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# Transversally CR foliations

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RIASSUNTO: *Si studiano foliazioni a struttura* CR *trasversa e la loro coomologia di Kohn-Rossi trasversa. Per* CR *foliazioni non degeneri a struttura pseudohermitiana trasversa si costruisce una connessione adattata che generalizza la connessione di Webster di una* CR *foliazione per punti. Si ottiene un risultato di immergibilità locale per* CR *foliazioni reali analitiche.*

ABSTRACT: *We study foliations with transverse* CR *structure and their transverse Kohn-Rossi cohomology. For nondegenerate* CR *foliations with transverse pseudohermitian structure we build an adapted connection which generalizes the Webster connection of a* CR *foliation by points. We establish a local embeddability result for real analytic* CR *foliations.*

## 1 – Introduction

The purpose of the present paper is to study the geometry of $\Gamma^r_{\mathrm{CR}}(N)$-foliations (with $r = \infty$ or $r = \omega$) where $\Gamma^r_{\mathrm{CR}}(N)$ is the pseudogroup of all local CR automorphisms of class $C^r$ of a given (model) CR manifold $N$. These are called CR *foliations* and CR manifolds correspond to the case of the trivial CR foliation by points.

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Our motivation comes from the theory of complex (and Levi) foliations on CR manifolds (cf. *e.g.* E.M. CHIRKA [5], p. 150). A complex foliation $\mathcal{F}$ of a CR manifold $(M, H(M))$ (where $H(M)$ is its Levi distribution) is one whose tangent bundle $P$ is a complex subbundle of $H(M)$ and whose foliated charts restricted to plaques give biholomorphisms (therefore such foliations occur on degenerate CR manifolds). The quotient $H(M)/P$ carries a natural complex structure $J$ and in cases of interest (cf. our Theorem 6) $J$ is parallel with respect to the Bott connection of $\mathcal{F}$, so that $\mathcal{H} = \text{Eigen}\ (i)$ (the eigenbundle of $J$ corresponding to the eigenvalue $i = \sqrt{-1}$) is a *transverse almost* CR *structure.* Moreover $\mathcal{H}$ is *integrable* (for any $x \in M$ there is an open neighborhood $U$ and an *admissible* frame $\{\zeta_\alpha\}$ of $\mathcal{H}$ on $U$, that is each $\zeta_\alpha$ is a transverse vector field and $[\zeta_\alpha, \zeta_\beta] \in \mathcal{H}$) in most examples at hand (cf. Section 6). To further motivate our line of thought, let us recall (cf. [5], p. 155, or S.I. PINCHUK S.I. TSYGANOV [14]) that a complex foliation $\mathcal{F}$ of complex dimension $k$ of a CR manifold $M$ is CR-*straightenable* if there are a domain $\Omega \subset \mathbf{C}^k$, a CR manifold $N$, and a CR diffeomorphism $\varphi : \Omega \times N \to M$ so that $\varphi(\Omega \times \{p\})$ is a leaf of $\mathcal{F}$, for any $p \in N$. If $\mathcal{F}$ is CR-straightenable then the transverse geometry of $\mathcal{F}$ is modelled on $N$, *i.e.* $\mathcal{F}$ is a $\Gamma^\infty_{\mathrm{CR}}(N)$-foliation.

In the end, it is worth mentioning that the notion of (transversally) CR foliation is implicit in [5], p. 157. There, a CR foliation is a foliation $\mathcal{F}$ of a CR manifold $M$ so that, for any defining local submersion $f : U \to U'$ (*i.e.* the leaves of $\mathcal{F}_U$ are the fibres of $f$) the local quotient manifold $U'$ is a CR manifold, $f$ is a CR map, and $f_* : H(U) \to H(U')$ is on-to. Such $\mathcal{F}$ carries a transverse CR structure. Yet, on one hand $\Gamma^\infty_{\mathrm{CR}}(N)$-foliations make sense on arbitrary $C^\infty$ manifolds (not just on CR manifolds); on the other, the requirement that $f : U \to U'$ be CR is somewhat misleading. Indeed, this yields a "tangential" CR structure (so that each leaf becomes a CR submanifold of $M$) thus prompting the choice of terminology (CR foliations) in [5] (the transverse CR structure is not looked at there). Also (at least in the CR codimension one case) the Levi form of $M$ must have a nontrivial kernel. Our point of view is, of course, that the tangential CR structure of $\mathcal{F}$ is only incidental, and that E. Chirka's approach to CR foliations (requiring that the local quotient manifolds possess some $G$-structure) is just the typical manner (cf. *e.g.* Proposition 2.6 in [13], p. 51-52) of assigning a transverse $G$-structure to $\mathcal{F}$.

Let $N$ be a CR manifold. The group $\mathrm{Aut}_{\mathrm{CR}}(N)$ of all global CR automorphisms of $N$ is a Lie transformation group (by a result of S.S. Chern and J. Moser [4]). In Section 2 we discuss foliations defined by suspension of a homomorphism $h : \pi_1(B, x_0) \to \mathrm{Aut}_{\mathrm{CR}}(N)$ (these turn out to be CR foliations (with nontrivial holonomy), cf. Theorem 3).

When the normal bundle of the given CR foliation has odd real rank we develop a foliated analogue of S. Webster's (cf. [15]) pseudohermitian geometry, cf. our Theorems 4 and 5. We introduce notions such as *transverse pseudohermitian structrure, transverse Levi form*, and *transverse Webster metric* $g_\theta$, as well as notions of (transverse) nondegeneracy and strict pseudoconvexity. In the nondegenerate case, the transverse Webster metric $g_\theta$ is a transverse metric (in the sense of [13], p. 77) for $\mathcal{F}$ and thus there is a bundle-like semi-Riemannian metric $g$ on $M$ inducing $g_\theta$. Our main result in this direction is that there is an adapted connection $\nabla$ in the normal bundle of the given nondegenerate CR foliation $\mathcal{F}$ which parallelizes both the transverse Levi form and the complex structure in the transverse Levi distribution ($\nabla$ is unique under some assumption on its torsion, cf. Theorem 10). In addition, $\nabla$ does not depend upon the choice of bundle-like semi-Riemannian metric $g$ (inducing the transverse Webster metric) used in its construction. For the case of a CR foliation by points $\nabla$ is the Webster connection (cf. [15]).

We show that any CR foliation comes equipped with a natural differential operator $\overline{\partial}_Q$ (a foliated analogue of the tangential Cauchy-Riemann operator in complex analysis) acting on transverse $(0, k)$-forms. We look at the cohomology of the resulting $\overline{\partial}_Q$-complex; for the case of a simple CR foliation defined by a submersion this cohomology turns out to be the Kohn-Rossi cohomology of the base CR manifold (cf. Theorems 7 and 8).

Let $\mathcal{F}$ be a CR foliation of type $(n, k)$ of $M$ and $\mathcal{H}$ its transverse CR structure. We introduce a concept of *embedding* of $(M, \mathcal{H})$. This is essentially an immersion $\psi : M \to \mathbf{C}^N$ for some $N > n + k$ which induces a bundle monomorphism $G$ of the normal bundle into $T(\mathbf{C}^{n+k})$ so that $G$ maps $\mathcal{H}$ into the holomorphic tangent bundle over $\mathbf{C}^{n+k}$. Any real analytic transverse CR structure is shown (cf. our Theorem 11) to be locally embeddable.

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## 2 – Transversally CR foliations

Let $M$ be a $C^\infty$ manifold and $\mathcal{F}$ a codimension $q$ foliation ($q = 2n+k$, $k \geq 1$) of class $C^\infty$ of $M$ thought of as (the collection of all connected maximal integral manifolds of) an integrable subbundle $P = T(\mathcal{F}) \subset T(M)$. Let $Q = \nu(\mathcal{F}) = T(M)/P$ be the normal (or transverse) bundle of $\mathcal{F}$. Let $\pi : T(M) \to Q$ be the natural bundle epimorphism. Let $\nabla^0$ be the *Bott connection* of $(M, \mathcal{F})$.

Let $\mathcal{H} \subset Q \otimes \mathbf{C}$ be a complex subbundle, of complex rank $n$. Set $H = \mathrm{Re}\{\mathcal{H} \oplus \overline{\mathcal{H}}\} \subset Q$. Throughout an overbar denotes complex conjugation. Then $H$ carries the complex structure $J : H \to H$ given by:

$$J(\alpha + \overline{\alpha}) = i(\alpha - \overline{\alpha}) \tag{1}$$

for any $\alpha \in \Gamma^\infty(\mathcal{H})$. Here $i = \sqrt{-1}$. The following notations are central for the rest of the present paper. We call $\mathcal{H}$ a *transverse almost* CR *structure* (of *transverse* CR *dimension* $n$) if 1) $\mathcal{H} \cap \overline{\mathcal{H}} = \{0\}$, 2) $H$ is parallel with respect to the Bott connection of $\mathcal{F}$ and 3) $\mathcal{L}_X J = 0$ for any $X \in \Gamma^\infty(P)$. Lie derivatives are defined with respect to $\nabla^0$.

Let $L(\mathcal{F}) = L(M, \mathcal{F}) \subset \mathcal{X}(M)$ be the Lie subalgebra of all *foliate vector fields* (or infinitesimal automorphisms of $\mathcal{F}$). Let $\ell(\mathcal{F}) = \ell(M, \mathcal{F}) \subset \Gamma^\infty(Q)$ be the Lie algebra of all *transverse vector fields* (*i.e.* $s \in \ell(\mathcal{F})$ iff $s = \pi Y$ for some $Y \in L(\mathcal{F})$). Let $\Gamma^\infty_B(Q)$ consist of all $s \in \Gamma^\infty(Q)$ with $\mathcal{L}_X s = 0$ for any $X \in \Gamma^\infty(P)$. Note that $\Gamma^\infty_B(Q) = \ell(\mathcal{F})$ (so that the Lie bracket $[s, r]$ of any $s, r \in \Gamma^\infty_B(Q)$ is well defined).

A transverse almost CR structure $\mathcal{H} \subset Q \otimes \mathbf{C}$ is termed *integrable* if for any $x \in M$ there is an open neighborhood $U \subseteq M$, $x \in U$, and there is a frame $\{\zeta_1, \cdots, \zeta_n\}$ of $\mathcal{H}$ on $U$ so that $\zeta_\alpha \in \Gamma^\infty_B(Q \otimes \mathbf{C})$ and $[\zeta_\alpha, \zeta_\beta] \in \Gamma^\infty(\mathcal{H})$ for any $1 \leq \alpha, \beta \leq n$. Such a (local) frame of $\mathcal{H}$ is termed *admissible*. An integrable transverse almost CR structure is referred to as a *transverse* CR *structure* on $(M, \mathcal{F})$.

A $\Gamma$-*foliation of codimension* $q$ *and class* $C^\infty$ on $M$ consists of the following data i) an open covering $\{U_i\}_{i \in I}$ of $M$, ii) an additional $C^\infty$ manifold $N$ and a pseudogroup $\Gamma$ of local transformations of $N$, iii) for each $i \in I$ a $C^\infty$ submersion $f_i : U_i \to N$, iv) for any $i, j \in I$ (with $U_{ji} = U_i \cap U_j \neq \emptyset$) an element $\gamma_{ji} \in \Gamma$ so that $f_j = \gamma_{ji} \circ f_i$ on $U_{ji}$. Cf. [8]. Let $x \in M$ and $i \in I$ with $x \in U_i$ and set $P_x = \mathrm{Ker}(d_x f_i)$. Then

$P \subset T(M)$ is a well defined (by iv)) integrable distribution so that any $\Gamma$-foliation gives rise to a foliation $\mathcal{F}$ of $M$.

Let $N$ be a $(2n+k)$-dimensional $C^\infty$ manifold. Let $T_{1,0}(N)$ be a CR *structure* (of CR *dimension* $n$) on $N$, *i.e.* a complex subbundle (of complex rank $n$) of the complexified tangent bundle $T(N) \otimes \mathbf{C}$ so that:

$$T_{1,0}(N) \cap T_{0,1}(N) = \{0\} \tag{2}$$

and

$$[\Gamma^\infty(T_{1,0}(N)), \Gamma^\infty(T_{1,0}(N))] \subset \Gamma^\infty(T_{1,0}(N)) \tag{3}$$

Here $T_{0,1}(N) = \overline{T_{1,0}(N)}$. Let $H(N) = \mathrm{Re}\{T_{1,0}(N) \oplus T_{0,1}(N)\}$ be the *Levi distribution.* It carries the complex structure $J_N : H(N) \to H(N)$ given by $J_N(Z + \overline{Z}) = i(Z - \overline{Z})$ for any $Z \in T_{1,0}(N)$. If $(N, T_{1,0}(N))$ and $(N', T_{1,0}(N'))$ are two CR manifolds then a $C^\infty$ map $\lambda : N \to N'$ is a CR *map* if $\lambda_* T_{1,0}(N) \subseteq T_{1,0}(N')$ (or equivalently $\lambda_* H(N) \subseteq H(N')$ and $\lambda_* \circ J_N = J_{N'} \circ \lambda_*$). A CR *automorphism* of $N$ is a $C^\infty$ diffeomorphisms and a CR map. Let $\Gamma^\infty_{\mathrm{CR}}(N)$ be the pseudogroup of all (local) CR automorphisms of $(N, T_{1,0}(N))$ (of class $C^\infty$). Let $\mathcal{F}$ be a $\Gamma^\infty_{\mathrm{CR}}(N)$-foliation of $M$. Then $\mathcal{F}$ is said to be a (*transversally*) CR *foliation* (of *transverse* CR *dimension* $n$ and *transverse* CR *codimension* $k$).

Let $\mathcal{F}$ be a CR foliation and $x \in U_i$. The differential $d_x f_i : T_x(M) \to T_{f_i(x)}(N)$ descends to a $\mathbf{R}$-linear isomorphism $F_{i,x} : Q_x \to T_{f_i(x)}(N)$ with $F_{i,x} \circ \pi_x = d_x f_i$. Set:

$$H_x = F_{i,x}^{-1} H(N)_{f_i(x)}$$

As $\gamma_{ji} \in \Gamma^\infty_{\mathrm{CR}}(N)$ one has in particular $(\gamma_{ji})_* H(N) = H(N)$ so that $H_x$ is well defined. It carries the complex structure $J_x$ given by:

$$J_x = F_{i,x}^{-1} \circ J_{N, f_i(x)} \circ F_{i,x}$$

Once again, as $\gamma_{ji} \in \Gamma^\infty_{\mathrm{CR}}(N)$, in particular $(\gamma_{ji})_* \circ J_N = J_N \circ (\gamma_{ji})_*$ so that $J_x$ is well defined. Then $H$ is referred as the *transverse Levi distribution* of $(M, \mathcal{F})$. We may state the following:

THEOREM 1. *Let $(N, T_{1,0}(N))$ be a* CR *manifold of class $C^\infty$ and type $(n,k)$. Let $\mathcal{F}$ be a* CR *foliation of $M$ whose transverse geometry is modelled on $(N, T_{1,0}(N))$. Let $H$ be the transverse Levi distribution of $\mathcal{F}$.*

*Extend $J$ to $H \otimes \mathbf{C}$ by $\mathbf{C}$-linearity and set $\mathcal{H} =$* Eigen (i)*. Then $\mathcal{H}$ is a transverse* CR *structure of transverse* CR *dimension $n$ (and $F_{i,x}(\mathcal{H}_x) = T_{1,0}(N)_{f_i(x)}$ for any $x \in U_i$).*

One may look at transverse (almost) CR structures as transverse $G$-structures, as well. Let $\mathcal{F}$ be a codimension $q$ foliation of $M$. Let:

$$B_T^1 = B_T^1(M, \mathcal{F})$$

be the (total space of the) principal $GL(q, \mathbf{R})$-bundle of transverse frames (cf. [13], p. 44). Let $\theta_T^1 \in \Gamma^\infty(T^*(B_T^1) \otimes \mathbf{R}^q)$ be the fundamental 1-form on $B_T^1$ (cf. (2.11) in [13], p. 45) and set:

$$P_T^1 = \{X \in T(B_T^1) : X \rfloor \theta_T^1 = X \rfloor d\theta_T^1 = 0\}$$

If $G \subset GL(q, \mathbf{R})$ is a Lie subgroup then a principal $G$-subbundle $E \subset B_T^1$ is a *transverse $G$-structure* if:

$$P_{T,z}^1 \subset T_z(E) \tag{4}$$

for any $z \in E$. The distribution $P_T^1$ is known to be integrable (cf. Proposition 2.4 in [13], p. 47) so that it gives rise to a foliation $\mathcal{F}_T^1$ of $B_T^1$ (the *lifted foliation*, cf. [13]) each leaf of which is a Galois covering of some leaf of $\mathcal{F}$. Then the geometric meaning of (4) is that $E$ is a union of leaves of $\mathcal{F}_T^1$. We call $E$ *locally flat* if for any $x \in M$ there is a foliated coordinate chart $(U, y^1, \cdots, y^q, x^1, \cdots, x^p)$ at $x$ so that $\sigma_T(U) \subset E$. Here $\sigma_T : U \to B_T^1$ is the natural field of transverse frames:

$$\sigma_T(x) = (x, \{(\pi \frac{\partial}{\partial y^j})_x\}_{1 \le j \le q})$$

Let $\mathcal{H}$ be a transverse almost CR structure of type $(n, k)$ and $(H, J)$ the corresponding transverse Levi distribution. Let $E_x$ consist of all $\mathbf{R}$-linear isomorphisms $z : \mathbf{R}^q \to Q_x$ with $z(e_\alpha) \in H_x$ and $J_x z(e_\alpha) = z(e_{\alpha+n})$. Here $q = 2n + k$ while $\{e_\alpha, e_{\alpha+n}, e_{j+2n}\}$ denotes the canonical linear basis in $\mathbf{R}^q$ (with $1 \le \alpha \le n$, $1 \le j \le k$). Let $G \subset GL(q, \mathbf{R})$ consist of all nonsingular matrices of the form:

$$\begin{bmatrix} g_\beta^\alpha & \Omega_\beta^\alpha & u_j^\alpha \\ -\Omega_\beta^\alpha & g_\beta^\alpha & v_j^\alpha \\ 0 & 0 & w_j^i \end{bmatrix}$$

THEOREM 2. *$E$ is a transverse $G$-structure. If $E$ is locally flat then $\mathcal{H}$ is integrable.*

We end this section by looking at an example of CR foliation. Let $B, T$ be two compact connected manifolds and $h : \pi_1(B, x_0) \to \mathrm{Diff}(T)$ a group homomorphism. Here $\pi_1(B, x_0)$ is the first homotopy group of $B$ (with base point $x_0 \in B$) and $\mathrm{Diff}(T)$ denotes the group of all global diffeomorphisms of $T$. Set $G = h(\pi_1(B, x_0))$. We may state the following:

THEOREM 3. *Let $\mathcal{F}$ be a* CR *foliation* (*whose transverse geometry is modelled on the* CR *manifold* $(N, T_{1,0}(N))$ *of the* $C^\infty$ *manifold* $M$. *If* $\mathcal{F}$ *is defined by suspension of* $h : \pi_1(B, x_0) \to \mathrm{Diff}(T)$ *then* $T$ *is a* CR *manifold of type* $(n, k)$ *and* $G$ *a group of global* CR *automorphisms of* $T$. *Conversely, if* $\mathrm{Aut}_{\mathrm{CR}}(T)$ *is the group of global* CR *automorphisms of a* CR *manifold* $(T, T_{1,0}(T))$ *then the foliation* $\mathcal{F}$ *defined by suspension of* $h : \pi_1(B, x_0) \to \mathrm{Aut}_{\mathrm{CR}}(T)$ *is a* CR *foliation.*

Let $\hat{p} : \hat{B} \to B$ be the universal cover of $B$. Set $\tilde{M} = \hat{B} \times T$. There is a natural action of $\pi_1(B, x_0)$ on $\tilde{M}$ given by:

$$(\hat{x}, y) \cdot [\gamma] = (\hat{x} \cdot [\gamma], h([\gamma]^{-1})(y))$$

for any $\hat{x} \in \hat{B}$, $y \in T$ and $[\gamma] \in \pi_1(B, x_0)$. Let:

$$\rho : \tilde{M} \to \tilde{M}/\pi_1(B, x_0)$$

be the canonical projection. Let $\tilde{\mathcal{F}}$ be the (simple) foliation of $\tilde{M}$ whose leaves are the fibres of:

$$p_2 : \tilde{M} \to T, \qquad p_2(\hat{x}, y) = y\,.$$

The asumption on $(M, \mathcal{F})$ in Theorem 3 amounts to $M = \tilde{M}/\pi_1(B, x_0)$ and $\tilde{\mathcal{F}} = \rho^*\mathcal{F}$ (that is $\tilde{\mathcal{F}}$ is the pullback of $\mathcal{F}$ by $\rho$, cf. [9] and [13], p. 28). Then:

$$T(\tilde{\mathcal{F}})_{\tilde{x}} = (d_{\tilde{x}}\rho)^{-1} T(\mathcal{F})_{\rho(\tilde{x})}$$

for any $\tilde{x} \in \tilde{M}$. Consequently $d_{\tilde{x}}\rho$ descends to an isomorphism:

$$\Pi_{\tilde{x}} : \nu(\tilde{\mathcal{F}})_{\tilde{x}} \to \nu(\mathcal{F})_{\rho(\tilde{x})}\,.$$

Let $H$ be the transverse Levi distribution of $\mathcal{F}$ and $J$ its complex structure. Set:

$$\begin{aligned}\tilde{H}_{\tilde{x}} &= \Pi_{\tilde{x}}^{-1} H_{\rho(\tilde{x})}\\ \tilde{J}_{\tilde{x}} &= \Pi_{\tilde{x}}^{-1} \circ J_{\rho(\tilde{x})} \circ \Pi_{\tilde{x}}\,.\end{aligned}$$

Then $(\tilde{H}, \tilde{J})$ makes $\mathcal{F}$ into a (transversally) CR foliation. This may be seen in yet another way. Let $\{U_i, f_i, \gamma_{ji}\}_{i,j\in I}$ be the data defining $\mathcal{F}$ (as a $\Gamma^\infty_{\mathrm{CR}}(N)$-foliation, for some given model CR manifold $N$). We may assume w.l.o.g. that $U_i = \rho(\hat{V}_i \times T)$ where $\hat{V}_i = \hat{p}^{-1}(V_i)$ for some simply connected open subset $V_i \subset B$. Let $\tilde{U}_i = \hat{V}_i \times T$ and define $\tilde{f}_i : \tilde{U}_i \to N$ by $\tilde{f}_i = f_i \circ \rho$. Then the data $\{\tilde{U}_i, \tilde{f}_i, \gamma_{ji}\}_{i,j\in I}$ determines $\tilde{\mathcal{F}}$ (that is $\mathcal{F}$ is a $\Gamma^\infty_{\mathrm{CR}}(N)$-foliation of $\tilde{M}$).

The differential $d_{\tilde{x}} p_2$ descends to an isomorphism:

$$A_{\tilde{x}} : \nu(\tilde{\mathcal{F}})_{\tilde{x}} \to T_y(T)$$

for any $\tilde{x} = (\hat{x}, y) \in \tilde{M}$. Let $y \in T$ and $\tilde{x} \in p_2^{-1}(y)$. Set by definition:

$$H(T)_y = A_{\tilde{x}}\, \tilde{H}_{\tilde{x}}\,.$$

As $\tilde{H}$ is invariant under sliding along the leaves of $\tilde{\mathcal{F}}$ (and $p_2^{-1}(y)$ is a leaf of $\tilde{\mathcal{F}}$) it follows that $H(T)_y$ is well defined (*i.e.* its definition does not depend upon the choice of $\tilde{x} \in p_2^{-1}(y)$). Similar considerations apply to the complex structure $J_{T,y}$ given by:

$$J_{T,y} = A_{\tilde{x}} \circ \tilde{J}_{\tilde{x}} \circ A_{\tilde{x}}^{-1}$$

for any $y \in T$ and some $\tilde{x} \in p_2^{-1}(y)$. Thus $T$ becomes a CR manifold (and $H(T)$ is its Levi distribution).

Let $g = h([\gamma]) \in G$. Consider $L_g : T \to T$ given by $L_g(y) = g^{-1}(y)$. Then:

$$p_2 \circ R_{[\gamma]} = L_g \circ p_2 \tag{5}$$

where $R_{[\gamma]} : \tilde{M} \to \tilde{M}$ is the right translation with $[\gamma] \in \pi_1(B, x_0)$. If $X \in T(\tilde{\mathcal{F}})_{\tilde{x}} = \mathrm{Ker}(d_{\tilde{x}} p_2)$ then (by (5)) we obtain:

$$(d_{\tilde{x}\cdot[\gamma]}\, p_2) \circ (d_{\tilde{x}} R_{[\gamma]})X = 0$$

that is $T(\tilde{\mathcal{F}})$ is $\pi_1(B, x_0)$-invariant. Thus $d_{\tilde{x}} R_{[\gamma]}$ descends to an isomorphism:

$$B_{\tilde{x},[\gamma]} : \nu(\tilde{\mathcal{F}})_{\tilde{x}} \to \nu(\tilde{\mathcal{F}})_{\tilde{x}\cdot[\gamma]} \, .$$

Furthermore:

$$(d_y L_g) \circ A_{\tilde{x}} = A_{\tilde{x}\cdot[\gamma]} \circ B_{\tilde{x},[\gamma]}$$

and:

$$\Pi_{\tilde{x}} = \Pi_{\tilde{x}\cdot[\gamma]} \circ B_{\tilde{x},[\gamma]}$$

yield:

$$(d_y L_g) H(T)_y = H(T)_{g^{-1}y} \, .$$

It may be shown similarly that $(L_g)_*$ comutes with $J_T$. Thus $L_g \in \mathrm{Aut}_{\mathrm{CR}}(T)$, Q.E.D.

Given a CR foliation $\mathcal{F}$ defined by suspension of the homomorphism $h : \pi_1(B, x_0) \to \mathrm{Aut}_{\mathrm{CR}}(T)$ one may attempt (in analogy with the case of Riemannian foliations, cf. [13], p. 97-99) to describe the closure of the leaf $L$ passing through a point of the fibre $p^{-1}(x_0) \approx T$, under the assumption that $\mathrm{Aut}_{\mathrm{CR}}(T)$ is compact.

## 3 – Transverse pseudohermitian geometry

Let $(N, T_{1,0}(N))$ be an orientable CR manifold of hypersurface type (*i.e.* $k = 1$). Set $E_p = \{\omega \in T_p^*(N) : \mathrm{Ker}(\omega) \supseteq H(N)_p\}$ for any $p \in N$. This gives a real line bundle $E \subset T^*(N)$. As $N$ is orientable and $H(N)$ is oriented by its complex structure $J_N$ it follows that $E$ admits nowhere zero globally defined sections $\theta_N \in \Gamma^\infty(E)$ each of which is referred to as a *pseudohermitian structure* on $(N, T_{1,0}(N))$. Cf. [15]. The *Levi form* $G_N$ is given by:

$$G_N(X, Y) = (d\theta_N)(X, J_N Y) \tag{6}$$

for any $X, Y \in \Gamma^\infty(H(N))$. Then $(N, T_{1,0}(N))$ is *nondegenerate* if $G_N$ is nondegenerate for some choice of pseudohermitian structure on $N$ (and thus for all).

The basic ideas of pseudohermitian geometry carry over to the context of CR foliations, as follows. Let $(\mathcal{F}, H, J)$ be a CR foliation (whose transverse geometry is modelled on the CR manifold $N$ (of hypersurface type)). On each $U_i$ one may consider the 1-form $\theta_{T,i}$ given by $\theta_{T,i} = f_i^*\theta_N$. Next, define $\theta_i \in \Gamma^\infty(Q^*)$ by $\theta_{i,x} \circ \pi_x = (\theta_{T,i})_x$ for any $x \in U_i$. Clearly $H_x = \mathrm{Ker}(\theta_{i,x})$ for any $x \in U_i$. Let $j \in I$ with $U_{ji} \neq 0$. As $\gamma_{ji} \in \Gamma^\infty_{\mathrm{CR}}(N)$ it follows in particular that $\gamma_{ji}$ is a contact transformation. Therefore:

$$\theta_j = (\lambda_{ji} \circ f_i)\theta_i \tag{7}$$

(on $U_{ji}$) for some nowhere vanishing $C^\infty$ functions $\lambda_{ji} : f_i(U_{ji}) \to \mathbf{R}$. To investigate the properties of $(U_i, \theta_i)$ we only need to look at the case of a simple foliation (defined by a submersion). We may state:

Theorem 4. *Let $f : M \to N$ be a $C^\infty$ submersion with connected fibres from a $C^\infty$ manifold $M$ on-to an orientable* CR *manifold $(N, T_{1,0}(N))$ (of hypersurface type) on which a pseudohermitian structure $\theta_N$ has been fixed. Let $\mathcal{F}$ be the foliation of $M$ tangent to the vertical bundle of $f$. Let $\theta_T = f^*\theta_N$ and $\theta \in \Gamma^\infty(Q^*)$ given by $\theta \circ \pi = \theta_T$. Then $\mathcal{L}_X\theta = 0$ for any $X \in \Gamma^\infty(P)$, i.e. $\theta \in \Gamma^\infty_B(Q^*)$. Assume that $(N, T_{1,0}(N))$ is nondegenerate and set $\xi_x = F_x^{-1}(\xi_{N,f(x)})$ where $\xi_N$ is the characteristic direction of $d\theta_N$ and $F_x : Q_x \to T_{f(x)}(N)$ is given by $F_x \circ \pi_x = d_x f$ for any $x \in M$. Then $\xi \in \Gamma^\infty_B(Q)$ and:*

$$\theta(\xi) = 1, \qquad \xi \rfloor d_Q\theta = 0 \tag{8}$$

Throughout $\Gamma^\infty_B(\Lambda^k Q^*)$ consists of all $\omega \in \Gamma^\infty(\Lambda^k Q^*)$ with $\mathcal{L}_X\omega = 0$ for any $X \in \Gamma^\infty(P)$.

Recall that a CR automorphism $\lambda : N \to N$ is *isopseudohermitian* if $\lambda^*\theta_N = \theta_N$. Clearly, if $\mathcal{F}$ is a $\Gamma$-foliation of $M$ where $\Gamma \subset \Gamma^\infty_{\mathrm{CR}}(N)$ is the subpseudogroup of all (local) isopseudohermitian (with respect to a fixed pseudohermitian structure $\theta_N$ on $N$) CR automorphisms of $N$, then the (local) sections $\theta_i$ glue up to a global section $\theta \in \Gamma^\infty_B(Q^*)$. We are led to the following general considerations. Let $(\mathcal{F}, H, J)$ be a CR foliation (of transverse CR codimension $k = 1$). A globally defined nowhere vanishing section $\theta \in \Gamma^\infty_B(Q^*)$ is a *transverse pseudohermitian structure* if $H = \mathrm{Ker}(\theta)$. By Theorem 4 any simple foliation given by a $C^\infty$ submersion on-to a CR hypersurface $N$ carries a transverse pseudohermitian structure (induced by a fixed pseudohermitian structure on $N$).

Let $\Omega^k_B(\mathcal{F}) = \Omega^k_B(M,\mathcal{F})$ be the $\Omega^0_B(\mathcal{F})$-module of basic $k$-forms, where $\Omega^0_B(\mathcal{F}) = \Omega^0_B(M,\mathcal{F})$ is the ring of all basic $C^\infty$ functions $\lambda$: $M \to \mathbf{R}$.

Let $\mathcal{F}$ be a CR foliation and $\theta$ a transverse pseudohermitian structure. Let $\theta_T = \theta \circ \pi$ be the corresponding basic 1-form. Set $D = \mathrm{Ker}(\theta_T)$. Then $H = D/P$. Next, set:

$$K_x = \{\alpha \in Q^*_x : \mathrm{Ker}(\alpha) \supseteq H_x\}$$

for any $x \in M$. This furnishes a real line subbundle $K \subset Q^*$ and any transverse pseudohermitian structure may be viewed as a (globally defined nowhere zero) section in $K$. Thus, if $\hat{\theta}$ is another transverse pseudohermitian structure then $\hat{\theta} = \lambda\theta$ for some nowhere vanishing basic function $\lambda$ on $M$.

With each $\omega \in \Gamma^\infty(\Lambda^k Q^*)$ we associate a differential $k$-form $\omega_T = \Phi_k\omega$ on $M$ given by:

$$\omega_T(Y_1, \cdots, Y_k) = \omega(\pi Y_1, \cdots, \pi Y_k)$$

for any $Y_1, \cdots, Y_k \in \mathcal{X}$. The map $\Phi_k$ yields a $\mathbf{R}$-linear isomorphism:

$$\Gamma^\infty_B(\Lambda^k Q^*) \approx \Omega^k_B(\mathcal{F})\,.$$

We shall need the differential operator:

$$d_Q = \Phi^{-1}_{k+1} \circ d \circ \Phi_k : \Gamma^\infty_B(\Lambda^k Q^*) \to \Gamma^\infty_B(\Lambda^{k+1} Q^*)\,.$$

Let $\mathcal{F}$ be a CR foliation and $\theta$ a transverse pseudohermitian structure. The *transverse Levi form* $G_\theta$ is defined by:

$$G_\theta(s,r) = (d_Q\theta)(s, Jr) \tag{9}$$

for any $s, r \in \Gamma^\infty(H)$. Then $G_{\lambda\theta} = \lambda G_\theta$. We need the following:

Lemma 1. *For any $\alpha, \beta \in \Gamma^\infty(\mathcal{H})$:*

$$(d_Q\theta)(\alpha,\beta) = (d_Q\theta)(\overline{\alpha},\overline{\beta}) = 0\,. \tag{10}$$

*That is $G_\theta$ (as a real $(0,2)$-tensor field) is symmetric and $G_\theta(Js, Jr) = G_\theta(s,r)$ for any $s, r \in \Gamma^\infty(H)$.*

A CR foliation $(\mathcal{F}, \mathcal{H})$ is *nondegenerate* if $G_\theta$ is nondegenerate for some transverse pseudohermitian structure $\theta$ (and thus for all). Any simple foliation given by a $C^\infty$ submersion on-to a nondegenerate CR manifold (of hypersurface type) is itself nondegenerate. We establish the following:

THEOREM 5. *For any transversally orientable nondegenerate* CR *foliation $(\mathcal{F}, \mathcal{H})$ on which a transverse pseudohermitian structure has been fixed, there is a globally defined nowhere vanishing section $\xi \in \Gamma^\infty(Q)$ so that $\xi \rfloor \theta = 1$ and $\xi \rfloor d_Q\theta = 0$. Such $\xi$ is unique and invariant under sliding along the leaves.*

Let $\mathrm{Null}(d_Q\theta)$ be the null space bundle of $d_Q\theta$, *i.e.* $z \in \mathrm{Null}(d_Q\theta)_x$ iff $z \in Q_x$ and $z \rfloor (d_Q\theta)_x = 0$. Then $\mathrm{rank}_{\mathbf{R}}\, \mathrm{Null}(d_Q\theta) = 1$ by the nondegeneracy of $d_Q\theta$ on $H$. Note that:

$$\mathrm{Null}(d_Q\theta) \approx Q/H \tag{11}$$

(a vector bundle isomorphism). To check (11) it suffices to show that the map $z \mapsto z + H_x$, $z \in \mathrm{Null}(d_Q\theta)_x$, is a bundle monomorphism. This follows from $\mathrm{Null}(d_Q\theta) \cap H = \{0\}$.

As $Q$ is orientable and $H$ oriented by its complex structure it follows that $Q/H$ admits a globally defined nowhere zero section $S$ ($M$ is assumed to be connected). Next there is $\gamma \in \Gamma^\infty(Q)$ so that $S(x) = \gamma(x) + H_x$ for any $x \in M$. Set $\lambda = \theta(\gamma) \in \Omega^0(M)$. Then $\lambda$ is nowhere zero. Indeed, if $\lambda(x) = 0$ for some $x \in M$ then $\gamma(x) \in \mathrm{Ker}(\theta_x) = H_x$, *i.e.* $S(x) = 0$, a contradiction. Set $s = (1/\lambda)\gamma$. By (11) there is $\xi \in \Gamma^\infty(\mathrm{Null}(d_Q\theta))$ so that $\xi(x) + H_x = s(x) + H_x$. Consequently $\xi(x) - s(x) \in H_x = \mathrm{Ker}(\theta_x)$, i.e. $\theta(\xi) = \theta(s) = 1$. To prove the last statement in Theorem 5, note that $\mathcal{L}_X\theta = 0$ and $\theta(\xi) = 1$ yield $0 = (\mathcal{L}_X\theta)\xi = X(\theta\xi) - \theta(\mathcal{L}_X\xi)$ that is $\mathcal{L}_X\xi \in \Gamma^\infty(H)$ for any $X \in \Gamma^\infty(P)$. Next, for any $s \in \Gamma^\infty(H)$ we have $0 = (\mathcal{L}_X d_Q\theta)(\xi, s) = X((d_Q\theta)(\xi, s)) - (d_Q\theta)(\mathcal{L}_X\xi, s) - (d_Q\theta)(\xi, \mathcal{L}_X s)$ so that (by the nondegeneracy of $d_Q\theta$ on $H$) we get:

$$\mathcal{L}_X\xi = 0 \tag{12}$$

for any $X \in \Gamma^\infty(P)$, *i.e.* $\xi \in \Gamma_B^\infty(Q)$.

The transverse vector field $\xi$ in Theorem 5 is referred to as the *characteristic direction* of $(\mathcal{F}, \mathcal{H}, \theta)$. Note that:

$$Q = H \oplus \mathbf{R}\xi \,. \tag{13}$$

By taking into account (13) we may extend the transverse Levi form $G_\theta$ to a semi-Riemannian holonomy invariant bundle metric $g_\theta$ in $Q$ by setting $g_\theta(s,r) = G_\theta(s,r)$, $g_\theta(s,\xi) = 0$ and $g_\theta(\xi,\xi) = 1$, for any $s, r \in \Gamma^\infty(H)$. The holonomy invariance of $g_\theta$ follows from: i) the fact that $H$ is parallel (with respect to the Bott connection of $\mathcal{F}$), ii) $\mathcal{L}_X d_Q\theta = 0$ for any $X \in \Gamma^\infty(P)$, and from (12). Then $g_\theta$ is referred to as the *transverse Webster metric* of $(M, (\mathcal{F}, \mathcal{H}, \theta))$.

The transverse Levi form may be viewed as a Hermitian form on $H \otimes \mathbf{C}$, *i.e.* let $L_\theta$ be given by:

$$\begin{aligned} L_\theta(\alpha, \overline{\beta}) &= -i(d_Q\theta)(\alpha, \overline{\beta}) \\ L_\theta(\alpha, \beta) &= L_\theta(\overline{\alpha}, \overline{\beta}) = 0 \\ L_\theta(\overline{\alpha}, \beta) &= \overline{L_\theta(\alpha, \overline{\beta})} \end{aligned}$$

for any $\alpha, \beta \in \Gamma^\infty(\mathcal{H})$. Then $L_\theta$ and the $\mathbf{C}$-linear extension of $G_\theta$ (to $H \otimes \mathbf{C}$) coincide. A CR foliation $(\mathcal{F}, \mathcal{H})$ is *strictly pseudoconvex* if $(L_\theta)_x(\sigma, \overline{\sigma}) > 0$ for any $\sigma \in \mathcal{H}_x - \{0\}$, $x \in M$, and some transverse pseudohermitian structure $\theta$. If this is the case then $(\mathcal{F}, g_\theta)$ is a Riemannian foliation.

Let $(\mathcal{F}, \mathcal{H})$ be a nondegenerate CR foliation carrying the transverse pseudohermitian structure $\theta$. Let $\{\zeta_\alpha\} \subset \Gamma^\infty_B(\mathcal{H})$ be an admissible frame of $\mathcal{H}$ on $U \subseteq M$. Set $h_{\alpha\overline{\beta}} = L_\theta(\zeta_\alpha, \zeta_{\overline{\beta}})$ where $\zeta_{\overline{\alpha}} = \overline{\zeta_\alpha}$. Then $h_{\alpha\overline{\beta}} : U \to \mathbf{C}$ are basic functions. Let $\lambda_\alpha(x)$ be the eigenvalues of $[h_{\alpha\overline{\beta}}(x)]$, $x \in U$. As $[h_{\alpha\overline{\beta}}]$ is Hermitian each $\lambda_\alpha$ is $\mathbf{R}$-valued and $C^\infty$ thus $L_\theta$ has constant index on $U$ (hence $G_\theta$ is a semi-Riemannian bundle metric in $H$). Assume the Levi form $L_\theta$ to have signature $(r, s)$. Consider the Hermitian form:

$$\langle z, w \rangle_r = \sum_{j=1}^{r} z_j \overline{w}_j - \sum_{j=r+1}^{n} z_j \overline{w}_j$$

where $n = r + s$ and $z, w \in \mathbf{C}^n$. Let:

$$U(r,s) = \{A \in GL(n, \mathbf{C}) : \langle Az, Aw \rangle_r = \langle z, w \rangle_r, \forall z, w \in \mathbf{C}^n\}$$

Let $G \subset GL(q, \mathbf{R})$ consist of all matrices of the form:

$$\begin{bmatrix} g^\alpha_\beta & \Omega^\alpha_\beta & 0 \\ -\Omega^\alpha_\beta & g^\alpha_\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with $[g^\alpha_\beta + i\Omega^\alpha_\beta] \in U(r,s)$. Here $q = 2n+1$. Let $E_{\theta,x}$ consist of all $z \in E_x$ with $z(e_{2n+1}) = \xi(x)$ and $g_\theta(z(e_\alpha), z(e_\beta)) = \epsilon_\alpha \delta_{\alpha\beta}$, $g_\theta(z(e_\alpha), z(e_{\beta+n})) = 0$. Here $\xi$ is the characteristic direction of $(\mathcal{F}, \theta)$. Also $\epsilon_\alpha = 1$ if $1 \le \alpha \le r$ and $\epsilon_\alpha = -1$ if $r+1 \le \alpha \le r+s$. Then $E_\theta$ is a transverse $G$-structure.

## 4 – Degenerate CR manifolds

Let $(M, T_{1,0}(M))$ and $(N, T_{1,0}(N))$ be two CR manifolds (of hypersurface type) of CR dimensions $N = n+k$ and $n$, respectively. Let $f : M \to N$ be a CR submersion (*i.e.* a $C^\infty$ submersion and a CR map).

PROPOSITION 1. *$(M, T_{1,0}(M))$ is degenerate.*

PROOF.

STEP 1. Let $\theta_M$ and $\theta_N$ be choices of pseudohermitian structures on $M$ and $N$, respectively. Then $f^*\theta_N = \lambda\theta_M$ for some nowhere vanishing $\lambda \in \Omega^0(M)$.

If $\omega$ is a differential form on $M$, we adopt the notation $\mathrm{Sing}(\omega) = \{x \in M : \omega_x = 0\}$. Let $x \in \mathrm{Sing}(f^*\theta_N)$. Then $\theta_{N,f(x)} \circ (d_x f) = 0$. On the other hand $d_x f : T_x(M) \to T_{f(x)}(N)$ is on-to. Thus $T_{f(x)}(N) \subseteq \mathrm{Ker}(\theta_{N,f(x)}) = H(N)_{f(x)}$, a contradiction. Thus:

$$\mathrm{Sing}(f^*\theta_N) = \emptyset\,. \tag{14}$$

Let $X \in H(M)$. Then (as $f$ is a CR map) $f_*X \in H(N)$. Hence:

$$(f^*\theta_N)X = \theta_N(f_*X) = 0$$

that is $X \in \mathrm{Ker}(f^*\theta_N)$. Let $x \in M$ and $d = \dim_{\mathbf{R}} \mathrm{Ker}(f^*\theta_N)_x$. Then $2N \le d \le 2N+1$. If $d = 2N+1$ then $x \in \mathrm{Sing}(f^*\theta_N)$, a contradiction. It remains that $d = 2N$, that is:

$$H(M) = \mathrm{Ker}(f^*\theta_N)\,. \tag{15}$$

By (14)-(15) it follows that $f^*\theta_N$ is a pseudohermitian structure on $M$ (and Step 1 is proved).

STEP 2. The Levi form $G_M$ is degenerate on the vertical bundle $P = \mathrm{Ker}(f_*)$.

Let $X \in P$. By Step 1 we have:

$$\theta_M X = \frac{1}{\lambda}(f^*\theta_N)X = \frac{1}{\lambda}\theta_N(f_*X) = 0$$

so that $P \subseteq H(M)$. Then Step 2 follows from the calculation:

$$\begin{aligned} G_M(X,Y) = (d\theta_M)(X, J_M Y) &= d\Big(\frac{1}{\lambda} f^*\theta_N\Big)(X, J_M Y) = \\ &= \left(\Big(d\frac{1}{\lambda}\Big) \wedge f^*\theta_N + \frac{1}{\lambda} d(f^*\theta_N)\right)(X, J_M Y) = \\ &= \frac{1}{\lambda}(d\theta_N)(f_*X, f_*J_M Y) = 0\,. \end{aligned}$$

Proposition 1 is proved. By our Theorems 1 and 4 the vertical bundle $P = \mathrm{Ker}(f_*)$ of $f$ is the tangent bundle of a CR foliation $\mathcal{F}$ of $M$ whose transverse geometry is that of $N$. Also $\mathcal{F}$ is nondegenerate if so is $(N, T_{1,0}(N))$. Its transverse Levi distribution is given by $H = H(M)/P$. One may say loosely that by passing to the quotient $H(M)/P$ one 'factors out' the degeneracy.

In general, let $(M, T_{1,0}(M))$ be a CR manifold (of hypersurface type) of CR dimension $n+k$. Given a pseudohermitian structure $\theta_M$ set:

$$P_{M,x} = \{v \in H(M)_x : (d\theta_M)_x(v,w) = 0, \quad \forall w \in H(M)_x\}$$

for any $x \in M$. Then $P_M$ is involutive and $J_M$-invariant (cf. *e.g.* [7]). Thus (by applying both the Frobenius and Newlander-Nirenberg theorems) if $\dim_{\mathbf{R}} P_{M,x} = 2k = \text{const.}$ then $M$ carries a foliation $\mathcal{F}$ by complex $k$-manifolds (with $T(\mathcal{F}) = P_M$). This is referred to as the *Levi foliation* of $M$. Set $H = H(M)/P_M \subset Q$. If $X \in \Gamma^\infty(P_M)$ and $s \in \Gamma^\infty(H)$ then:

$$0 = (d\theta_M)(X, Y_s) = -\frac{1}{2}\theta_M([X, Y_s])$$

(for some $Y_s \in \Gamma^\infty(H(M))$ with $\pi Y_s = s$) yields $[X, Y_s] \in \Gamma^\infty(H(M))$ and thus:

$$\nabla^0_X s = \pi[X, Y_s] \in \Gamma^\infty(H)$$

*i.e.* $H$ is parallel with respect to the Bott connection of $\mathcal{F}$. Define $J : H \to H$ by setting:

$$Js = \pi J_M Y_s \tag{16}$$

for any $s \in \Gamma^\infty(H)$. As $J_M$ descends to a complex structure in $P_M$ we may extend it (by $\mathbf{C}$-linearity) to $P_M \otimes \mathbf{C}$ and let $P_M^{1,0} = \mathrm{Eigen}(i)$ be the eigenbundle corresponding to the eigenvalue $i$. Similarly, extend $J$ to $H \otimes \mathbf{C}$ and set $\mathcal{H} = \mathrm{Eigen}(i)$.

PROPOSITION 2. $\mathcal{H} = T_{1,0}(M)/P_M^{1,0}$

PROOF. Let $\sigma \in \mathcal{H}$ and $Y_\sigma \in H(M) \otimes \mathbf{C}$ with $\pi Y_\sigma = \sigma$. Then $J\sigma = i\sigma$ yields:

$$J_M Y_\sigma - i Y_\sigma = Z \tag{17}$$

for some $Z \in P_M \otimes \mathbf{C}$. By applying $J_M$ to (17) one gets $J_M = -iZ$, that is $Z \in P_M^{0,1}$ (we set $P_M^{0,1} = \overline{P_M^{1,0}}$). Let us define $W \in H(M) \otimes \mathbf{C}$ by setting:

$$W = Y_\sigma + \frac{i}{2}(\overline{Z} - Z)\,. \tag{18}$$

Then $W \in T_{1,0}(M)$ and (by (18)) $\pi W = \sigma$, *i.e.* $\sigma \in T_{1,0}(M)/P_M^{1,0}$.

As $P_M \rfloor \theta_M = 0$ there is a unique $\theta \in \Gamma^\infty(Q^*)$ so that $\theta \circ \pi = \theta_M$. We have:

THEOREM 6. *Assume $\mathcal{L}_X J_M = 0$ and $\mathcal{L}_X \theta_M = 0$ for any $X \in \Gamma^\infty(P_M)$. Then* i) *$\mathcal{H}$ is a transverse almost* CR *structure, and* ii) *$\theta$ is a transverse pseudohermitian structure.*

REMARK 1. 1) The Lie derivative $\mathcal{L}_X J_M$ is well defined because $[X, Y] \in \Gamma^\infty(H(M))$ for any $X \in \Gamma^\infty(P_M)$ and $Y \in \Gamma^\infty(H(M))$.
2) Due to $(\mathcal{L}_X \theta_M)Y = X(\theta_M Y) - \theta_M([X, Y]) = 0$ for any $Y \in \Gamma^\infty(H(M))$, the hypothesis $\mathcal{L}_X \theta_M = 0$ in Theorem 6 may be weakened to

$$F \rfloor \mathcal{L}_X \theta_M = 0$$

for some complement $F$ of $H(M)$ in $T(M)$.
3) Recall that any CR-straightenable complex foliation is a CR foliation. Thus (by a result of [14]) if $M$ is realized in $\mathbf{C}^{N+1}$ then a sufficient condition for the integrability of $\mathcal{H}$ in Theorem 6 is that the Gauss map $x \in M \mapsto P_{M,x} \in G(k, N+1)$ is a CR map. Here $G(k, N+1)$ denotes the (complex) Grassmann manifold of all complex $k$-subspaces of $\mathbf{C}^{N+1}$.

To prove Theorem 6 let $s \in \Gamma^\infty(H)$. Then:

$$Y_{Js} - J_M Y_s = V$$

for some $V \in \Gamma^\infty(P_M)$. Thus $[X, V] \in \Gamma^\infty(P_M)$ and we may conduct the calculation:

$$(\mathcal{L}_X J)s = \pi\{[X, Y_{Js}] - J_M[X, Y_s]\} = \pi(\mathcal{L}_X J_M)Y_s = 0$$

Clearly $H = \mathrm{Ker}(\theta)$. Also $\mathcal{L}_X \theta = 0$ iff $\mathcal{L}_X \theta_M = 0$, Q.E.D.

REMARK 2. 1) Let $f : M \to N$ be a CR submersion and $P = \mathrm{Ker}(f_*)$. Then:

$$P \subseteq P_M \tag{19}$$

Note that $d_x f : H(M)_x \to H(N)_{f(x)}$ is on-to. Indeed, let $v \in H(N)_{f(x)}$. As $f$ is a submersion, there is $u \in T_x(M)$ so that $(d_x f)u = v$. Then:

$$0 = \theta_{N,f(x)}(v) = \theta_{N,f(x)}(d_x f)u = (f^*\theta_N)_x(u) = \lambda(x)\theta_{M,x}(u)$$

yields $u \in H(M)_x$, Q.E.D. Let $X \in P_M$. Then $f_* X \in H(N)$ and:

$$\begin{aligned}(d\theta_N)(f_*X, H(N)) &= (d\theta_N)(f_*X, f_*H(M)) = \\ &= (d\lambda \wedge \theta_M + \lambda d\theta_M)(X, H(M)) = 0\,.\end{aligned}$$

Consequently:

$$f_* P_M \subseteq P_N$$

Assume $(N, T_{1,0}(N))$ to be nondegenerate. Then $P_N = \{0\}$. Hence $P_M \subseteq \mathrm{Ker}(f_*) = P$. Thus, if $N$ is nondegenerate, then (by (19)) $P$ is the Levi foliation of $M$.
2) In practice, the hypothesis of our Theorem 6 often hold good. For instance, let $(N, T_{1,0}(N))$ be a nondegenerate CR manifold (of hypersurface type). The product manifold $M = N \times \mathbf{C}^k$ carries the complex foliation $\mathcal{F}$ whose leaves are $\{y\} \times \mathbf{C}^k$, $y \in N$. It is easy to see that $M$ is a CR manifold and $\mathcal{F}$ its Levi foliation. Indeed, if $x = (y, \zeta) \in M$ let $\phi_\zeta : N \to M$ and $\psi_y : \mathbf{C}^k \to M$ given by $\phi_\zeta(y) = \psi_y(\zeta) = x$. Then:

$$(d_y \phi_\zeta) T_{1,0}(N)_y \oplus (d_\zeta \psi_y) T^{1,0}(\mathbf{C}^k)_\zeta, \quad (y, \zeta) \in M$$

is a CR structure on $M$ (and $\phi_\zeta$ is a CR immersion). Also, the natural projection $f : M \to N$ is a CR submersion and $\mathcal{F}$ is tangent to $P_M = \mathrm{Ker}(f_*)$. Then (by Theorem 1) $\mathcal{F}$ is a CR foliation of $M$. In particular $(\mathcal{L}_X J_M) Y = (\mathcal{L}_X J) \pi Y = 0$ for any $X \in P_M$ and $Y \in T(M)$. Next, let $\theta_M = f^* \theta_N$ (where $\theta_N$ is a pseudohermitian structure on $N$) and $\theta \in \Gamma^\infty(Q^*)$ given by $\theta \circ \pi = \theta_M$. Then (by Theorem 4) $(\mathcal{L}_X \theta_M) Y = (\mathcal{L}_X \theta) \pi Y = 0$. Clearly $\mathcal{F}$ is a CR-straightenable complex foliation of $M$.

A *realized* CR manifold is a real submanifold $M \subset \mathbf{C}^{N+1}$ whose CR structure is given by $T_{1,0}(M) = T^{1,0}(\mathbf{C}^{N+1}) \cap [T(M) \otimes \mathbf{C}]$. A CR manifold is *realizable* if it is CR diffeomorphic to a realized one. Assume that the Levi form of $M$ has a nontrivial kernel $P_M$ of constant dimension and the transverse Levi form is positive definite (*i.e.* one may factor out the degeneracy toward a strictly pseudoconvex transverse structure). The (local) embeddability problem for degenerate CR manifolds is open (in the $C^\infty$ category). However, in the light of our Theorem 11 it is tempting to conjecture that "foliated" versions of known embeddability results (*e.g.* any strictly pseudoconvex CR manifold $M$ is locally realizable if $M$ is compact (cf. [3]) or if $M$ is noncompact yet of CR dimension $\geq 3$ (cf. [12] and [1])) should hold for strictly pseudoconvex transverse CR structures (occurring on degenerate CR manifolds).

## 5 – The transverse CR complex

Let $(\mathcal{F},\mathcal{H})$ be a CR foliation. We consider the differential operator:

$$\overline{\partial}_Q : \Gamma^\infty_B(\Lambda^k\overline{\mathcal{H}}^*) \to \Gamma^\infty_B(\Lambda^{k+1}\overline{\mathcal{H}}^*)$$

defined by the following considerations. Let $\omega \in \Gamma^\infty_B(\Lambda^k\overline{\mathcal{H}}^*)$ and $\alpha_j \in \Gamma^\infty(\overline{\mathcal{H}})$, $1 \le j \le k+1$. Let $Y_j \in \Gamma^\infty(T(M)\otimes \mathbf{C})$ so that $\pi Y_j = \alpha_j$, $1 \le j \le k+1$. Finally, set:

$$\begin{aligned}(\overline{\partial}_Q\omega)(\alpha_1,\cdots,\alpha_{k+1}) &= \sum_{j=1}^{k+1}(-1)^{j+1}Y_j(\omega(\alpha_1,\cdots,\hat{\alpha}_j,\cdots,\alpha_{k+1}))+\\ &+\sum_{1\le i<j\le k+1}(-1)^{i+j}\omega(\pi[Y_i,Y_j],\alpha_1,\cdots,\hat{\alpha}_i,\cdots,\hat{\alpha}_j,\cdots,\alpha_{k+1})\end{aligned} \tag{20}$$

The definition (20) does not depend upon the choice of representatives $Y_j$ of $\alpha_j$.

Theorem 7. *Let $(\mathcal{F},\mathcal{H})$ be a* CR *foliation of $M$. Then:*

$$\Omega^0_B(\mathcal{F})\otimes\mathbf{C} \xrightarrow{\overline{\partial}_Q} \Gamma^\infty_B(\overline{\mathcal{H}}^*) \xrightarrow{\overline{\partial}_Q} \Gamma^\infty_B(\Lambda^2\overline{\mathcal{H}}^*) \xrightarrow{\overline{\partial}_Q} \cdots \tag{21}$$

*is a cochain complex, i.e.*

$$\overline{\partial}_Q\circ\overline{\partial}_Q = 0 \tag{22}$$

This follows from (20) and the integrability property of $\mathcal{H}$. We refer to (21) as the *transverse Cauchy-Riemann complex* of $(M,\mathcal{F},\mathcal{H})$.

Let $\mathcal{F}$ be nondegenerate. Let $\theta$ be a transverse pseudohermitian structure. Let $\xi$ be the corresponding characteristic direction. Then each $\eta \in \Gamma^\infty_B(\Lambda^kQ^*\otimes\mathbf{C})$ with $\mathcal{H}\rfloor\eta = 0$ and $\xi\rfloor\eta = 0$ may be regarded as an element of $\Gamma^\infty_B(\Lambda^k\mathcal{H}^*)$, and conversely. Then $\eta$ (respectively $\Phi_k\eta$) is referred to as a *transverse* $(0,k)$*-form* (respectively as a *basic* $(0,k)$*-form*) on $M$. In this pseudohermitian setting the complex (21) may be redefined by declaring $\overline{\partial}_Q\eta$ to be the unique transverse $(0,k+1)$-form which coincides with $d_Q\eta$ on $\overline{\mathcal{H}}\otimes\cdots\otimes\overline{\mathcal{H}}$ ($k+1$ factors). By taking into account Proposition 3.11 in [10], vol. I, p. 36, it follows that the two definitions of $\overline{\partial}_Q$ are equivalent. Let $H^k_{\overline{\partial}_Q}(\mathcal{F}) = H^k_{\overline{\partial}_Q}(M, \mathcal{F})$ be the cohomology groups of the complex (21). These are referred to as the *transverse Kohn-Rossi cohomology groups* of $(M, \mathcal{F}, \mathcal{H})$.

THEOREM 8. *Let $\mathcal{F}$ be the simple* CR *foliation on $M$ defined by a $C^\infty$ submersion $f : M \to N$ on-to a nondegenerate* CR *manifold $N$ (of hypersurface type). Then the transverse Kohn-Rossi cohomology groups of $(M, \mathcal{F})$ are isomorphic to the Kohn-Rossi cohomology groups of the base manifold $N$.*

Let $\theta_N$ be a fixed pseudohermitian structure on $N$ and $\xi_N$ the characteristic direction of $(N, \theta_N)$. Let $\alpha$ be a $(0, k)$-form on $N$, *i.e.* $\alpha \in \Gamma^\infty(\Lambda^k T^*(N) \otimes \mathbf{C})$, $T_{1,0}(N) \rfloor \alpha = 0$, and $\xi_N \rfloor \alpha = 0$. Pullbacks of forms on $N$ via $f$ are basic. Thus $f^*\alpha \in \Omega^k_B(\mathcal{F}) \otimes \mathbf{C}$. Set $\Psi_k = \Phi_k^{-1}$ and:

$$\alpha_f = \Psi_k f^* \alpha\,.$$

If $\sigma \in \Gamma^\infty(\mathcal{H})$ then $f_* Y_\sigma \in \Gamma^\infty(T_{1,0}(N))$ (for some $Y_\sigma \in \mathcal{X}(M)$ with $\pi Y_\sigma = \sigma$) and:

$$\sigma \rfloor \alpha_f = \Psi_{k-1} f^* \left[ (f_* Y_\sigma) \rfloor \alpha \right] = 0\,.$$

Similarly $f_* Y_\xi = \xi_N \circ f$ yields:

$$\xi \rfloor \alpha_f = \Psi_{k-1} f^* \left[ (f_* Y_\xi) \rfloor \alpha \right] = \Psi_{k-1} f^* (\xi_N \rfloor \alpha) = 0$$

where $\xi \in \Gamma^\infty_B(Q)$ is the characteristic direction of $(M, \mathcal{F}, \theta)$ and $\theta \circ \pi = f^*\theta_N$. Thus $\alpha_f \in \Gamma^\infty_B(\Lambda^k \overline{\mathcal{H}}^*)$. Assume that $\overline{\partial}_N \alpha = 0$ where:

$$\overline{\partial}_N : \Gamma^\infty(\Lambda^k T_{0,1}(N)^*) \to \Gamma^\infty(\Lambda^{k+1} T_{0,1}(N)^*)$$

is the tangential Cauchy-Riemann operator of $(N, T_{1,0}(N))$. Note that:

$$d_Q \alpha_f = (d\alpha)_f\,.$$

Thus:

$$(\overline{\partial}_Q \alpha_f)(\sigma_1, \cdots, \sigma_{k+1}) = (\overline{\partial}_N \alpha)(f_* Y_{\sigma_1}, \cdots, f_* Y_{\sigma_{k+1}}) = 0$$

for any $\sigma_j \in \Gamma^\infty(\overline{\mathcal{H}})$. Let $H^{0,k}_{KR}(N)$ be the Kohn-Rossi cohomology groups (of the tangential Cauchy-Riemann complex of $N$, cf. J.J. Kohn [11], p. 83). Define $\phi : H^{0,k}_{KR}(N) \to H^k_{\overline{\partial}_Q}(\mathcal{F})$ by setting $\phi([\alpha]) = [\alpha_f]$. To see that $\phi([\alpha])$ is well defined let $\alpha' = \alpha + \overline{\partial}_N \beta$ for some $\beta \in \Gamma^\infty(\Lambda^{k-1} T_{0,1}(N)^*)$. Then $\alpha'_f = \alpha_f + \Psi_k f^* \overline{\partial}_N \beta = \alpha_f + \overline{\partial}_Q \beta_f$. Already $\alpha \mapsto \alpha_f$ is on-to (indeed $f^*\alpha = \Phi_k \omega$ with $\omega \in \Gamma^\infty_B(\Lambda^k \overline{\mathcal{H}}^*)$ may be solved for $\alpha$ as follows. Set $\alpha(Z_1, \cdots, Z_k) = \omega(\pi Y_1, \cdots, \pi Y_k)$ for any $Z_j \in \Gamma^\infty(T_{0,1}(N))$ and some $Y_j \in \mathcal{X}(M) \otimes \mathbf{C}$ with $f_* Y_j = Z_j$. Then $\alpha(Z_1, \cdots, Z_k)$ is well defined because $X \rfloor \Phi_k \omega = 0$ for each $X \in \Gamma^\infty(P)$). To check that $\phi$ is a monomorphism assume that $\alpha_f = \overline{\partial}_Q \eta$ for some $\eta \in \Gamma^\infty_B(\Lambda^{k-1} \overline{\mathcal{H}}^*)$. As $\Omega^{k-1}_B(\mathcal{F}) \approx \Gamma^\infty(\Lambda^{k-1} T^*(N))$ there is a unique $\gamma \in \Gamma^\infty(\Lambda^{k-1} T^*(N) \otimes \mathbf{C})$ so that $f^*\gamma = \Phi_{k-1}\eta$ and therefore $\gamma$ is a $(0, k-1)$-form on $N$ and $\alpha = \overline{\partial}_N \gamma$, Q.E.D.

In analogy with the study of the basic cohomology of foliated manifolds, and encouraged by the substantial progress there (cf. *e.g.* A. El-Kacimi and G. Hector [6]) one may raise several questions related to the cohomology of the complex (21) (*e.g.* existence of spectral sequences abutting on $H^*_{\overline{\partial}_Q}(\mathcal{F})$, finitude and vanishing theorems, etc.). However, we expect a lack of relationship between $H^*_{\overline{\partial}_Q}(\mathcal{F})$ and the basic cohomology of $(M, \mathcal{F})$ (as a foliated counterpart of the - not sufficiently understood as yet - lack of relationship between the Kohn-Rossi and De Rham cohomologies of a CR manifold).

Let $(M, T_{1,0}(M))$ be a CR manifold and CR$(M)$ the set of all CR functions on $M$ (*i.e.* $\lambda \in \mathrm{CR}(M)$ iff $\overline{\partial}_M \lambda = 0$). We establish the following:

Theorem 9. *Let $(M, T_{1,0}(M))$ be a* CR *manifold whose Levi form has a nontrivial kernel $P_M$ with $\dim_{\mathbf{R}} P_{M,x} = 2k$, $x \in M$, and let $\mathcal{F}$ be the foliation of $M$ by complex $k$-manifolds tangent to $P_M$. Assume that $\mathcal{L}_X J_M = 0$, for any $X \in \Gamma^\infty(P_M)$, and that $\mathcal{H}$ is integrable. Then each $\lambda \in H^0_{\overline{\partial}_Q}(\mathcal{F})$ is a basic* CR *function on $M$. Also, there is a natural injection of $H^1_{\overline{\partial}_Q}(\mathcal{F})$ into the first Kohn-Rossi cohomology group of $M$.*

As $\mathcal{L}_X J_M = 0$ it follows (by Theorem 6) that $\mathcal{H}$ is a transverse almost CR structure. Next $\mathcal{H}$ is assumed to be integrable, so that $\mathcal{F}$ is a CR foliation. Let $\lambda \in \Omega^0_B(\mathcal{F}) \otimes \mathbf{C}$ with $\overline{\partial}_Q \lambda = 0$. As (by (20)) $\overline{\partial}_M \lambda = (\overline{\partial}_Q \lambda) \circ \pi$

it follows that:

$$H^0_{\overline{\partial}_Q}(\mathcal{F}) = [\Omega^0_B(\mathcal{F}) \otimes \mathbf{C}] \cap \mathrm{CR}(M)$$

Next, let $\eta$ be a transverse $(0,1)$-form with $\overline{\partial}_Q \eta = 0$. Let $\tilde{\eta}$ be given by (in view of Proposition 2):

$$\tilde{\eta}(Z) = \eta(\pi Z)$$

for any $Z \in T_{0,1}(M)$. Then $(\overline{\partial}_M \tilde{\eta})(Z,W) = (\overline{\partial}_Q \eta)(\pi Z, \pi W)$ for any $Z, W \in T_{0,1}(M)$. The map $H^1_{\overline{\partial}_Q}(\mathcal{F}) \to H^{0,1}_{KR}(M)$ defined by $[\eta] \mapsto [\tilde{\eta}]$ is one-to-one. Indeed, if $[\tilde{\eta}] = 0$ there is $\lambda \in \Omega^0(M) \otimes \mathbf{C}$ so that $\tilde{\eta} = \overline{\partial}_M \lambda$. For any $X \in P_{0,1}$ we see that $0 = X \rfloor \tilde{\eta} = X(\lambda)$ so that $\lambda$ is basic. Finally $\overline{\partial}_Q \lambda = \eta$, Q.E.D.

The authors hope that a theory of CR foliations may lead to a better understanding of degenerate CR manifolds.

## 6 – The transverse Webster connection

Let $(\mathcal{F}, \mathcal{H})$ be a nondegenerate CR foliation endowed with the transverse pseudohermitian structure $\theta$. Let $\xi$ be the characteristic direction of $(\mathcal{F}, \theta)$. Let $H$ be the transverse Levi distribution. Let us extend its complex structure $J : H \to H$ to a bundle morphism $J : Q \to Q$ by requesting $J\xi = 0$. If $g_\theta$ is the transverse Webster metric set:

$$g_{\theta,T}(Y,Z) = g_\theta(\pi Y, \pi Z)$$

for any $Y, Z \in \mathcal{X}(M)$. Let $g$ be a semi-Riemannian (*i.e.* nondegenerate and of constant index) metric on $M$. Assume $g$ is nondegenerate on $P$. Then $T(M) = P \oplus P^\perp$ where $P^\perp$ is the orthogonal complement of $P$ in $T(M)$ with respect to $g$. Let $\sigma_g : Q \to P^\perp$ be the natural isomorphism. Then $\pi(\sigma_g s) = s$ for any $s \in Q$. Also $g$ induces a bundle metric $g_Q$ in $Q$ given by $g_Q(s,r) = g(\sigma_g s, \sigma_g r)$ for any $s, r \in Q$. Then $g$ is referred to as *bundle-like* if $g_Q$ is holonmy invariant (i.e. $\mathcal{L}_X g_Q = 0$ for any $X \in \Gamma^\infty(P)$). Also $g_T(Y,Z) = g_Q(\pi Y, \pi Z)$ is the *associated transverse metric* of $g$. By slightly generalizing Proposition 3.3 in [13], p. 80, we see that there is a bundle-like semi-Riemannian metric $g$ on $M$ whose associated transverse metric is precisely $g_{\theta,T}$. Indeed, let $h$ be just any Riemannian metric on $M$ and $P_h$ the orthogonal complement of $P$ in $T(M)$ with respect to $h$. If $Y \in \mathcal{X}(M)$ then $Y_P$ and $Y_{P_h}$ denote respectively its components with respect to the direct sum decomposition $T(M) = P \oplus P_h$. Then we define $g$ by setting:

$$g(Y,Z) = h(Y_P, Z_P) + g_{\theta,T}(Y_{P_h}, Z_{P_h}) \tag{23}$$

for any $Y, Z \in \mathcal{X}(M)$. If $L_\theta$ has signature $(r,s)$, $r+s=n$, then $g_\theta$ has signature $(2r+1, s)$. Hence $g$ (given by (23)) has signature $(2r+p+1, 2s)$ where $p = \dim_{\mathbf{R}} P_x$, $x \in M$.

Let $\nabla$ be a connection in $Q \to M$ and $T_\nabla$ its torsion tensor field (*i.e.* $T_\nabla(Y,Z) = \nabla_Y \pi Z - \nabla_Z \pi Y - \pi[Y,Z]$ for any $Y, Z \in \mathcal{X}(M)$). If $\nabla$ is *adapted* (*i.e.* $\nabla_X = \nabla^0_X$ for any $X \in \Gamma^\infty(P)$) then we define Tor and $\tau : Q \to Q$ by setting:

$$\operatorname{Tor}(s,t) = T_\nabla(Y_s, Y_t)$$
$$\tau(s) = \operatorname{Tor}(\xi, s)$$

for any $s, t \in \Gamma^\infty(Q)$ and some $Y_s, Y_t \in \mathcal{X}(M)$ with $\pi Y_s = s$, $\pi Y_t = t$. It is easy to check that $\operatorname{Tor}(s,t)$ is well defined. Indeed, for any $X, X' \in \Gamma^\infty(P)$ we have $T_\nabla(X, X') = 0$ due to the integrability of $P$ and $T_\nabla(Y_s, X) = 0$ because $\nabla$ is adapted. We may state

THEOREM 10. *Let $(\mathcal{F}, \mathcal{H})$ be a nondegenerate* CR *foliation and $\theta$ a fixed transverse pseudohermitian structure. Then there is a unique adapted connection $\nabla$ in $Q$ satisfying the following axioms:*
i) *$H$ is parallel with respect to $\nabla$*
ii) *$\nabla J = 0$, $\nabla g_\theta = 0$*
iii) *$\tau J + J\tau = 0$*
iv) *$\forall \alpha, \beta \in \Gamma^\infty(\mathcal{H}) : \operatorname{Tor}(\alpha,\beta) = 0$, $\operatorname{Tor}(\alpha, \overline{\beta}) = 2i L_\theta(\alpha, \overline{\beta})\xi$.*

Let $g$ be a bundle-like semi-Riemannian metric on $M$ whose associated transverse metric is $g_{\theta,T}$. To establish uniqueness, let $\nabla$ be an adapted connection in $Q$ obeying to i)-iv). By i) and $\nabla J = 0$ it follows that:

$$\nabla_X \Gamma^\infty(\mathcal{H}) \subset \Gamma^\infty(\mathcal{H}), \quad \nabla_X \Gamma^\infty(\overline{\mathcal{H}}) \subset \Gamma^\infty(\overline{\mathcal{H}})$$

for any $X \in \mathcal{X}(M)$. Let $\rho_+ : Q \otimes \mathbf{C} \to \mathcal{H}$ and $\rho_- : Q \otimes \mathbf{C} \to \overline{\mathcal{H}}$ be the natural projections associated with the direct sum decomposition:

$$Q \otimes \mathbf{C} = \mathcal{H} \oplus \overline{\mathcal{H}} \oplus \mathbf{C}\xi \,.$$

By iv) we have:

$$\mathrm{Tor}(\overline{\alpha}, \beta) = -2iL_\theta(\beta, \overline{\alpha})\xi$$

or:

$$\nabla_{\sigma_g \overline{\alpha}}\beta - \nabla_{\sigma_g \beta}\overline{\alpha} - \pi[\sigma_g \overline{\alpha}, \sigma_g \beta] = -2iL_\theta(\beta, \overline{\alpha})\xi$$

which yields:

$$\nabla_{\sigma_g \overline{\alpha}}\beta = \rho_+ \pi[\sigma_g \overline{\alpha}, \sigma_g \beta] \tag{24}$$

for any $\alpha, \beta \in \Gamma^\infty(\mathcal{H})$. Let $\omega$ be given by:

$$\omega = -d_Q\theta \,.$$

The axiom $\nabla g_\theta = 0$ may be written:

$$X(g_\theta(s, r)) = g_\theta(\nabla_X s, r) + g_\theta(s, \nabla_X r)$$

for any $X \in T(M)$, $s, r \in Q$. In particular for $s = \xi$ one has:

$$(\nabla_X \theta)r = g_\theta(\nabla_X \xi, r) \,. \tag{25}$$

If $r \in H$ then (25) gives $g_\theta(\nabla_X \xi, r) = 0$ or $\pi_H \nabla_X \xi = 0$ (where $\pi_H : Q \to H$ is the natural projection associated with (13)). Similarly, if $r = \xi$ then (25) becomes $\theta(\nabla_X \xi) = 0$. Therefore:

$$\nabla \xi = 0 \,. \tag{26}$$

Note that:

$$\nabla \omega = 0 \tag{27}$$

as a consequence of ii) and $\omega(\alpha, \overline{\beta}) = -ig_\theta(\alpha, \overline{\beta})$. Let us take the complex conjugate of (24) and use (27) so that to obtain:

$$\omega(\nabla_{\sigma_g \alpha}\beta, \overline{\gamma}) = (\sigma_g \alpha)(\omega(\beta, \overline{\gamma})) - \omega(\beta, \rho_- \pi[\sigma_g \alpha, \sigma_g \overline{\gamma}]) \tag{28}$$

for any $\alpha, \beta, \gamma \in \mathcal{H}$. Set:

$$S(g) = \sigma_g(\xi)\,.$$

Next, define $T_{S(g)}$ by setting:

$$T_{S(g)} = -\frac{1}{2} J \circ (\mathcal{L}_{S(g)} J)\,.$$

Then (by (26)):

$$\nabla_{S(g)} r = \mathcal{L}_{S(g)} r + \tau(r)\,. \tag{29}$$

By ii) and (29) we have:

$$\begin{aligned} 0 = (\nabla_{S(g)} J) r &= \nabla_{S(g)} J r - J \nabla_{S(g)} r = \\ &= \pi[S(g), \sigma_g J r] + \tau(Jr) - J\pi[S(g), \sigma_g r] - J\tau(r) = \\ &= \mathcal{L}_{S(g)} J r - J \mathcal{L}_{S(g)} r + (\tau J - J\tau) r \end{aligned}$$

so that (by iii)):

$$\tau = T_{S(g)}\,. \tag{30}$$

Summing up (by (24), (26), and (28)-(30)) we have:

$$\begin{cases} \nabla_{\sigma_g \overline{\alpha}} \beta & = \rho_+ \pi[\sigma_g \overline{\alpha}, \sigma_g \beta] \\ \nabla_{\sigma_g \alpha} \beta & = U_{\alpha\beta}(g) \\ \nabla_{S(g)} \beta & = \mathcal{L}_{S(g)} \beta + T_{S(g)} \beta \\ \nabla \xi & = 0 \end{cases} \tag{31}$$

where $U_{\alpha\beta}(g) \in \mathcal{H}$ is defined by:

$$\omega(U_{\alpha\beta}(g), \overline{\gamma}) = (\sigma_g \alpha)(\omega(\beta, \overline{\gamma})) - \omega(\beta, \rho_- \pi[\sigma_g \alpha, \sigma_g \overline{\gamma}])\,.$$

The uniqueness statement in Theorem 10 is completely proved. To establish existence, let $g$ be a bundle-like semi-Riemannian metric on $M$ inducing the transverse Webster metric in $Q$. Let $\nabla : \Gamma^\infty(T(M) \otimes \mathbf{C}) \times \Gamma^\infty(Q \otimes \mathbf{C}) \to \Gamma^\infty(Q \otimes \mathbf{C})$ be defined by (31) together with:

$$\begin{cases} \nabla_{\sigma_g \alpha} \overline{\beta} & = \overline{\nabla_{\sigma_g \overline{\alpha}} \beta} \\ \nabla_{\sigma_g \overline{\alpha}} \overline{\beta} & = \overline{\nabla_{\sigma_g \alpha} \beta} \\ \nabla_{S(g)} \overline{\beta} & = \overline{\nabla_{S(g)} \beta} \end{cases} \tag{32}$$

and:

$$\nabla_X = \nabla^0_X$$

for any $\alpha, \beta \in \mathcal{H}$, $X \in P$. Before going any further, note that the definition (31) does not depend upon the choice of $g$. Indeed, if $g'$ is another bundle-like semi-Riemannian metric inducing $g_\theta$ then there is a natural bundle morphism:

$$\epsilon = \epsilon_{g,g'} : Q \to P$$

given by $\epsilon_{g,g'}(s) = \sigma_{g'}(s) - \sigma_g(s)$ for any $s \in Q$. Then:

$$T_{S(g')}\beta = T_{S(g)}\beta + \frac{1}{2} J\{\nabla^0_{\epsilon(J\beta)}\xi - \nabla^0_{\epsilon(\xi)} J\beta - J\nabla^0_{\epsilon(\beta)}\xi + J\nabla^0_{\epsilon(\xi)}\beta\}\,.$$

Using $\nabla^0 J = 0$ (as $\mathcal{H}$ is a transverse CR structure) and $\xi \in \Gamma^\infty_B(Q)$ we obtain:

$$T_{S(g')}\beta = T_{S(g)}\beta \tag{33}$$

for any $\beta \in \mathcal{H}$. At this point we may use:

$$[\epsilon(\overline{\alpha}), \epsilon(\beta)] \in P \otimes \mathbf{C}$$
$$\nabla^0_{\epsilon(\beta)}\overline{\alpha} \in \overline{\mathcal{H}}$$

and the first identity in (31) to conduct the following calculation:

$$\nabla_{\sigma_{g'}\overline{\alpha}}\beta = \nabla_{\sigma_g\overline{\alpha}}\beta + \nabla^0_{\epsilon(\overline{\alpha})}\beta = \rho_+\pi[\sigma_g\overline{\alpha}, \sigma_g\beta] + \nabla^0_{\epsilon(\overline{\alpha})}\beta = \rho_+\pi[\sigma_{g'}\overline{\alpha}, \sigma_{g'}\beta]\,.$$

Next, using $\nabla^0\omega = 0$ one may derive:

$$U_{\alpha\beta}(g') = U_{\alpha\beta}(g) + \nabla^0_{\epsilon(\alpha)}\beta$$

so that (by the second identity in (31)):

$$\nabla_{\sigma_{g'}\alpha}\beta = \nabla_{\sigma_g\alpha}\beta + \nabla^0_{\epsilon(\alpha)}\beta = U_{\alpha\beta}(g')\,.$$

Finally:

$$\nabla_{S(g')}\beta = \nabla_{S(g)}\beta + \nabla^0_{\epsilon(\xi)}\beta = \mathcal{L}_{S(g)}\beta + T_{S(g)}\beta + \nabla^0_{\epsilon(\xi)}\beta = \mathcal{L}_{S(g')}\beta + T_{S(g')}\beta\,.$$

Taking into account (33) we adopt the notation $T_\xi = T_{S(g)}$. Note that:

$$J^2 = -I + \theta \otimes \xi\,. \tag{34}$$

Also:

$$\mathcal{L}_{S(g)}\theta = 0 \tag{35}$$

as a consequence of:

$$(\mathcal{L}_{S(g)}\theta)r = S(g)(\theta r) - \theta(\pi[S(g), \sigma_g r]) = 2(d_Q\theta)(\xi, r) = 0\,.$$

It is straightforward that:

$$J \circ (\mathcal{L}_{S(g)}J) + (\mathcal{L}_{S(g)}J) \circ J = 0 \tag{36}$$

$$\mathcal{L}_{S(g)}\xi = 0, \qquad \theta(\mathcal{L}_{S(g)}r) = 0 \tag{37}$$

for any $r \in H$. Next (36) yields:

$$JT_\xi + T_\xi J = 0\,. \tag{38}$$

Let $\beta \in \mathcal{H}$. We have (by (36) and (38)):

$$\begin{aligned}
J(\mathcal{L}_{S(g)}\beta + T_\xi\beta) &= \mathcal{L}_{S(g)}J\beta - (\mathcal{L}_{S(g)}J)\beta + JT_\xi\beta = \\
&= \mathcal{L}_{S(g)}J\beta + (\mathcal{L}_{S(g)}J)J^2\beta + JT_\xi\beta = \\
&= \mathcal{L}_{S(g)}J\beta + JT_\xi\beta - J(\mathcal{L}_{S(g)}J)J\beta = i(\mathcal{L}_{S(g)}\beta + T_\xi\beta)
\end{aligned}$$

that is:

$$\beta \in \mathcal{H} \Longrightarrow \mathcal{L}_{S(g)}\beta + T_\xi\beta \in \mathcal{H} \tag{39}$$

We may conduct the following calculation:

$$\begin{aligned}
0 &= (d_Q^2\theta)(\xi, \beta, \overline{\gamma}) = -(d_Q\omega)(\xi, \beta, \overline{\gamma}) = \\
&= -\frac{1}{3}\{S(\omega(\beta, \overline{\gamma}))+ \\
&\quad + (\sigma_g\beta)(\omega(\overline{\gamma}, \xi)) + (\sigma_g\overline{\gamma})(\omega(\xi, \beta)) - \omega(\pi[S(g), \sigma_g\beta], \overline{\gamma})+ \\
&\quad - \omega(\pi[\sigma_g\beta, \sigma_g\overline{\gamma}], \xi) - \omega(\pi[\sigma\overline{\gamma}, S(g)], \beta)\} = \\
&= -\frac{1}{3}\{S(g)(\omega(\beta, \overline{\gamma})) - \omega(\mathcal{L}_S\beta, \overline{\gamma}) + \omega(\mathcal{L}_{S(g)}\overline{\gamma}, \beta)\}
\end{aligned}$$

which yields:

$$\mathcal{L}_{S(g)}\omega = 0\,. \tag{40}$$

Next, the following calculation:

$$\begin{aligned}\omega(T_\xi r, t) + \omega(r, T_\xi t) &= \frac{1}{2}\{\omega((\mathcal{L}_{S(g)}J)r, Jt) + \omega(Jr, (\mathcal{L}_{S(g)}J)t)\} = \\ &= \frac{1}{2}\{(\mathcal{L}_{S(g)}\omega)(r,t) - (\mathcal{L}_{S(g)}\omega)(Jr, Jt)\}\end{aligned}$$

leads (by (40)) to:

$$\omega(T_\xi r, t) + \omega(r, T_\xi t) = 0 \tag{41}$$

for any $r, t \in Q$. Finally:

$$(d_Q\omega)(\alpha, \beta, \overline{\gamma}) = 0$$

yields:

$$\begin{aligned}&(\sigma_g\alpha)(\omega(\beta,\overline{\gamma})) + (\sigma_g\beta)(\omega(\overline{\gamma},\alpha)) + \omega(\overline{\gamma}, \pi[\sigma_g\alpha, \sigma_g\beta]) + \\ &\qquad + \omega(\alpha, \rho_-\pi[\sigma_g\beta, \sigma_g\overline{\gamma}]) + \omega(\beta, \rho_-\pi[\sigma_g\overline{\gamma}, \sigma_g\alpha]) = 0\end{aligned} \tag{42}$$

for any $\alpha, \beta, \gamma \in \mathcal{H}$. At this point one may check the axioms. Firstly (by (31)-(32)):

$$\begin{aligned}\mathrm{Tor}(\alpha, \overline{\beta}) &= \nabla_{\sigma_g\alpha}\overline{\beta} - \nabla_{\sigma_g\overline{\beta}}\alpha - \pi[\sigma_g\alpha, \sigma_g\overline{\beta}] = \\ &= \rho_-\pi[\sigma_g\alpha, \sigma_g\overline{\beta}] - \rho_+\pi[\sigma_g\overline{\beta}, \sigma_g\alpha] - \pi[\sigma_g\alpha, \sigma_g\overline{\beta}] = \\ &\quad - \theta(\pi[\sigma_g\alpha, \sigma_g\overline{\beta}])\xi = 2(d_Q\theta)(\alpha, \overline{\beta})\xi = 2iL_\theta(\alpha, \overline{\beta})\xi\end{aligned}$$

Lemma 2.

$$\alpha, \beta \in \mathcal{H} \Longrightarrow \pi[\sigma_g\alpha, \sigma_g\beta] \in \mathcal{H}$$

As $\mathrm{Tor}(\alpha,\beta)=U_{\alpha\beta}(g)-U_{\beta\alpha}(g)-\pi[\sigma_g\alpha,\sigma_g\beta]$ it follows (by Lemma 2) that $\mathrm{Tor}(\alpha,\beta)\in\mathcal{H}$. Then (by (42)) we have $\omega(\mathrm{Tor}(\alpha,\beta),\overline{\gamma})=0$ and axiom iv) is checked. Using (31) we have $\mathrm{Tor}(\xi,\alpha)=\nabla_{S(g)}\alpha-\pi[S(g),\sigma_g\alpha]=T_\xi\alpha$. Noting also that:

$$T_\xi\xi=0 \tag{43}$$

we may conclude that:

$$\mathrm{Tor}(\xi,r)=T_\xi r \tag{44}$$

for any $r\in Q$. Then (38) yields iii) in Theorem 10. Finally:

$$\begin{aligned}(\nabla_{\sigma_g\alpha}\omega)(\beta,\overline{\gamma})&=(\sigma_g\alpha)(\omega(\beta,\overline{\gamma}))-\omega(\nabla_{\sigma_g\alpha}\beta,\overline{\gamma})-\omega(\beta,\nabla_{\sigma_g\alpha}\overline{\gamma})=\\&=(\sigma_g\alpha)(\omega(\beta,\overline{\gamma}))-\omega(U_{\alpha\beta}(g),\overline{\gamma})-\omega(\beta,\rho_-\pi[\sigma_g\alpha,\sigma_g\overline{\gamma}])=0\end{aligned}$$

and:

$$\begin{aligned}(\nabla_{S(g)}\omega)(\beta,\overline{\gamma})&=S(g)(\omega(\beta,\overline{\gamma}))-\omega(\nabla_{S(g)}\beta,\overline{\gamma})-\omega(\beta,\nabla_{S(g)}\overline{\gamma})=\\&=S(g)(\omega(\beta,\overline{\gamma}))+\\&\quad-\omega(\mathcal{L}_{S(g)}\beta+T_\xi\beta,\overline{\gamma})-\omega(\beta,\mathcal{L}_{S(g)}\overline{\gamma}+T_\xi\overline{\gamma})=\\&=(\mathcal{L}_{S(g)}\omega)(\beta,\overline{\gamma})-\{\omega(T_\xi\beta,\overline{\gamma})+\omega(\beta,T_\xi\overline{\gamma})\}=0\end{aligned}$$

(by (40)-(41)) yield $\nabla\omega=0$ (which together with $\nabla\xi=0$ implies $\nabla g_\theta=0$).

It is known that with the Webster connection of a nondegenerate CR manifold $M$ (of CR dimension $n$) one may associate the pseudoconformal curvature tensor $S^\alpha_{\beta\rho\overline{\sigma}}$ (cf. (3.8) in [15], p. 35) and $S^\alpha_{\beta\rho\overline{\sigma}}=0$ iff $M$ is locally CR equivalent to the sphere $S^{2n+1}\subset\mathbf{C}^{n+1}$ (cf. [4]). In view of our Theorem 10 it is tempting to look for a foliated analogue of this result.

## 7 – Embedding transverse CR structures

Let $M$ be a $m$-dimensional $C^\infty$ manifold, $m=2n+k+p$, and $(\mathcal{F},\mathcal{H})$ a CR foliation of $M$ of codimension $q=2n+k$ and transverse CR dimension $n$. An *embedding* of $(M,\mathcal{H})$ is a $C^\infty$ immersion:

$$\psi=(g_1,\cdots,g_{n+k},f_1,\cdots,f_r):M\to\mathbf{C}^N$$

with $N = n + k + r$, $r \geq p$, so that the following conditions are satisfied:
i) $g_j \in \Omega^0_B(\mathcal{F}) \otimes \mathbf{C}$,
ii) $\overline{\partial}_Q g_j = 0\,,\ 1 \leq j \leq n+k$.
Set $g = (g_1, \cdots, g_{n+k}) : M \to \mathbf{C}^{n+k}$. As each $g_j$ is basic the differential $d_x g$ induces a map $G_x : Q_x \to T_{g(x)}(\mathbf{C}^{n+k})$. Finally, we request:
iii) for any $x \in M$, $G_x$ is one-to-one.
The embedding $\psi$ is *generic* if $r = p$. A pair $(M, \mathcal{H})$ for which an embedding $\psi$ exists is termed *embeddable.*

Let $\psi : M \to \mathbf{C}^N$ be an embedding of $(M, \mathcal{H})$. Then:

$$G_x(\mathcal{H}_x) \subset T^{1,0}(\mathbf{C}^{n+k})_{g(x)}\,. \tag{45}$$

Here $T^{1,0}(\mathbf{C}^{n+k})$ denotes the holomorphic tangent bundle of $\mathbf{C}^{n+k}$. To prove (45) let $\{\zeta_\alpha\}$ be an admissible frame and $T_\alpha$ some foliate complex vector fields with $\pi T_\alpha = \zeta_\alpha$. Then (by ii)) in the definition of $\psi$) we have $T_{\overline{\alpha}}(g_j) = 0$ where $T_{\overline{\alpha}} = \overline{T_\alpha}$. Let $(\zeta^1, \cdots, \zeta^{n+k})$ be the natural complex coordinates on $\mathbf{C}^{n+k}$. Finally:

$$\begin{aligned} G_x \zeta_{\overline{\alpha}}(x) &= (d_x g) T_{\overline{\alpha}}(x) = \\ &= T_{\overline{\alpha}}(g_j) \frac{\partial}{\partial \zeta^j} + T_{\overline{\alpha}}(\overline{g_j}) \frac{\partial}{\partial \overline{\zeta}^j} \in T^{1,0}(\mathbf{C}^{n+k})_{g(x)}\,. \end{aligned}$$

If additionally $q = 2n + 1$ then:

$$G_x \mathcal{H}_x = T^{0,1}(\mathbf{C}^{n+1})_{g(x)} \cap G_x(Q_x \otimes \mathbf{C}) \tag{46}$$

for any $x \in M$. Indeed, let $d$ be the complex dimension of the right hand term in (46). Then (by (45)) $n \leq d \leq n + 1$. If $d = n + 1$ then $T^{1,0}(\mathbf{C}^{n+1})_{g(x)} \subset G_x(Q_x \otimes \mathbf{C})$ and (by taking complex conjugates) one gets a contradiction. Thus $d = n$, Q.E.D.

We call $(M, \mathcal{H})$ *locally embeddable* if for any $x \in M$ there is an open neighborhood $U$ of $x$ in $M$ and an embedding $\psi : U \to \mathbf{C}^N$ of $(U, \mathcal{H}_U)$, where $\mathcal{H}_U$ denotes the portion of $\mathcal{H}$ over $U$. We may state the following:

THEOREM 11. *Let $M$ be a $m$-dimensional real analytic manifold, $m = 2n + k + p$, and $(N, T_{1,0}(N))$ a real analytic* CR *manifold of type $(n, k)$. Let $\mathcal{F}$ be a $\Gamma^\omega_{\mathrm{CR}}(N)$-foliation of $M$ of codimension $q = 2n + k$ and $\mathcal{H}$ its* (*real analytic*) *transverse* CR *structure. Then $(M, \mathcal{H})$ is locally embeddable.*

Near $x_0 \in M$, $\mathcal{H}$ is generated by $n$ real analytic sections $\zeta_\alpha \in \Gamma^\omega_B(\mathcal{H})$ so that $[\zeta_\alpha, \zeta_\beta] \in \Gamma^\omega(\mathcal{H})$. Using a real analytic foliated coordinate system $(y^1, \cdots, y^q, x^1, \cdots, x^p)$ for $M$, we may assume that $M$ is an open subset of $\mathbf{R}^{q+p}$ containing the origin and that $\zeta_\alpha = \pi L_\alpha$ for some real analytic foliate vector field $L_\alpha$ in $T(\mathbf{R}^{q+p}) \otimes \mathbf{C}$. We write:

$$\overline{L}_\alpha = \sum_{j=1}^{q} a_{\alpha j}(y, x) \frac{\partial}{\partial y^j}, \qquad 1 \le \alpha \le n$$

for some $C^\omega$ functions $a_{\alpha j} : \mathbf{R}^{q+p} \to \mathbf{C}$. As both $\overline{L}_\alpha$ and $\partial/\partial y^j$ are foliate, $a_{\alpha j}$ are basic, *i.e.* $a_{\alpha j} = a_{\alpha j}(y)$. Since $\{\zeta_{\overline{\alpha}}\}_{1 \le \alpha \le n}$ are linearly independent and $\{\pi(\partial/\partial y^j)\}_{1 \le j \le q}$ a frame of $Q$, the matrix $[a_{\alpha j}(0)]$ has complex rank $n$. By reordering the coordinates if needed, we may assume the $n \times n$ block $A = [a_{\alpha\beta}]$ is nonsingular in a neighborhood $U$ of $0 \in \mathbf{R}^{q+p}$. Set $y = (t, u)$, $t \in \mathbf{R}^n$, $u \in \mathbf{R}^{n+k}$. By multiplying the coefficients of $\{\zeta_{\overline{\alpha}}\}$ with $A^{-1}$ we obtain another admissible frame of $\overline{\mathcal{H}}$ (over $U$) of the form $\zeta_{\overline{\alpha}} = \pi \overline{L}_\alpha$ where:

$$\overline{L}_\alpha = \frac{\partial}{\partial t^\alpha} + \sum_{j=1}^{n+k} \lambda_{\alpha j}(t, u) \frac{\partial}{\partial u^j}, \qquad 1 \le \alpha \le n$$

for some $C^\omega$ functions $\lambda_{\alpha j} : U \to \mathbf{C}$ (depending only on $y = (t, u)$). The Lie product $[\overline{L}_\alpha, \overline{L}_\beta]$ has no $(\partial/\partial t^\alpha)$-component. Also:

$$[\zeta_{\overline{\alpha}}, \zeta_{\overline{\beta}}] = \sum_{\gamma=1}^{n} C^\gamma_{\alpha\beta} \zeta_{\overline{\gamma}}$$

(because $\{\zeta_\alpha\}$ is admissible) so that:

$$[\zeta_{\overline{\alpha}}, \zeta_{\overline{\beta}}] = 0, \qquad 1 \le \alpha, \beta \le n. \tag{47}$$

Let $\zeta \in \mathbf{C}^n$, $w \in \mathbf{C}^{n+k}$ and $z \in \mathbf{C}^p$ be the complexifications of $t \in \mathbf{R}^n$, $u \in \mathbf{R}^{n+k}$ and $x \in \mathbf{R}^p$, respectively (so that $t = \mathrm{Re}(\zeta)$, $u = \mathrm{Re}(w)$ and $x = \mathrm{Re}(z)$). By replacing $t$ and $u$ by $\zeta$ and $w$ in the power series expansion of $\lambda_{\alpha j}$ (about 0) we get functions $\tilde{\lambda}_{\alpha j} : \mathbf{C}^{2n+k} \to \mathbf{C}$ which are holomorphic in a neighborhood $\tilde{U}$ of $0 \in \mathbf{C}^{2n+k}$ and $\tilde{\lambda}_{\alpha j}(t, u) = \lambda_{\alpha j}(t, u)$. Define:

$$\tilde{L}_\alpha = \frac{\partial}{\partial \zeta^\alpha} + \sum_{j=1}^{n+k} \tilde{\lambda}_{\alpha j}(\zeta, w) \frac{\partial}{\partial w^j}, \qquad 1 \le \alpha \le n.$$

Note that:

$$
\frac{\partial\tilde{\lambda}_{\alpha j}}{\partial\zeta^{\alpha}}(t,u)=\frac{\partial\lambda_{\alpha j}}{\partial t^{\alpha}}(t,u),\ \frac{\partial\tilde{\lambda}_{\alpha j}}{\partial w^{\ell}}(t,u)=\frac{\partial\lambda_{\alpha j}}{\partial u^{\ell}}(t,u) \tag{48}
$$

(because $\tilde{\lambda}_{\alpha j}(\zeta,w)$ are holomorphic in $\zeta$ and $w$). At this point (47)-(48) and the identity theorem for holomorphic functions yield:

$$
[\tilde{L}_{\alpha},\tilde{L}_{\beta}]=0
$$

on $\tilde{U}\subset\mathbf{C}^{2n+k}$. Therefore we may apply Lemma 1 in A. Boggess, [2], p. 56, to conclude that there is a holomorphic map $(W_1,\cdots,W_{n+k}):\mathbf{C}^n\times\mathbf{C}^{n+k}\to\mathbf{C}^{n+k}$ (defined on a possibly smaller neighborhood $\tilde{U}$ of $0\in\mathbf{C}^{2n+k}$) so that:

$$
\begin{gathered}
\tilde{L}_{\alpha}W_j=0\\
W_j(0,w)=w_j,\quad (0,w)\in\tilde{U}\,.
\end{gathered}
$$

Similarly (*i.e.* again by Lemma 1 in [2], p. 56, for the operators:

$$
\tilde{L}_{\alpha}+\sum_{s=1}^{p}\mu_{\alpha s}\frac{\partial}{\partial z^s}
$$

with $\mu_{\alpha s}=0$) there is a holomorphic function $(V_1,\cdots,V_{n+k},\tilde{f}_1,\cdots,\tilde{f}_p):\mathbf{C}^n\times\mathbf{C}^{n+k}\times\mathbf{C}^p\to\mathbf{C}^{n+k}\times\mathbf{C}^p$ so that:

$$
\begin{aligned}
\tilde{L}_{\alpha}V_j&=0, &\qquad \tilde{L}_{\alpha}\tilde{f}_s&=0\\
V_j(0,w,z)&=w_j, &\qquad \tilde{f}_s(0,w,z)&=z_s\,.
\end{aligned}
$$

Let $\rho$ be the projection $(\zeta,w,z)\mapsto(\zeta,w)$ and set $\tilde{g}_j=W_j\circ\rho$ for $1\leq j\leq n+k$. Next consider $\tilde{\psi}=(\tilde{g}_1,\cdots,\tilde{g}_{n+k},\tilde{f}_1,\cdots,\tilde{f}_p)$ and define the $C^{\omega}$ map $\psi=(g_1,\cdots,g_{n+k},f_1,\cdots,f_p)$ by setting:

$$
\psi(t,u,x)=\tilde{\psi}(t,u,x)
$$

for $(t,u,x)\in\mathbf{R}^n\times\mathbf{R}^{n+k}\times\mathbf{R}^p$. Note that $\overline{L}_{\alpha}g_j=\tilde{L}_{\alpha}\tilde{g}_j=0$ on $\tilde{U}\cap(\mathbf{R}^n\times\mathbf{R}^{n+k})$. Also $g_j$ are basic. Moreover $\psi(0,u,x)=\tilde{\psi}(0,u,x)=(u,x)$. Let us show that $\psi$ is a generic embedding. As $\overline{L}_\alpha g_j = 0$ yields $\overline{\partial}_Q g_j = 0$ one should only check that $d_0\psi$ has real rank $2n + k + p$. Set $\psi = X + iY$. Note that $(u, x) = \psi(0, u, x) = X(0, u, x) + iY(0, u, x)$ yields:

$$d_0\psi = \begin{pmatrix} (\partial X/\partial t)(0) & I_{n+k+p} \\ (\partial Y/\partial t)(0) & 0 \end{pmatrix}.$$

Also, the imaginary parts of $\overline{L}_\alpha g_j = 0$ and $\overline{L}_\alpha f_s = 0$ may be written in the following matrix form:

$$\frac{\partial Y}{\partial t}(0) = -\begin{pmatrix} I_{n+k} \\ 0 \end{pmatrix} \cdot (\mathrm{Im}\ \lambda)^t(0)$$

where $\mathrm{Im}\ \lambda = [\mathrm{Im}(\lambda_{\alpha j})]$. Next (by $\mathcal{H} \cap \overline{\mathcal{H}} = \{0\}$) $\{\zeta_1 - \zeta_{\overline{1}}, \cdots, \zeta_n - \zeta_{\overline{n}}\}$ are linearly independent (over $\mathbf{C}$) and therefore:

$$\frac{1}{2i}(\zeta_{\overline{\alpha}} - \zeta_\alpha) = \sum_{j=1}^{n+k} \mathrm{Im}(\lambda_{\alpha j})\pi\frac{\partial}{\partial u^j}$$

shows that $(\mathrm{Im}\ \lambda)(0)$ has rank $n$. Set $g = (g_1, \cdots, g_{n+k})$. Similarly, to show that $\mathrm{rank}(d_0 g) = 2n + k$ set $g = U + iV$. Then $g(0, u, x) = u$ yields:

$$d_0 g = \begin{pmatrix} (\partial U/\partial t)(0) & I_{n+k} & 0 \\ (\partial V/\partial t)(0) & 0 & 0 \end{pmatrix}$$

and the imaginary part of $\overline{L}_\alpha g_j = 0$ may be written:

$$\frac{\partial V}{\partial t} = -(\mathrm{Im}\ \lambda)^t(0)$$

so that $\mathrm{rank}(\partial V/\partial t)(0) = n$, Q.E.D.

To give an example of embedded transverse CR structure, let $N \subset \mathbf{C}^{n+1}$ be a nondegenerate real hypersurface and $M = N \times \mathbf{C}^{k+1}$ with the natural complex foliation $\mathcal{F}$. Let $\mathcal{H}$ be the transverse CR structure of $\mathcal{F}$. Then:

$$\psi = (g_1, \cdots, g_{n+1}, f_1, \cdots, f_{2k}) : M \to \mathbf{C}^{n+1+2k}$$

given by $g_j(z,\zeta) = z^j$ and $f_a(z,\zeta) = \zeta^a$, $f_{k+a}(z,\zeta) = 0$ for $1 \leq j \leq n+1$ and $1 \leq a \leq k$, is a generic embedding of $(M, \mathcal{H})$. Indeed, let $\{T_\alpha\}$ be a (local) frame of $T_{1,0}(N)$. Thus $\{T_\alpha, \partial/\partial\zeta^a\}$ is a (local) frame of $T_{1,0}(M)$. The coordinate functions $z^j$ are holomorphic so that $z^j_{|N} \in \mathrm{CR}(N)$ and hence $g_j$ are CR functions on $M$. For each leaf $S = \{z\} \times \mathbf{C}^k$ of $\mathcal{F}$ we have $(g_j)_{|S} = \text{const.}$ so that $g_j \in \Omega^0_B(\mathcal{F})$. Then (by Theorem 9) $\overline{\partial}_Q g_j = 0$. If $E$ is a vector bundle over $\mathbf{C}^k$ let $g^*E$ be the pullback of $E$ by $g : M \to \mathbf{C}^{n+1}$, $g(z,\zeta) = z$. Finally, we need to check that $G : Q \to g^*T(\mathbf{C}^{n+1})$ is a bundle monomorphism. To this end let $G_x s = 0$. There is $Y = (V, W)$ so that $\pi_x Y = s$ and $V \in T_z(N)$, $W \in T_\zeta(\mathbf{C}^k)$. Then $V = (d_x g)Y = Gs = 0$ so that $Y = (0, W) \in P_{M,x}$, Q.E.D.

## REFERENCES


[1] T. Akahori: *A new approach to the local embedding theorem of* CR *structures, the local embedding theorem for n* $\geq 4$, Memoires A.M.S., vol. 366, 1987.

[2] A. Boggess: CR *manifolds and the tangential Cauchy-Riemann complex*, Studies in Advanced Mathematics, CRC Press, Boca Raton-Ann Arbor-Boston-London, 1991.

[3] L. Boutet de Monvel: *Intégration des équations de Cauchy-Riemann induites formelles*, Sém. Goulaouic-Lions-Schwartz, (1974-75), Centre Math. École Politech., Paris, 1975.

[4] S.S. Chern – J. Moser: *Real hypersurfaces in complex manifolds*, Acta Math., **133** (1974), 219-271.

[5] E.M. Chirka: *Introduction to the geometry of* CR *manifolds*, Russian Math. Surveys, (1) **46** (1991), 95-197.

[6] A. El-Kacimi – G. Hector: *Décomposition de Hodge basique pour un feuilletage Riemannien*, Ann. Inst. Fourier, (3) **36** (1986), 207-227.

[7] M. Freeman: *The Levi form and local complex foliations*, Trans. A.M.S., **57** (1976), 369-370.

[8] A. Haefliger: *Structures feuilletées et cohomologie à valeur dans un faisceau de groupoïdes*, Comment. Math. Helvet., **32** (1958), 248-329.

[9] A. Haefliger: *Variétés feuilletées*, Ann. Scuola Norm. Sup., Pisa, **16** (1962), 367-397.

[10] S. Kobayashi – K. Nomizu: *Foundations of differential geometry*, Interscience Publishers, New York, vol. I, 1963, vol. II, 1969.

[11] J.J. Kohn: *Boundaries of complex manifolds*, Proc. Conf. on Complex Analysis, Minneapolis, 1964, 81-94, Springer-Verlag, New York.

[12] M. Kuranishi: *Strongly pseudoconvex structures over small balls*, I-II-III, Ann. of Math., **115** (1982), 451-500, ibid., **116** (1982), 1-64, ibid., **116** (1982), 249-330.

[13] P. Molino: *Riemannian foliations*, Progress in Math., vol. 73, edited by J. Oesterlé and A. Weinstein (Birkhäuser, Boston-Basel) 1988.

[14] S.I. Pinchuk – S.I. Tsyganov: CR-*straightening of real manifolds in* $\mathbf{C}^n$, Mat. Zametki, to appear.

[15] S. Webster: *Pseudohermitian structures on a real hypersurface*, J. Diff. Geometry, (1) **13** (1978), 25-41.

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