^2$, $0 < \alpha \le 1$, $M(\rho)$ is a positive continuous function on $[0,\infty)$. Under further assumptions ($M \in C^1$ and $1/2 \le \alpha \le 1$), they also obtained the uniqueness.

Vasconcellos and Teixeira [21] considered the following nonlinear damped hyperbolic problem:

$$(1.2)\quad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - M\left(\displaystyle\int_\Omega |\nabla u(x,t)|^2\,dx\right)\Delta u - \Delta\dfrac{\partial u}{\partial t} + F(u) = 0 \text{ in } \Omega\times(0,\infty)\,, \\ u(x,0) = u_0(x),\ \dfrac{\partial u}{\partial t}(x,0) = u_1(x) \text{ in } \Omega\,, \\ u = 0 \qquad \text{on } \Gamma\times(0,\infty)\,, \end{cases}$$

where Ω, Γ, M are as above, while $F(u)$ is a suitable nonlinear function. They obtained some global existence and uniqueness theorems for strong solutions. Such theorems still hold in the case where the damping term in (1.2) is $(-\Delta)^\alpha \partial u/\partial t$ ($1/2 \le \alpha \le 1$) and when the equation in (1.2) is inhomogeneous like the one in (1.1). The proofs in [16] and in [21] are carried out in an abstract spaces framework, by using, as a main tool, the finite dimensional Galerkin approximation. The exponential decay of the energy is also obtained in both papers. When $N = 1$, problems like (1.1) and (1.2) arise in the study of vibrations of an elastic string (see, *e.g.*, Carrier [7]) with damping.

As far as we know, an obstacle problem associated to the previous Cauchy-Dirichlet ones has not been investigated yet. Let us consider the following unilateral hyperbolic mixed problem:

$$(1.3)\quad \begin{cases} a)\ \dfrac{\partial^2 u}{\partial t^2} - M\left(\displaystyle\int_\Omega |\nabla u(x,t)|^2\,dx\right)\Delta u - \Delta\dfrac{\partial u}{\partial t} \ge 0 \text{ and } u \ge 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. in } \Omega\times(0,T)\,, \\ b)\ \left(\dfrac{\partial^2 u}{\partial t^2} - M\left(\displaystyle\int_\Omega |\nabla u(x,t)|^2\,dx\right)\Delta u - \Delta\dfrac{\partial u}{\partial t}\right) u = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. in } \Omega\times(0,T)\,, \\ c)\ u(\cdot,0) = u_0 \ge 0,\ \left(\dfrac{\partial u}{\partial t}(\cdot,0) - u_1\right) \ge 0,\ \left(\dfrac{\partial u}{\partial t}(\cdot,0) - u_1\right) u_0 = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. in } \Omega\,, \\ d)\ u = 0 \qquad \text{on } \Gamma\times(0,T)\,; \end{cases}$$

here $0 < T < +\infty$; Ω, Γ, M are as above. The conditions (1.3) a) and (1.3) b) together describe the following situation: the unilateral constraint is $u(x,t) \geq 0$ and, when $u(x,t) > 0$, the model is governed by the equation

$$\frac{\partial^2 u}{\partial t^2} - M\left(\int_\Omega |\nabla u(x,t)|^2\,dx\right)\Delta u - \Delta\frac{\partial u}{\partial t} = 0 \quad \text{a.e. in } \Omega\times(0,T)\,,$$

while (1.3) c) accounts for the initial conditions accordingly with the unilateral problem.

The aim of this paper is to establish the existence of weak solutions (the definition will be given in the following section) to the obstacle problem (1.3), under the same assumptions on M, u_0, u_1, Ω taken in [21].

In the field of hyperbolic unilateral problems, the closest reference to (1.3) is the paper [13] by Jarušek, Málex, Nečas and Šverák; they proved the existence of weak solutions to the following hyperbolic obstacle problem (where n denotes the outward normal unitary vector to Γ):

$$(1.4)\quad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u - \Delta\dfrac{\partial u}{\partial t} \geq 0 \text{ and } u \geq 0 \text{ a.e. in } \Omega\times(0,T)\,, \\[2ex] \left(\dfrac{\partial^2 u}{\partial t^2} - \Delta u - \Delta\dfrac{\partial u}{\partial t}\right) u = 0 \text{ a.e. in } \Omega\times(0,T)\,, \\[2ex] u(\cdot,0) = u_0 \geq 0, \quad \left(\dfrac{\partial u}{\partial t}(\cdot,0) - u_1\right) \geq 0,\ \left(\dfrac{\partial u}{\partial t}(\cdot,0) - u_1\right) u_0 = 0 \\ \hfill \text{a.e. in } \Omega\,, \\[2ex] \dfrac{\partial}{\partial n}\left(\dfrac{\partial u}{\partial t} + u\right) \geq 0,\ u \geq 0,\ \left(\dfrac{\partial}{\partial n}\left(\dfrac{\partial u}{\partial t} + u\right)\right) u = 0 \text{ on } \Gamma\times(0,T)\,, \end{cases}$$

an existence result was also established in the case of homogeneous Dirichlet boundary conditions. This problem is a model for a viscous drum vibrating in the presence of an obstacle. Let us remark that (1.3) concerns the same phenomenon, but the nonlinear model can be deduced under less restrictive assumptions (also see Remark 3.2 below). In these models the unknown $u(x,t)$ denotes the (transversal) elongation of the point x at the time t.

The results in [13] are obtained by using, in particular, an approximation by a special penalty operator and a compactness lemma. We adapt here the procedure used in [13] with some modifications, due to the presence of the nonlinear function M.

We can put the problem (1.3) in the form of a variational inequality of hyperbolic type, with unilateral constraints on the unknown function u. Hyperbolic variational inequalities have been extensively investigated in the case where the unilateral constraints concern the time derivative of the unknown function (see, *e.g.*, [5], [6], [15]); on the other hand, some results are known in the case where the constraints are imposed on the unknown function: we mention, *e.g.*, [1], [2], [8], [18], [4].

The paper is structured as follows. The next section deals with the notation, the assumptions, the weak formulation of the problem (1.3), the statement of the existence result (Theorem 2.1) and of the main (abstract) result of [21], together with the corresponding abstract setting. In the Section 3, we carry out the proof of Theorem 2.1 in some steps: penalization method, a priori estimates, passage to the limit. Finally, we point out some direct extensions of our result and some open problems related to (1.3).

2 – Weak formulation. Statement of the main result

Let Ω be an open bounded subset of $\mathbb{R}^N$, with a boundary $\partial\Omega = \Gamma$ of class C^2. Let T be given, such that $0 < T < +\infty$. Let moreover M be a function such that:

$$M \in C^1([0,+\infty); \mathbb{R}) \text{ and } M(\rho) \geq m_0 > 0,\ \forall \rho \in [0,+\infty). \tag{2.1}$$

In the sequel we will denote $u' = \partial u/\partial t$, $u'' = \partial^2 u/\partial t^2$ and $\|\cdot\| = \|\nabla u\|_{L^2(\Omega)}$.

Definition 2.1. Let us suppose that $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$ and $u_1 \in H^1_0(\Omega)$. We say that the pair $(u(x,t), y(x))$ is a weak solution to the problem (1.3), if it satisfies the following conditions:

$$\begin{aligned} &\text{a) } u \in L^\infty(0,T; H^2(\Omega) \cap H^1_0(\Omega)), \\ &\text{b) } u' \in L^2(0,T; H^1_0(\Omega)) \cap L^\infty(0,T; L^2(\Omega))\,, \\ &\text{c) } y \in L^2(\Omega)\,, \\ &\text{d) } u \geq 0 \text{ a.e. in } \Omega \times (0,T)\,, \end{aligned} \tag{2.2}$$

$$
\begin{aligned}
&\int_0^T \int_\Omega [M(\|u\|^2)\nabla u\cdot\nabla(v-u)+\nabla u'\nabla(v-u)-u'(v'-u')]\,dx\,dt+\\
(2.3)\qquad &+\int_\Omega [y(x)(v(x,T)-u(x,T))-u_1(x)(v(x,0)-u_0(x))]\,dx\ge 0\,,\\
&\qquad\qquad\qquad\qquad\qquad\qquad \forall v\in K\,,
\end{aligned}
$$

where

$$
\begin{aligned}
(2.4)\qquad K=\{v|v\in L^2(0,T;H_0^1(\Omega)), v'\in L^2(0,T;L^2(\Omega))\\
\text{and } v\ge 0 \text{ a.e. in } \Omega\times(0,T)\}\,.
\end{aligned}
$$

Remark 2.1. The weak formulation (2.3) has been obtained from the "strong" formulation (1.3) in the following way: multiply the first inequality in (1.3) a) by $v\in K$, consider (1.3) b), integrate on Ω and from 0 to T, using the Green formula, the integration by parts, and the conditions in (1.3) c), d), and finally put $u'(x,T)=y(x)$. On the other hand, it can be seen that if $u(x,t)$, given in the Definition 2.1, is smooth enough, then it satisfies (1.3).

Now we give the statement of our main result; the proof will be carried out in the next section.

Theorem 2.1. *Let the previous assumptions on Ω and M hold. Let $u_0\in H^2(\Omega)\cap H_0^1(\Omega)$ and $u_1\in H_0^1(\Omega)$ be given, with $u_0\ge 0$ a.e. in Ω. Then there exists a weak solution $(u(x,t),y(x))$ of the problem (1.3), in the sense of the Definition 2.1 above.*

For the sake of completeness, we recall the precise statement of Theorem 2.1 of [21] (see Proposition 2.1 below); towards this aim we also have to introduce the abstract setting and some assumptions given in [21].

Let H be a real separable Hilbert space with inner product denoted by $(\cdot|\cdot)$ and norm $|\cdot|_H$. We consider a linear operator A in H, with the following properties:

$$(2.5)\qquad \text{The domain of } A,\ D(A), \text{ is dense in } H,$$

A is a self-adjoint operator and there exists a constant $C_0>0$ such that

$$(2.6)\qquad (Au|u)\ge C_0|u|_H^2\,,\quad \forall u\in H\,.$$

The Hilbert space $V = D(A^{1/2})$, with norm denoted by $\|v\|_V = |A^{1/2}v|_H$, for v in V, is compactly embeded on H. The inner product in V is denoted by $[\cdot|\cdot]$.

We consider a continuous mapping $F : V \to H$ satisfying:

$$\begin{aligned} &\forall c > 0 \ \exists \alpha_c \ \text{ such that} \\ &|F(u) - F(v)|_H \le \alpha_c \|u - v\|_V \, , \\ &\text{ if } u, v \in V \text{ and } \|u\|_V^2 + \|v\|_V^2 \le c \, . \end{aligned} \tag{2.7}$$

There exist constants $k_0 > 0\,, \ \ \mu > 0$ such that

$$\begin{aligned} &\int_0^t (F(u(s)), u'(s)) ds \ge -k_0 \|u(0)\|_V^\mu \, , \\ &\forall t > 0 \, , \ \forall u \in C^1([0, \infty); V) \, . \end{aligned} \tag{2.8}$$

For every $(u, v) \in D(A) \times V$ we have

$$F(u(\cdot)) \in V \, , \tag{2.9}$$

$$\begin{aligned} &\forall c > 0 \ \exists \beta_c \ \text{ such that} \\ &|[F(u(\cdot))|v]| \le \beta_c |Au|_H \, \|v\|_V \, , \quad \text{ if } \|u\|_V \le c \, , \end{aligned} \tag{2.10}$$

$$\begin{aligned} &\forall c > 0 \ \exists \gamma_c \ \text{ such that} \\ &\|F(u(\cdot))\|_V \le \gamma_c \, , \quad \text{ if } |Au|_H \le c \, . \end{aligned} \tag{2.11}$$

Now we are able to give the precise statement of the main result of [21].

PROPOSITION 2.1. *Let* (2.1) *and* (2.5-11) *hold. If* $(u_0, u_1) \in D(A) \times V$ *and* $\forall T > 0$, *there exists a unique* $u : [0, T] \to H$ *such that:*

$$u \in C^0([0, T]; D(A)) \cap C^1([0, T]; V) \, , \tag{2.12}$$

$$\forall t \in [0, T], \quad (u(t), u'(t), F(u(t))) \in D(A) \times D(A) \times V \, , \tag{2.13}$$

$$u' + Au \in C^1([0, T]; H) \ \textit{and} \ (u' + Au)' = u'' + Au' \, , \tag{2.14}$$

$$u'' \in L^2(0, T; H) \, , \tag{2.15}$$

and u *satisfies*

$$(2.16)\ \begin{cases} u''(t)+M(\|u(t)\|_V^2)Au(t)+F(u(t))+Au'(t)=0 \textit{ in } H\,,\ \forall t>0\,,\\ u(0)=u_0\,, \qquad u'(0)=u_1\,. \end{cases}$$

3 – Proof of Theorem 2.1

As we have already pointed out in the Introduction, the proof of Theorem 2.1 consists of the following steps: 1) penalization method; 2) a priori estimates; 3) passage to the limit.

1) We approximate our weak unilateral problem by a countable family of penalized equations (see (3.1) below) together with initial and boundary conditions (see (3.2) and (3.3) below). We establish an existence and uniqueness result for each penalized mixed problem (see Proposition 3.1 below), by applying Proposition 2.1.

2) We get some a priori estimates, concerning the boundedness of the solution u_k of the problem (3.1)-(3.2)-(3.3) in some suitable spaces, independently of the penalization parameter k.

3) From the boundedness established in the step 2), we deduce some weak and strong convergences; by using a compactness result given in [13], we get some further strong convergences, which allow us to complete the passage to the limit as $k\to+\infty$. The limit $\big(u(x,t),y(x)\big)$, obtained with this procedure, satisfies (2.2) and (2.3), and hence it is a weak solution of the problem (1.3), in the sense of the Definition 2.1.

3.1 – Penalization method

Let us take any integer $k\geq 1$. We approximate our unilateral problem by the following penalized one:

$$(3.1)\qquad u_k''-M(\|u_k\|^2)\Delta u_k-\Delta u_k'-ku_k^-=0 \quad \text{in } \Omega\times(0,T),$$

$$(3.2)\qquad u_k(\cdot,0)=u_0,\qquad u_k'(\cdot,0)=u_1 \qquad \text{in } \Omega\,,$$

$$(3.3)\qquad u_k=0 \qquad \text{on } \Gamma\times(0,T)\,,$$

where $\xi^-=\max\{-\xi,0\}$.

Now we want to apply the abstract result of the Proposition 2.1, in order to obtain the existence and the uniqueness of solutions to the mixed problem (3.1)-(3.2)-(3.3), *for every fixed* k (see Proposition 3.1 below). The application is as follows. Take $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, $A = -\Delta$, and, hence, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Clearly (2.6) is satisfied. Finally, we have to take $F(u_k) = -ku_k^-$; we remark at once that such $F(\cdot)$ is a continuous mapping from $H_0^1(\Omega)$ in $L^2(\Omega)$. It remains to show that $-ku_k^-$ satisfies (2.7-11) *for every fixed* k. Firstly, thanks to the Lipschitz continuity of $(\xi)^-$ with respect to ξ and to a straightforward application of the Green formula in (2.6) (or considering some results concerning equivalent norms in $H_0^1(\Omega)$), we have that (2.7) holds with $\alpha_c = C_0$ (and hence α_c is actually independent of c). Analogously, (2.8) holds with, *e.g.*, $\mu = 2$ and $k_0 = kC_0/2$, taking into account the identity $-u_k^- u_k' = 1/2[(u_k^-)^2]'$. If $u_k \in H^2(\Omega) \cap H_0^1(\Omega)$ clearly $-ku_k^- \in H_0^1(\Omega)$ and hence (2.9) holds. The use of the following inequality

$$\exists \tilde{c} \text{ such that } \quad \|u\|_{H_0^1(\Omega)} \leq \tilde{c} \|\Delta u\|_{L^2(\Omega)}$$

(which can be obtained, *e.g.*, applying the Green formula and the Hölder inequality) allows to prove (2.10) and (2.11) directly.

So, Proposition 2.1 yields, in particular, the following:

PROPOSITION 3.1. *Under the previous assumptions on* Ω *and* M, *if* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ *and* $u_1 \in H_0^1(\Omega)$, *then,* $\forall k \geq 1$, *there exists a unique* u_k, *such that:*

$$u_k \in C^0([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T]; H_0^1(\Omega)), \tag{3.4}$$

$$\forall t \in [0,T], \quad (u_k'(\cdot,t), u_k^-(\cdot,t)) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega), \tag{3.5}$$

$$(u_k' - \Delta u_k)' = u_k'' - \Delta u_k' \in C^0([0,T]; L^2(\Omega)), \tag{3.6}$$

$$u_k'' \in L^2(0,T; L^2(\Omega)), \tag{3.7}$$

and u_k *satisfies* (3.1)-(3.2)-(3.3).

3.2 – A priori estimates

I) We multiply the equation (3.1) by u_k', and we integrate on Ω and from 0 to t ($t \leq T$); defining $\hat{M}(\lambda) = \int_0^\lambda M(\rho)\,d\rho$, $\forall \lambda \geq 0$, after some calculations we obtain:

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega |u_k'(x,t)|^2\,dx + \frac{1}{2}\hat{M}\left(\int_\Omega |\nabla u_k(x,t)|^2\,dx\right) + \\
&+\int_0^t\int_\Omega |\nabla u_k'(x,\tau)|^2\,dx\,d\tau + \frac{k}{2}\int_\Omega [u_k(x,t)^-]^2\,dx \leq \\
&\leq \frac{1}{2}\int_\Omega |u_1|^2\,dx + \frac{1}{2}\hat{M}\Big(\int_\Omega |\nabla u_0|^2\,dx\Big) + \frac{k}{2}\int_\Omega [u_0^-]^2\,dx\,, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \forall t \in [0,T].
\end{aligned}
\tag{3.8}
$$

Recall that $u_0 \geq 0$ and remark that $\hat{M}(\lambda) \geq m_0\lambda$; thanks to the assumptions on u_0, u_1 and M, from (3.8) we deduce that there exists a positive constant c_1, $c_1 = c_1(u_0, u_1, M)$, but c_1 independent of k, such that:

$$\int_\Omega |\nabla u_k(x,t)|^2\,dx \leq c_1, \quad \forall t \in [0,T]\,; \tag{3.9}$$

$$\int_\Omega |u_k'(x,t)|^2\,dx \leq c_1, \quad \forall t \in [0,T]\,; \tag{3.10}$$

$$\int_0^t\int_\Omega |\nabla u_k'(x,\tau)|^2\,dx\,d\tau \leq c_1, \quad \forall t \in [0,T]\,; \tag{3.11}$$

$$k\int_\Omega [u_k(x,t)^-]^2\,dx \leq c_1, \quad \forall t \in [0,T]\,. \tag{3.12}$$

II) We multiply the equation (3.1) by $-\Delta u_k$; we integrate on Ω and from 0 to t ($t \leq T$); after some calculations we obtain:

$$
\begin{aligned}
&\int_0^t\int_\Omega M(\|u_k\|^2)|\Delta u_k(x,\tau)|^2\,dx\,d\tau + \frac{1}{2}\int_\Omega |\Delta u_k(x,t)|^2\,dx + \\
&-k\int_0^t\int_\Omega \nabla\big(u_k(x,\tau)^-\big)\cdot\nabla u_k(x,\tau)\,dx\,d\tau = \\
&= \frac{1}{2}\int_\Omega |\Delta u_0|^2\,dx + \int_\Omega [u_k'(x,t)\Delta u_k(x,t) - u_1\Delta u_0]\,dx + \\
&+\int_0^t\int_\Omega |\nabla u_k'(x,\tau)|^2\,dx\,d\tau, \quad \forall t \in [0,T]..
\end{aligned}
\tag{3.13}
$$

Since $-\int_\Omega \nabla(u_k^-)\cdot\nabla u_k\,dx = \int_\Omega \nabla(u_k^-)\cdot\nabla(u_k^-)\,dx$, and thanks to (3.10), (3.11), and to the assumptions on u_0, u_1 and M, from (3.13) we deduce that there exists a positive constant c_2, $c_2 = c_2(u_0, u_1, M)$, but c_2 independent of k, such that:

$$\int_\Omega |\Delta u_k(x,t)|^2\,dx \le c_2, \quad \forall t \in [0,T]\,. \tag{3.14}$$

III) Our goal is to establish the boundedness of $\{u_k''\}$ in a suitable space; our strategy is to look for the boundedness of $\{ku_k^-\}$, and then to use the equation (3.1).

Let us consider the equation (3.1); since the conclusions of Proposition 3.1 hold, in particular (3.6), we have:

$$\begin{aligned} 0 \le \int_0^t\int_\Omega ku_k(x,\tau)^-\,dx\,d\tau = \int_\Omega \big[u_k'(x,t) - u_1 +& \\ -\int_0^t M(\|u_k\|^2)\Delta u_k(x,\tau)\,d\tau - \Delta u_k(x,t) + \Delta u_0\big]\,dx\,&, \\ \forall t\in[0,T]\,&. \end{aligned} \tag{3.15}$$

Thanks to (3.10), (3.9), (3.14), and to the assumptions on u_0, u_1 and M, from (3.15) we deduce that there exists a positive constant c_3, $c_3 = c_3(u_0, u_1, M)$, but c_3 independent of k, such that:

$$\int_0^t\int_\Omega |ku_k(x,t)^-|\,dx\,d\tau \le c_3, \quad \forall t\in[0,T]\,. \tag{3.16}$$

Let us go back to the equation (3.1); we can deduce the boundedness of $\{u_k''\}$ from the just established boundedness of the other terms: if we take $s > N/2$, thanks to (3.9), (3.14), (3.11), (3.16) and to the continuity of M, we have that:

$$\{u_k''\}_{k\ge1} \quad \text{is bounded in } L^1(0,T;H^{-s}(\Omega))\,, \tag{3.17}$$

where, of course, $H^{-s}(\Omega) = \big(H_0^s(\Omega)\big)^*$, and since $L^1(\Omega) \hookrightarrow H^{-s}(\Omega)$, when $s > N/2$.

3.3 – Passage to the limit

Thanks to (3.9), (3.14), (3.10), (3.11), we can extract from $\{u_k\}_{k\geq 1}$ a subsequence, still denoted by $\{u_k\}$, such that, as $k \to +\infty$:

$$u_k \to u \quad \text{weakly star in } L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega))\,; \tag{3.18}$$

$$u_k' \to u' \quad \text{weakly star in } L^\infty(0,T;L^2(\Omega)) \text{ and weakly in } L^2(0,T;H_0^1(\Omega)); \tag{3.19}$$

$$u_k'(x,T) \to y(x) \quad \text{weakly in } L^2(\Omega); \tag{3.20}$$

$$u_k(x,T) \to u(x,T) \quad \text{weakly in } H_0^1(\Omega). \tag{3.21}$$

By using a classical compactness argument, from (3.18) and (3.19) we have that

$$u_k \to u \quad \text{strongly in } L^2(0,T;H_0^1(\Omega))\,; \tag{3.22}$$

thanks to (3.22) and to (2.1), we also have that:

$$M(\|u_k\|^2) \to M(\|u\|^2) \quad \text{strongly in } L^2(0,T)\,. \tag{3.23}$$

Now we recall a quite general compactness result proved in [13] (see Theorem 3.1).

Proposition 3.2. *Let $B_0 \hookrightarrow\hookrightarrow B \hookrightarrow B_1$ be Banach spaces, with B_0 reflexive and separable. Let $1 < p < \infty$, $1 \leq q < \infty$. Then:*

$$W \equiv \{w | w \in L^p(0,T;B_0),\ w' \in L^q(0,T;B_1)\} \hookrightarrow\hookrightarrow L^p(0,T;B), \tag{3.24}$$

where $\hookrightarrow\hookrightarrow$ denotes compact embedding.

We apply Proposition 3.2 in the case where: $B_0 = H_0^1(\Omega)$, $B = L^2(\Omega)$, $B_1 = H^{-s}(\Omega)$, $p = 2$, $q = 1$; thanks to (3.19) and (3.17), we get that:

$$u_k' \to u' \quad \text{strongly in } L^2(0,T;L^2(\Omega)); \tag{3.25}$$

from (3.25) we also have that

$$\begin{aligned} &u_k(\cdot,t) \to u(\cdot,t) \quad \text{strongly in } L^2(\Omega), \quad \forall t \in [0,T] \\ &(\text{in particular: } u_k(x,T) \to u(x,T) \quad \text{strongly in } L^2(\Omega))\,. \end{aligned} \tag{3.26}$$

Now we multiply the penalized equation (3.1) by $v - u_k$, with $v \in K$ (see (2.4)); we integrate on Ω and from 0 to T. Since $v^- = 0$ a.e. in $\Omega \times (0,T)$, we have that:

$$\begin{aligned} 0 \le \int_0^T \int_\Omega k u_k^- (v - u_k)\, dx\, dt = \int_0^T \times \\ \times \int_\Omega \Big[u_k''(v-u_k) - M(\|u_k\|^2)\Delta u_k (v-u_k) - \Delta u_k'(v-u_k)\Big]\, dx\, dt\,. \end{aligned} \tag{3.27}$$

We take the lim inf, as $k \to +\infty$, in (3.27); after some calculations, taking into account the properties of u_k, we obtain:

$$\begin{aligned} \liminf_{k\to+\infty} \Big\{ \int_0^T \int_\Omega &[|u_k'|^2 - u_k' v' + \\ &+ M(\|u_k\|^2)\nabla u_k \cdot \nabla v - M(\|u_k\|^2)\nabla u_k \cdot \nabla u_k + \\ &+ \nabla u_k' \cdot \nabla v]\, dx\, dt + \\ &+ \int_\Omega [u_k'(x,T)(v(x,T) - u_k(x,T)) - u_1(v(x,0) - u_0)]\, dx \Big\} \ge \\ &\ge \liminf_{k\to+\infty} \int_\Omega \Big[\frac{1}{2}|\nabla u_k(x,T)|^2 - \frac{1}{2}|\nabla u_0|^2\Big]\, dx\,. \end{aligned} \tag{3.28}$$

Thanks to (3.25), (3.23), (3.18), (3.22), (3.19), (3.20), (3.26), the left hand side of (3.28) (is, in fact, a limit, and it) is equal to

$$\begin{aligned} \int_0^T \int_\Omega &[|u'|^2 - u'v' + M(\|u\|^2)\nabla u \cdot \nabla v - M(\|u\|^2)\nabla u \cdot \nabla u + \\ &+ \nabla u' \cdot \nabla v]\, dx\, dt + \\ &+ \int_\Omega [y(x)(v(x,T) - u(x,T)) - u_1(v(x,0) - u_0)]\, dx\,; \end{aligned} \tag{3.29}$$

as for the right hand side of (3.28), we remark that, thanks to (3.21),

$$\liminf_{k\to+\infty} \int_\Omega \frac{1}{2}|\nabla u_k(x,T)|^2\, dx \ge \int_\Omega \frac{1}{2}|\nabla u(x,T)|^2\, dx\,. \tag{3.30}$$

Now, taking into account (3.29), (3.30), from (3.28) we deduce that:

$$
(3.31)\quad
\begin{aligned}
&\int_0^T \int_\Omega [|u'|^2 - u'v' + M(\|u\|^2)\nabla u \cdot \nabla v - M(\|u\|^2)\nabla u \cdot \nabla u+\\
&+ \nabla u' \cdot \nabla v] \times dx\,dt+\\
&+ \int_\Omega [y(x)(v(x,T) - u(x,T)) - u_1(v(x,0) - u_0)]\,dx \geq\\
&\geq \int_\Omega \left[\frac{1}{2}|\nabla u(x,T)|^2 - \frac{1}{2}|\nabla u_0|^2\right] dx = \int_0^T \int_\Omega \nabla u' \cdot \nabla u\,dx\,dt\,;
\end{aligned}
$$

from (3.31) we can easily obtain (2.3).

From (3.18), (3.19), (3.20) we have, respectively, (2.2) a), (2.2) b), (2.2) c); in order to obtain (2.2) d), by (3.12), (3.22) and the Fatou Lemma, we have that:

$$
(3.32)\qquad 0 = \liminf_{k\to+\infty} \int_0^T \int_\Omega [u_k(x,t)^-]^2\,dx\,dt \geq \int_0^T \int_\Omega [u(x,t)^-]^2\,dx\,dt\,;
$$

hence the proof of Theorem 2.1 is complete.

Remark 3.1. Let us consider the following obstacle problem:

$$
(3.33)\quad
\begin{cases}
\dfrac{\partial^2 u}{\partial t^2} - M\left(\displaystyle\int_\Omega |\nabla u(x,t)|^2\,dx\right)\Delta u - \Delta\dfrac{\partial u}{\partial t} \leq 0 \text{ and } u \leq 0 \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. in } \Omega \times (0,T)\,,\\
\left(\dfrac{\partial^2 u}{\partial t^2} - M\left(\int_\Omega |\nabla u(x,t)|^2\,dx\right)\Delta u - \Delta\dfrac{\partial u}{\partial t}\right) u = 0\\
\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. in } \Omega \times (0,T)\,,\\
u(\cdot,0) = u_0 \leq 0,\ \left(\dfrac{\partial u}{\partial t}(\cdot,0) - u_1\right) \leq 0,\ \left(\dfrac{\partial u}{\partial t}(\cdot,0) - u_1\right) u_0 = 0\\
\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. in } \Omega\,,\\
u = 0 \text{ on } \Gamma \times (0,T)\,;
\end{cases}
$$

clearly, with some obvious changes in the preceding proof and in the weak formulation, we can obtain an existence result for weak solutions to the problem (3.33), like the one in Theorem 2.1. In the same way, we can also obtain an analogous result, when the main unilateral constraint is expressed by $u \geq c_1(x)$ or $u \leq c_2(x)$ (where $c_1(x) \leq 0$ or $c_2(x) \geq 0$ is suitably regular in Ω, e.g. belongs to $H^2(\Omega)$), and the other unilateral conditions are modified accordingly.

Moreover, we observe that our existence result still holds in other inhomogeneous cases, i.e. when a (suitably regular) datum $f(x,t)$ also appears in (1.3) a) and in (1.3) b).

REMARK 3.2. As we have already remarked in the Introduction, equations like the ones in (1.1) and (1.2) are modifications —via the addition of a damping term— of the following equation:

$$\frac{\partial^2 u}{\partial t^2} - M\left(\int_\Omega |\nabla u(x,t)|^2\, dx\right)\Delta u = 0, \tag{3.34}$$

which was proposed by Kirchhoff (see, *e.g.*, [19]) in order to describe the small, transversal vibrations of an elastic string and it can be deduced under the assumption that Hooke's law holds and that the stress can be approximated by its x-average. Cauchy and mixed problems associated to (3.34) have been extensively studied by many authors in various settings: we mention, among the other references, [3], [9], [19], [11], [17]. Since the presence of the damping term was crucial to obtain the strong convergence of $\{u'_k\}$, of course, the method used in the proof of Theorem 2.1 doesn't apply to variational inequalities associated to equation (3.34), when the unilateral constraints concern the unknown function. However, obstacle problems, related to (3.34), could be considered, if the unilateral constraints concern the time derivative of the unknown function or if they concern the behaviour of the unknown function on the boundary, as done by KIM[14] for the wave equation.

On the other hand, a memory term, instead of the damping, is considered in [20] and in [10], for problems like (1.1), and in [12], for an obstacle problem like (1.4); then an interesting matter would be to investigate the existence of (weak) solutions to an obstacle problem like (1.3), when a memory term (as in [20] and in [10]) takes the place of the damping.

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