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Existence for quasilinear elliptic systems due to a small L^∞-bound

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RIASSUNTO: *Si considera un sistema ellittico quasilineare a crescita quadratica nel gradiente. Si dimostra l'esistenza di almeno una soluzione ottenuta come limite in* $H_0^1(\Omega)^m$ *di soluzioni approssimanti, perchè su di esse si riesce a provare una stima* L^∞ *abbastanza piccola. D'altra parte tale stima sussiste se ciascuna equazione del sistema ha la stessa parte principale e se il termine non lineare ha una struttura particolare (di tipo one-sided).*

ABSTRACT: *We consider a quasilinear elliptic system with a nonlinear term with quadratic growth in the gradient. Assuming that a small* L^∞*-estimate on the unknown function is known, we prove the existence of at least one solution. On the other hand, assuming a one-sided condition on the nonlinear term, we prove a* L^∞*-estimate for at least one solution of the system.*

1 – Statement of the results

We consider in this paper a quasilinear elliptic system whose principal part is in diagonal form:

$$(1.1)\begin{cases} -\displaystyle\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\Big(a_{ij}^{\gamma}(x,u)\frac{\partial u^{\gamma}}{\partial x_j}\Big)+f^{\gamma}(x,u,\nabla u)=0 \text{ in } \mathcal{D}'(\Omega) \quad 1\le\gamma\le m\\ u\in\ (H_0^1(\Omega)\cap L^\infty(\Omega))^m;\end{cases}$$

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here u^γ $(1 \leq \gamma \leq m)$ are the components of the vector $u = (u^1, ..., u^m)$, ∇u is the $m \times N$ matrix whose γ-th row is $Du^\gamma = (\frac{\partial u^\gamma}{\partial x_1}, ..., \frac{\partial u^\gamma}{\partial x_N})$; Ω is a bounded open subset of $\mathbb{R}^N$, with boundary $\partial\Omega$ (no smothness is assumed on $\partial\Omega$); the coefficients a^γ_{ij} $(1 \leq i, j \leq N, \quad 1 \leq \gamma \leq m)$ are Carathéodory functions defined on $\Omega \times \mathbb{R}^m$, such that for some $\alpha > 0$ and some $\beta > 0$:

$$(1.2) \qquad \begin{cases} \text{a.e.} \quad x \in \Omega, \forall s \in \mathbb{R}^m, \forall \xi \in \mathbb{R}^N \\ \displaystyle\sum_{i,j=1}^{N} a^\gamma_{ij}(x,s)\xi_i\xi_j \geq \alpha \mid \xi \mid^2 \qquad 1 \leq \gamma \leq m \\ \mid a^\gamma_{ij}(x,s) \mid \leq \beta; \end{cases}$$

the nonlinearities f^γ $(1 \leq \gamma \leq m)$ are Carathéodory functions defined on $\Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times N}$ such that:

$$(1.3) \qquad \begin{cases} \text{a.e.} \quad x \in \Omega, \forall s \in \mathbb{R}^m, \forall \Xi \in \mathbb{R}^{m\times N} \\ \mid f^\gamma(x,s,\Xi) \mid \leq C_0(1+ \mid \Xi \mid^2). \end{cases}$$

For the sake of simplicity we denote by $a^\gamma(x,s)$ the matrix whose components are $a^\gamma_{ij}(x,s)$, in such a way that (1.1) reads as:

$$(1.4) \quad \begin{cases} -\operatorname{div}\Big(a^\gamma(x,u)Du^\gamma\Big) + f^\gamma(x,u,\nabla u) = 0 \text{ in } \mathcal{D}'(\Omega), \quad 1 \leq \gamma \leq m \\ u \in \ (H^1_0(\Omega) \cap L^\infty(\Omega))^m. \end{cases}$$

Our goal in the present paper is twofold:

i) On the first hand, we consider an equation which approximates (1.4) (see (1.7)). We assume that the solutions of this approximated equation are bounded in $(L^\infty(\Omega))^m$, with an L^∞-norm which is sufficiently small. We then prove that we can pass to the limit in the approximated equation. This implies the existence of at least one solution of (1.4).

ii) On the other hand, we consider again the equation (1.7) which approximates (1.4) but we now assume that each equation of the system has the same principal part and that the nonlinearities satisfy a one-sided condition (see (1.12)). We then prove that any solution of the approximated equation satisfies some L^∞-estimate.

Approximation

For $\epsilon > 0$, let f_ϵ^γ be Carathéodory functions defined on $\Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times N}$ such that:

$$\text{(1.5)} \qquad \begin{cases} \text{a.e.} \quad x \in \Omega,\ \forall s \in \mathbb{R}^m,\ \forall \Xi \in \mathbb{R}^{m\times N} \\ \mid f_\epsilon^\gamma(x,s,\Xi) \mid \leq \dfrac{1}{\epsilon} \\ |f_\epsilon^\gamma(x,s,\Xi) \mid \leq |f^\gamma(x,s,\Xi)| \end{cases}$$

and

$$\text{(1.6)} \qquad \begin{cases} f_\epsilon^\gamma(x,s_\epsilon,\Xi_\epsilon) \longrightarrow f^\gamma(x,s,\Xi) \\ \quad \text{if} \quad s_\epsilon \longrightarrow s \quad \text{in} \quad \mathbb{R}^m, \quad \Xi_\epsilon \longrightarrow \Xi \quad \text{in} \quad \mathbb{R}^{m\times N}. \end{cases}$$

Note that hypotheses (1.5) and (1.6) are fulfilled in the case where:

$$f_\epsilon^\gamma(x,s,\Xi) = \frac{f^\gamma(x,s,\Xi)}{1+\epsilon \mid f^\gamma(x,s,\Xi) \mid}.$$

We now consider the approximated problem:

$$\text{(1.7)} \qquad \begin{cases} -\operatorname{div}\Big(a^\gamma(x,u_\epsilon)Du_\epsilon^\gamma\Big) + f_\epsilon^\gamma(x,u_\epsilon,\nabla u_\epsilon) = 0 \ \text{ in } \ \mathcal{D}'(\Omega) \quad 1 \leq \gamma \leq m \\ u^\epsilon \in (H_0^1(\Omega))^m. \end{cases}$$

Since for all $w \in \Big(H_0^1(\Omega)\Big)^m$, we have

$$\|f_\epsilon^\gamma(x,w,\nabla w)\|_{L^\infty(\Omega)} \leq \frac{1}{\epsilon} \quad 1 \leq \gamma \leq m$$

the mapping which associates to each $w \in \Big(H_0^1(\Omega)\Big)^m$ the unique solution w_ϵ^γ of the problem:

$$\begin{cases} -\operatorname{div}\Big(a^\gamma(x,w)\nabla w_\epsilon^\gamma\Big) + f_\epsilon^\gamma(x,w,\nabla w) = 0 \ \text{ in } \mathcal{D}'(\Omega) \qquad 1 \leq \gamma \leq m \\ w_\epsilon \in (H_0^1(\Omega))^m \end{cases}$$

satisfies the hypotheses of Schauder's fixed point theorem, and problem (1.7) has at least one solution u_ε. Moreover, since for each γ, u_ε^γ is the solution of a scalar second order equation whose source term $f_\varepsilon^\gamma(x, u_\varepsilon, \nabla u_\varepsilon)$ belongs to $L^\infty(\Omega)$ (with L^∞-norm less than $\frac{1}{\varepsilon}$, (see (1.5)) we have

$$u_\varepsilon^\gamma \in L^\infty(\Omega) \quad \text{with} \quad \|u_\varepsilon^\gamma\|_{L^\infty(\Omega)} \leq \frac{C}{\varepsilon}.$$

As said before our first aim is to prove that under the sole hypothesis (1.3) on f^γ, there exists at least one solution of problem (1.4) if u_ϵ^γ satisfies a (small enough) L^∞-estimate:

Theorem I.1. *Assume that:*

$$\|u_\varepsilon^\gamma\|_{L^\infty(\Omega)} \leq M \quad \textit{where} \quad M < \frac{\alpha}{4C_0} \log \frac{m}{m-1} \quad 1 \leq \gamma \leq m, \tag{1.8}$$

where α and C_0 appear in (1.2) *and* (1.3). *Then we can extract a subsequence* (*still denoted by u_ε*) *such that*

$$u_\varepsilon \longrightarrow u \quad \textit{strongly in} \quad (H_0^1(\Omega))^m \tag{1.9}$$

and u is a solution of problem (1.4) *such that:*

$$\|u^\gamma\|_{L^\infty(\Omega)} \leq M. \tag{1.10}$$

On the other hand, any solution of (1.7) satisfies an L^∞-estimate if each equation of the system has the same principal part and if the nonlinearities satisfy a one-sided condition:

Theorem I.2. *Assume that:*

$$a^\gamma(x,s) = a(x,s) \quad 1 \leq \gamma \leq m \tag{1.11}$$

and that

$$\sum_{\gamma=1}^{m} f_\varepsilon^\gamma(x,s,\Xi)s^\gamma \geq -\lambda|\Xi|^2 - h(x) \tag{1.12}$$

where $h \in L^p(\Omega)$, $p > \frac{N}{2}$, and $\lambda \leq \alpha$ (with α the coercivity constant).

Then any solution u^ϵ of (1.7) *satisfies the L^∞-estimate*

$$\|u_\epsilon\|_{(L^\infty(\Omega))^m} \leq M_1 \tag{1.13}$$

where M_1 only depends on $|\Omega|$, α, p, N and $\|h\|_{L^p(\Omega)}$, and is small if $\|h\|_{L^p(\Omega)}$ is small (*see* (3.5)).

Remark I.1. To prove both theorems we will use nonlinear (in u_ε) test functions, a method of proof which is inspired by the works of J. Frehse [3], A. Bensoussan-J. Frehse [1] and L. Boccardo-F. Murat-J.P. Puel [2]. In particular Theorem I.1 generalizes to the vector-valued case of the result obtained in [2] for a scalar equation.

Remark I.2. The L^∞-estimate (1.8) on u_ε^γ is assumed to be small, but it is larger than the naïve one, which would consist in using a linear test function (and not a nonlinear one), see Remark II.1 below.

REMARK I.3. The principal part of the system is in a diagonal form, but in Theorem I.1 the matrix $a^\gamma(x,s)$ can vary from an equation to another.

2 – Proof of theorem I.1

The proof of theorem 1.1 will be performed in three steps: we will first prove a $\left(H_0^1(\Omega)\right)^m$-estimate for u_ϵ, then the strong convergence in $\left(H_0^1(\Omega)\right)^m$ of u_ϵ, and finally we pass to the limit in the approximated equation (1.7).

For this purpose we need the following lemma, the proof of which is obvious

Lemma II.1. *For all $\alpha > 0$ for all $C_0 > 0$ and for all M such that:*

$$0 < M < \frac{\alpha}{4C_0} \log \frac{m}{m-1} \qquad m > 1$$

the function φ defined by:

$$\varphi(t) = \begin{cases} e^{\frac{2C_0 t}{\alpha}} - 1, & \text{if} \quad t \geq 0 \\ 1 - e^{-\frac{2C_0 t}{\alpha}}, & \text{if} \quad t < 0 \end{cases} \tag{2.1}$$

satisfies:

$$\begin{cases} \varphi(0) = 0 \\ \varphi'(t) \geq 0, \\ \alpha\varphi'(t) - 2C_0|\varphi(t)| - 2(m-1)C_0\varphi(2M) \geq \mu > 0 \end{cases}$$

where μ depends only on C_0, M and m.

Now we have the following:

Proposition II.1. *Under hypotheses* (1.2), (1.3), (1.5), (1.6) *and* (1.8), *any solution u_ε of* (1.7) *satisfies the estimate*

$$\|u_\varepsilon\|_{(H_0^1(\Omega))^m} \leq C \tag{2.2}$$

where C only depends on α, C_0, M, m and $|\Omega|$.

Proof of Proposition II.1 Since u_ε^γ belongs to $L^\infty(\Omega) \cap H_0^1(\Omega)$, the function $\varphi(u_\varepsilon^\gamma)$, where φ is defined by (2.1), belongs to $H_0^1(\Omega)$ and is an admissible test function for the γ-th equation of system (1.7). We obtain:

$$\sum_{\gamma=1}^{m} \int_\Omega a^\gamma(x, u_\varepsilon) Du_\varepsilon^\gamma Du_\varepsilon^\gamma \varphi'(u_\varepsilon^\gamma)\, dx = -\sum_{\delta=1}^{m} \int_\Omega f_\varepsilon^\delta(x, u_\varepsilon, \nabla u_\varepsilon)\varphi(u_\varepsilon^\delta)\, dx. \tag{2.3}$$

From the coercivity condition (1.2) and the growth conditions (1.3) and (1.5) we have:

$$\begin{aligned} \alpha \sum_{\gamma=1}^{m} \int_\Omega |Du_\varepsilon^\gamma|^2 \varphi'(u_\varepsilon^\gamma)\, dx &\leq \int_\Omega C_0(1 + |\nabla u_\varepsilon|^2) \sum_{\delta=1}^{m} |\varphi(u_\varepsilon^\delta)|\, dx = \\ &= \int_\Omega C_0(1 + \sum_{\gamma=1}^{m} |Du_\varepsilon^\gamma|^2) \sum_{\delta=1}^{m} |\varphi(u_\varepsilon^\delta)|\, dx. \end{aligned} \tag{2.4}$$

Since in view of (1.8) we have

$$\sum_{\delta=1}^{m} |\varphi(u_\varepsilon^\delta)| \leq |\varphi(u_\varepsilon^\gamma)| + (m-1)\varphi(M),$$

inequality (2.4) implies that

$$\sum_{\gamma=1}^{m}\int_{\Omega}[\alpha\varphi'(u_\varepsilon^\gamma) - C_0|\varphi(u_\varepsilon^\gamma)| - (m-1)C_0\varphi(M)]|Du_\varepsilon^\gamma|^2\,dx \leq \tag{2.5}$$

$$\leq C_0 m|\Omega|\varphi(M).$$

Using Lemma II.1, inequality (2.5) implies, since $C_0 < 2C_0$ and $\varphi(M) < \varphi(2M)$

$$\mu\int_{\Omega}|\nabla u_\varepsilon|^2\,dx \leq C_0 m|\Omega|\varphi(M),$$

i.e. (2.2).

In view of estimate (2.2) we can extract a subsequence, still denoted by u_ε, such that

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in} \quad (H_0^1(\Omega))^m. \tag{2.6}$$

In view of (1.8) we have

$$u^\gamma \in L^\infty(\Omega) \quad \text{with} \quad \|u^\gamma\|_{L^\infty(\Omega)} \leq M \qquad 1 \leq \gamma \leq m.$$

Actually we have the strong convergence:

PROPOSITION II.2. *Under hypotheses* (1.2),(1.3),(1.5),(1.6) *and* (1.8), *we have*

$$u_\varepsilon \longrightarrow u \qquad \textit{strongly in} \quad (H_0^1(\Omega))^m. \tag{2.7}$$

PROOF OF THE PROPOSITION II.2. Let $\overline{u}_\varepsilon^\gamma = u_\varepsilon^\gamma - u^\gamma$. Then the approximated equation (1.7) can be written in the form:

$$-\operatorname{div}\Big(a^\gamma(x,u_\varepsilon)D\overline{u}_\varepsilon^\gamma\Big) - \operatorname{div}\Big(a^\gamma(x,u_\varepsilon)Du^\gamma\Big) + f_\varepsilon^\gamma(x,u_\varepsilon,\nabla u_\varepsilon) = 0 \tag{2.8}$$

$$\text{in } \mathcal{D}'(\Omega) \quad 1 \leq \gamma \leq m.$$

We use in the γ-th equation of system (2.8) the admissible test function $\varphi(\overline{u}_\varepsilon^\gamma)$ which belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$. We obtain:

$$\begin{aligned}
&\sum_{\gamma=1}^{m} \int_\Omega a^\gamma(x, u_\varepsilon) D\overline{u}_\varepsilon^\gamma D\overline{u}_\varepsilon^\gamma \varphi'(\overline{u}_\varepsilon^\gamma)\, dx = \\
&= -\sum_{\gamma=1}^{m} \int_\Omega a^\gamma(x, u_\varepsilon) Du^\gamma D\overline{u}_\varepsilon^\gamma \varphi'(\overline{u}_\varepsilon^\gamma)\, dx \\
&\quad -\sum_{\delta=1}^{m} \int_\Omega f_\varepsilon^\delta(x, u_\varepsilon, \nabla u_\varepsilon) \varphi(\overline{u}_\varepsilon^\delta)\, dx.
\end{aligned} \tag{2.9}$$

We use the coercivity (1.2), the growth conditions (1.3) and (1.5) and the following estimate:

$$|\nabla u_\varepsilon|^2 \leq 2(|\nabla \overline{u}_\varepsilon|^2 + |\nabla u|^2)$$

to obtain:

$$\begin{aligned}
\sum_{\gamma=1}^{m} \int_\Omega \alpha |D\overline{u}_\varepsilon^\gamma|^2 \varphi'(\overline{u}_\varepsilon^\gamma)\, dx \leq &-\sum_{\gamma=1}^{m} \int_\Omega a^\gamma(x, u_\varepsilon) Du^\gamma D\overline{u}_\varepsilon^\gamma \varphi'(\overline{u}_\varepsilon^\gamma)\, dx + \\
&+ \int_\Omega C_0(1 + 2|\nabla u|^2) \sum_{\delta=1}^{m} |\varphi(\overline{u}_\varepsilon^\delta)|\, dx + \\
&+ 2C_0 \int_\Omega \sum_{\gamma=1}^{m} |D\overline{u}_\varepsilon^\gamma|^2 \sum_{\delta=1}^{m} |\varphi(\overline{u}_\varepsilon^\delta)|\, dx.
\end{aligned} \tag{2.10}$$

Since in view of (1.8) and of $\|u^\gamma\|_{L^\infty(\Omega)} \leq M$ we have

$$\sum_{\delta=1}^{m} |\varphi(\overline{u}_\varepsilon^\delta)| \leq |\varphi(\overline{u}_\varepsilon^\gamma)| + (m-1)\varphi(2M),$$

inequality (2.10) implies that

$$\begin{aligned}
&\sum_{\gamma=1}^{m} \int_\Omega [\alpha\varphi'(\overline{u}_\varepsilon^\gamma) - 2C_0|\varphi(\overline{u}_\varepsilon^\gamma)| - 2(m-1)C_0\varphi(2M)] |D\overline{u}_\varepsilon^\gamma|^2\, dx \leq \\
&\leq -\sum_{\gamma=1}^{m} \int_\Omega a^\gamma(x, u_\varepsilon) Du^\gamma D\overline{u}_\varepsilon^\gamma \varphi'(\overline{u}_\varepsilon^\gamma)\, dx + \\
&\quad + \int_\Omega C_0(1 + 2|\nabla u|^2) \sum_{\delta=1}^{m} |\varphi(\overline{u}_\varepsilon^\delta)|\, dx.
\end{aligned} \tag{2.11}$$

The weak convergence (2.6), Rellich's compactness theorem, and Lebesgue's dominated convergence theorem prove that the right hand side of (2.11) tends to zero. In view of Lemma II.1 we thus have

$$\limsup_{\varepsilon} \ \mu \sum_{\gamma=1}^{m} \int_\Omega |Du_\varepsilon^\gamma - Du^\gamma|^2 \, dx \le 0, \tag{2.12}$$

i.e. (2.7), which completes the proof of Proposition II.2.

Applying Proposition II.2, Vitali's theorem and hypotheses (1.5) and (1.6) on f_ε^γ we pass to the limit in each term of (1.7). We obtain the existence of at least one solution of (1.4).

Remark II.1: About the smallness of the L^∞-estimate. The L^∞-estimate (1.8) used in the above proof is assumed to be small, in such a way that for the nonlinear function φ defined by (2.1) the inequality

$$\alpha\varphi^{'}(t) - 2C_0|\varphi(t)| - 2(m-1)C_0\varphi(2M) \ge \mu > 0.$$

holds true. This implies that M is sufficiently small with respect to $\frac{\alpha}{C_0}$.

In place of the nonlinear test functions $\varphi(u_\varepsilon^\gamma)$ and $\varphi(\overline{u}_\varepsilon^\gamma)$ that we used, we could use the linear test functions u_ε^γ and $\overline{u}_\varepsilon^\gamma$ but this would lead to assume an L^∞-estimate stronger that (1.8), as the following computation shows. Indeed if we use $\overline{u}_\varepsilon^\gamma$ as test function in (2.8), we obtain in place of (2.10):

$$\begin{aligned}\sum_{\gamma=1}^{m}\int_\Omega \alpha|D\overline{u}_\varepsilon^\gamma|^2 \le &-\sum_{\gamma=1}^{m}\int_\Omega a^\gamma(x,u_\varepsilon)Du^\gamma D\overline{u}_\varepsilon^\gamma dx+\\ &+\int_\Omega C_0(1+2|\nabla u|^2)\sum_{\delta=1}^{m}|\overline{u}_\varepsilon^\delta|dx+\\ &+2C_0\int_\Omega \sum_{\gamma=1}^{m}|D\overline{u}_\varepsilon^\gamma|^2\sum_{\delta=1}^{m}|\overline{u}_\varepsilon^\delta|dx.\end{aligned} \tag{2.13}$$

If we now make the assumption that

$$\|u_\varepsilon^\gamma\|_{L^\infty(\Omega)} \le M' \qquad 1 \le \gamma \le m,$$

which implies $\sum_{\delta=1}^{m}|\overline{u}_\varepsilon^\delta|\le 4M'$, inequality (2.13) implies

$$\sum_{\gamma=1}^{m}\int_\Omega[\alpha-4mC_0M']|Du_\varepsilon^\gamma|^2dx\le-\sum_{\gamma=1}^{m}\int_\Omega a^\gamma(x,u_\varepsilon)Du^\gamma D\overline{u}_\varepsilon^\gamma dx+$$

$$+\int_\Omega C_0(1+2|\nabla u|^2)\sum_{\gamma=1}^{m}|\overline{u}_\varepsilon^\gamma|dx$$

which leads to assume that

$$M'<\frac{\alpha}{4C_0}\frac{1}{m}.$$

In hypothesis (1.8) we have assumed that:

$$\|u_\varepsilon^\gamma\|_{L^\infty(\Omega)}\le M\quad\text{with}\quad M<\frac{\alpha}{4C_0}\log\frac{m}{m-1}\qquad 1\le\gamma\le m,$$

and we remark that

$$\frac{\alpha}{4C_0}\frac{1}{m}<\frac{\alpha}{4C_0}\log\frac{m}{m-1}\quad\text{when}\quad m>1.$$

The L^∞-estimate (1.8) used in Propositions II.1 and II.2 is thus larger the naïve L^∞-estimate which would be necessary to obtain a similar result by using linear test functions.

3 – Proof of Theorem I.2

For the proof of Theorem II.2, we will use the following lemma, the proof of which is due to G. Stampacchia [5] (see also S. Hildebrant-K.O. Widman [4] and W. Wieser [6]).

Lemma III.1. *If $v\in L^1(\Omega)$ and if*

$$\forall k\ge 0\quad\int_\Omega(v-k)^+\,dx\le M_0|A(k)|^\theta,\quad\theta>1\tag{3.1}$$

where $A(k)=\{x\in\Omega:\quad v(x)>k\}$ and $M_0>0$, then

$$v(x)\le M_0|\Omega|^{\theta-1}(\frac{\theta}{\theta-1}).\tag{3.2}$$

PROOF OF THEOREM I.2. Since u_ε^γ belongs to $L^\infty(\Omega)\cap H_0^1(\Omega)$, the function:

$$W_\varepsilon^\gamma=\sum_{\delta=1}^m\Big(\frac{(u_\varepsilon^\delta)^2}{2}-k\Big)^+u_\varepsilon^\gamma=\psi_\varepsilon^+u_\varepsilon^\gamma,\quad\text{where}\quad\psi_\varepsilon=\Big(\frac{|u_\varepsilon|^2}{2}-k\Big)$$

which belongs to $L^\infty(\Omega)\cap H_0^1(\Omega)$, is an admissible test function and we can use it in the γ-th equation of (1.7). Summing up from $\gamma=1$ to m, we obtain, since $a^\gamma(x,s)=a(x,s)$ by hypothesis (1.11):

$$\begin{aligned}\sum_{\gamma=1}^m\int_\Omega a(x,u_\varepsilon)Du_\varepsilon^\gamma Du_\varepsilon^\gamma\psi_\varepsilon^+\,dx+\sum_{\gamma=1}^m\int_\Omega a(x,u_\varepsilon)Du_\varepsilon^\gamma D\psi_\varepsilon^+u_\varepsilon^\gamma\,dx+\\ +\sum_{\gamma=1}^m\int_\Omega f_\varepsilon^\gamma(x,u_\varepsilon,\nabla u_\varepsilon)u_\varepsilon^\gamma\psi_\varepsilon^+\,dx=0.\end{aligned}\tag{3.3}$$

Using the coercivity condition (1.2), the fact that $D\psi_\varepsilon=\sum_{\gamma=1}^m u_\varepsilon^\gamma Du_\varepsilon^\gamma$ and the one-sided condition (1.12), (3.3) becomes:

$$\begin{aligned}\alpha\int_\Omega|\nabla u_\varepsilon|^2\psi_\varepsilon^+\,dx+\int_\Omega a(x,u_\varepsilon)D\psi_\varepsilon D\psi_\varepsilon^+\,dx+\\ -\lambda\int_\Omega|\nabla u_\varepsilon|^2\psi_\varepsilon^+\,dx-\int_\Omega h(x)\psi_\varepsilon^+\,dx\le 0,\end{aligned}$$

which using again the coercivity condition (1.2) and $\alpha-\lambda\ge 0$ becomes:

$$\alpha\int_\Omega|D\psi_\varepsilon^+|^2\,dx\le\int_\Omega h(x)\psi_\varepsilon^+\,dx.\tag{3.4}$$

If $N>2$, Sobolev's inequality asserts that

$$\|\psi_\varepsilon^+\|_{L^{2^\star}(\Omega)}\le C\|D\psi_\varepsilon^+\|_{L^2(\Omega)}\qquad\text{with}\quad\frac{1}{2^\star}=\frac12-\frac1N,$$

while if $N=2$ the same inequality holds true for any $2^\star$ such that $2\le 2^\star<+\infty$. Using Hölder's inequality with r defined by $\frac1p+\frac{1}{2^\star}+\frac1r=1$ and defining

$$v_\varepsilon=\frac{|u_\varepsilon|^2}{2}\quad\text{and}\quad A_\varepsilon(k)=\{x\in\Omega:\quad v_\varepsilon(x)>k\}$$

we thus have:

$$\alpha \|\psi_\varepsilon^+\|^2{}_{L^{2^\star}(\Omega)} \leq \int_{A_\varepsilon(k)} h(x)\psi_\varepsilon^+ \, dx \leq \|h\|_{L^p(\Omega)} \|\psi_\varepsilon^+\|_{L^{2^\star}(\Omega)} |A_\varepsilon(k)|^{\frac{1}{r}}.$$

Therefore

$$\int_\Omega \left(\frac{|u_\varepsilon|^2}{2} - k\right)^+ dx = \|\psi_\varepsilon^+\|_{L^1(\Omega)} \leq$$
$$\leq \|\psi_\varepsilon^+\|_{L^{2^\star}(\Omega)} |A_\varepsilon(k)|^{\frac{1}{(2^\star)'}} \leq \frac{1}{\alpha} \|h\|_{L^p(\Omega)} |A_\varepsilon(k)|^{\frac{1}{r} + \frac{1}{(2^\star)'}}$$

where

$$\frac{1}{r} + \frac{1}{(2^\star)'} = 1 - \frac{1}{p} - \frac{1}{2^\star} + 1 - \frac{1}{2^\star} = 2 - \frac{1}{p} - \frac{2}{2^\star}.$$

When $N > 2$, the latest number is nothing but $1 - \frac{1}{p} + \frac{2}{N}$, which is strictly greater than 1 when $p > \frac{N}{2}$, a result which also holds true when $N = 2$ and $p > 1$ if $2^\star$ is choosen sufficiently large. Lemma III.1 therefore implies that:

$$v_\varepsilon = \frac{|u_\varepsilon|^2}{2} \leq \frac{M_1^2}{2}$$

where M_1 is a constant which only depends on $|\Omega|$, α, p, N and $\|h\|_{L^p(\Omega)}$; in the case where $N > 2$, this constant is given by:

$$M_1^2 = \frac{2}{\alpha} |\Omega|^{\frac{2}{N} - \frac{1}{p}} \left(\frac{1 + \frac{2}{N} - \frac{1}{p}}{\frac{2}{N} - \frac{1}{p}}\right) \|h\|_{L^p(\Omega)}. \tag{3.5}$$

This proves Theorem I.2.

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