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Approximation of linearizable mappings of (C,0) through resonant diffeomorphisms

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Riassunto: *Dato un diffeomorfismo analitico linearizzabile* F *di* $(C, 0)$, *caratterizzato da un autovalore* $\lambda = e^{i\omega}$, *ove* $\omega/(2\pi) \in \mathbb{R}$ *è un numero di Brjuno, si considera una successione di diffeomorfismi* $F^{(q)}$ *che converge ad* F. *Gli autovalori di* $F^{(q)}$ *sono* $e^{2i\pi p/q}$ *ove* $q \in \mathbb{N}$ *e* p *è la parte intera di* $q\omega/(2\pi)$. *I diffeomorfismi* $F^{(q)}$ *sono perturbazioni di una forma normale che converge alla parte lineare di* F *per* $q \to \infty$.

La funzione $\Psi^{(q)}$ *che coniuga* $F^{(q)}$ *alla sua forma normale è analizzata al primo ordine perturbativo: si dimostra che è sommabile secondo Borel e analitica su settori. Si dimostra inoltre che una sottosuccessione di* $\Psi^{(q)}$ *converge in un intorno dell'origine alla funzione che coniuga* F *con la sua parte lineare.*

Abstract: *Given a linearizable analytic diffeomorphism* F *of* $(C, 0)$ *with eigenvalue* $\lambda = e^{i\omega}$, *where* $\omega/(2\pi) \in \mathbb{R}$ *is a Brjuno number, we consider a sequence of diffeomorphisms* $F^{(q)}$ *converging to* F. *The eigenvalues of* $F^{(q)}$ *are* $e^{2i\pi p/q}$ *where* $q \in \mathbb{N}$ *and* p *is the integer part of* $q\omega/(2\pi)$. *The diffeomorphisms* $F^{(q)}$ *are perturbations of a normal form converging to the linear part of* F *for* $q \to \infty$.

The conjugation function $\Psi^{(q)}$ *of* $F^{(q)}$ *with its normal form is analysed at the first perturbation order, and shown to be Borel summable and sectorially analytical. We prove the convergence of a sub-sequence of* $\Psi^{(q)}$ *in a neighbourhood of the origin to the function that conjugates* F *with its linear part.*

Key Words and Phrases: *Resurgent functions – Brjuno numbers – Resonant limit – Small divisors – Siegel problem – Sectorial analyticity*
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1 – Introduction

During the last decades, the analysis of the dynamical and geometrical properties of resonant holomorphic mappings has been succesfully accomplished [1]-[5]. Let us consider the set of all holomorphic mappings of $(\mathbf{C},\mathbf{0})$, whose linear part is $z' = \lambda_q z$, with $\lambda_q = \exp(2\pi i p/q)$: then, the flower theorem [1] states that the dynamics of a map belonging to this set is homeomorphic to the dynamics of its normal form

$$(1.1) \qquad y' = \frac{\lambda_q y}{(1+y^{kq})^{1/kq}} \qquad y \in \mathbf{C}\,,$$

i.e. the motion takes place on $2kq$ petals (see fig. 1). Correspondingly, if we consider the function Ψ that conjugates the resonant diffeomorphism to its normal form (1.1), one has that the analytic structure of Ψ is given by a collection of $2kq$ functions, analytic on sectors of aperture smaller than $2\pi/q$, which differ by terms that are exponentially small in the distance from the origin [2], [3] ,[6], [7].

On the other hand, in the diophantine case ($\lambda = \exp(i\omega)$, with $\omega/2\pi$ Brjuno number [8]) the dynamics of the mappings in the neighbourhood of the origin takes place on closed curves that are analytically diffeomorphic to circles [9]-[12]. Correspondingly, the conjugation function Ψ to the normal form (*i.e.* the linear map) is analytic on a closed neighbourhood of the origin.

The aim of this paper is to investigate the transition from resonant orbits, which have the flower structure with $2kq$ petals, to nonresonant orbits, which are diffeomorphic to circles. The considered model is a sequence of analytic mappings $F^{(q)}(x)$ of $(\mathbf{C},\mathbf{0})$

$$(1.2) \qquad F^{(q)}(x) = \frac{\lambda_q x}{(1+x^q)^{1/q}} + \epsilon f(x) \qquad x \in \mathbf{C}$$

with resonant eigenvalues λ_q converging to a limit $\lambda_\infty = e^{i\omega}$, where $\omega/2\pi$ is a Brjuno number. This dynamically nontrivial limit $q \to \infty$ is examined by proving that the sequence of the normalizing transformations $\Psi^{(q)}(x)$ converges to the function that linearizes the limit map $F^{\infty}(x)$. The type of convergence is specified in the followings. The present proof applies to the homologic equation.

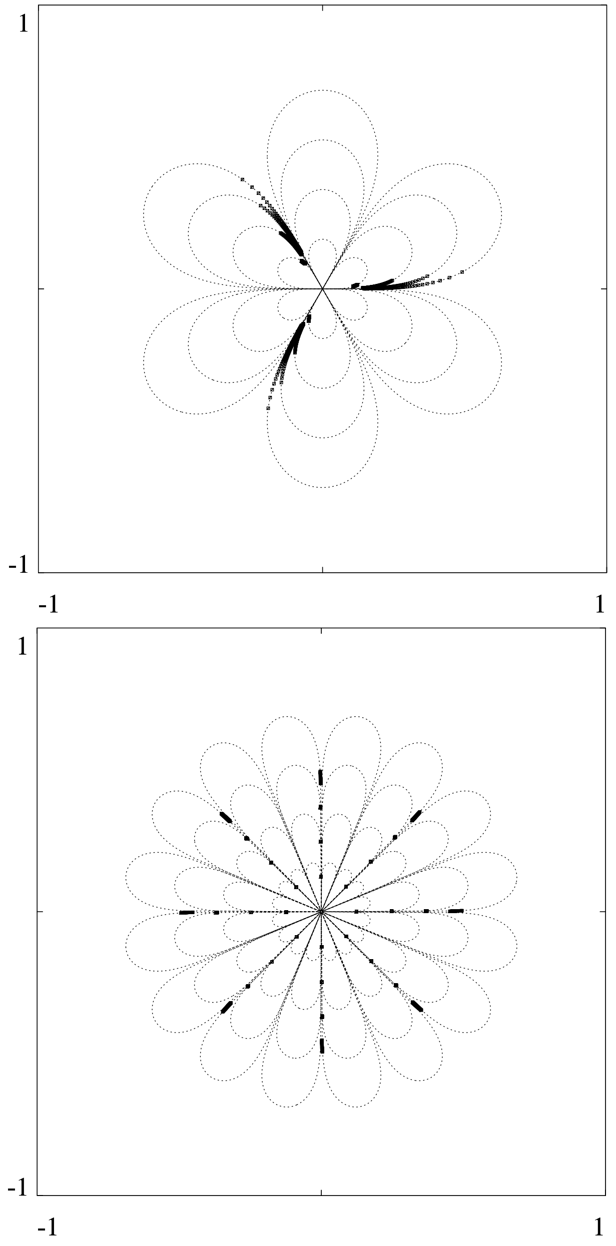

Fig. 1. — Phase portrait of the interpolating flow (solid line) and of the first 1000 iterates (dots) relative to four different initial conditions of the normal form (1.1) with $kq = 3$ (up) and $kq = 8$ (down).

The geometric meaning is quite evident and can be read on the sequence of normal forms (1.1) (see fig. 1): as q increases the radial motion on each of the q petals is slowing down and ceases in the limit where it is a pure rotation; the conjugation $\Psi^{(q)}(x)$ becomes analytic via a mechanism that is clear in the Borel plane where the residues of the poles (at the homologic level) lying on the $2q$ anti-Stokes lines [7] vanish. A key point consists in choosing the sequence where the rationals p_r/q_r are the truncations of the continued fraction of the Brjuno number $\omega/2\pi$.

In order to analyse the limit, we consider the conjugation of $F^{(q)}$ to a more general normal form $U^{(q)}$, called preliminary, that differs from the normal form (1.1) in terms of order y^{q+2}. The preliminary normal form is expressed by a divergent power series that can be resummed to $2q$ functions analytic on sectors of aperture smaller than $2\pi/q$, centered in the origin. In this way we considerably simplify the functional equation that defines $\Psi^{(q)}$, and the analysis of the limit. The main result of this paper is expressed in the following propositions, where both the formal and the analytic properties of the conjugating function $\Psi^{(q)}$ and of the preliminary normal form $U^{(q)}$ in its dependence on the parameter q are analyzed.

PROPOSITION 1. *We consider the subsequence of p_r, q_r given by the continued fraction expansion of ω [8]; if we expand the conjugating function in a power series in the small parameter ϵ, $\Psi^{(q_r)}(x) = x + \epsilon\psi^{(q_r)}(x) + O(\epsilon^2)$, then for $r \to \infty$ the first order term $\psi^{(q_r)}$ formally converges to the solution of the homologic Siegel problem $\psi^{(\infty)}$. Moreover, the preliminary normal form $U^{(q_r)}$ formally converges to the linear part (i.e. ,to the normal form) of $F^{(\infty)}$.*

PROPOSITION 2. *For each q_r, the function $\psi^{(q_r)}$ can be split into two parts: $\psi^{(q_r)} = \psi_I^{(q_r)} + \psi_{II}^{(q_r)}$; the first one is analytic on a neighbourhood of the origin and in the limit uniformly converges to the solution of the homologic Siegel problem. The second part $\psi_{II}^{(q_r)}$ can be resummed to $2q_r$ functions analytic on sectors Σ_{j_r} (where $j_r = 1, \ldots, 2q_r$) of aperture smaller than 2π; even though each domain has a measure that tends to zero for $r \to \infty$, the union of these domains always includes an open neighbourhood of the origin. In the limit $r \to \infty$, each one of the $2q_r$ resummed functions converges to zero on the domain of analyticity.*

Proposition 3. *Using the same subsequence p_r, q_r of Proposition 2, the preliminary normal form $U^{(q_r)}$ converges in a neighbourhood of the origin to the linear part of $F^{(\infty)}$ according to the same properties given for the conjugating function $\psi^{(q_r)}$.*

The proof is based on the evaluation of the conjugating function according to the resurgent methods of the Borel resummation [2], [5], which are standards of the resonant holomorphic theory, and through a direct estimate of $\psi^{(q)}$ in its representation of the Laplace transform.

The plan of the paper is the following: in section 2 we write down the functional equation that defines $\Psi^{(q)}$ and $U^{(q)}$; in section 3 we recall some properties of the Borel transform in $\mathbf{C}_q$ and we explicit the formal solution. The convergence proof for both functions is given in section 4.

2 – Functional equation

We consider a sequence of mappings of $(\mathbf{C}, 0)$ that are small perturbations of the resonant standard shift (1.1)

$$F^{(q)}(x) = \frac{\lambda_q x}{(1+x^q)^{1/q}} + \epsilon f(x) \qquad f(x) = \sum_{i=2}^{\infty} f_i x^i \,, \tag{2.1}$$

where $f(x)$ is taken independent of q and analytic on the disc $D_R = \{|x| \leq R\}$ with $R < 1$. The eigenvalue $\lambda_q = \exp(2\pi i p/q)$ tends to $\lambda_\infty = \exp(i\omega)$, and satisfies the Cremer condition

$$\sup_{r\in\mathbf{N}} \frac{\log q_{r+1}}{q_r} < +\infty \,, \tag{2.2}$$

where q_r are the denominators of the approximations of ω given by the continued fraction [1]. When $q \to \infty$ the resonant map $F^{(q)}$ tends uniformly in D_R to the mapping

$$F^{(\infty)}(x) = \lambda_\infty x + \epsilon f(x) \,. \tag{2.3}$$

The orbits of $F^{(\infty)}(x)$ in the neighbourhood of the origin are deformed closed circles, and one can build a convergent transformation $\Psi^{(\infty)}$ that conjugates the map to its linear part [9] $y' = \lambda_\infty y$. If one expands the transformation in a power series of ϵ, $y = \Psi^{(\infty)}(x) = x + \epsilon\psi^{(\infty)}(x) + O(\epsilon^2)$, the first order term satisfies the homologic equation $\psi^{(\infty)}(\lambda_\infty x) - \lambda_\infty\psi^{(\infty)}(x) = -f(x)$ whose solution is

$$\psi^{(\infty)}(x) = \sum_{i=2}^{+\infty} \frac{f_i}{\lambda_\infty - (\lambda_\infty)^i} x^i\,; \tag{2.4}$$

using the Brjuno condition one can prove that $\psi^{(\infty)}$ is analytic in a neighbourhood of the origin.

On the other hand, one can conjugate the resonant map $F^{(q)}$ to a preliminary normal form at order one in ϵ, defined according to

$$\begin{aligned} U^{(q)}(y) &= \frac{\lambda_q y}{(1+y^q)^{1/q}} + \epsilon\sum_{k=1}^{\infty} f_{kq+1} y^{kq+1} + \epsilon u^{(q)}(y) \\ u^{(q)}(y) &= \sum_{j=q+2}^{\infty} u_j y^j \end{aligned} \tag{2.5}$$

through a transformation

$$y = \Psi^{(q)}(x) = x + \epsilon\psi^{(q)}(x) + O(\epsilon^2) \tag{2.6}$$

that contains only nonresonant terms

$$\psi^{(q)}(x) = \sum_{j\neq kq+1, j=2}^{\infty} \psi_j x^j \equiv \sum_j^{*} \psi_j x^j\,. \tag{2.7}$$

The functional equation reads

$$\begin{aligned} f(x) + \psi^{(q)}\left(\frac{\lambda_q x}{(1+x^q)^{1/q}}\right) = {} & \frac{\lambda_q}{(1+x^q)^{1+1/q}}\psi^{(q)}(x) + \\ & + \sum_{k=1}^{\infty} f_{kq+1} x^{kq+1} + u(x)\,. \end{aligned} \tag{2.8}$$

We choose $\Psi^{(q)}$ to satisfy the functional equation

$$\sum_j^{*} f_j x^j + \psi^{(q)}\left(\frac{\lambda_q x}{(1+x^q)^{1/q}}\right) - \lambda_q\psi^{(q)}(x) = 0 \tag{2.9}$$

so that $u^{(q)}$ is given by

$$u^{(q)}(x) = \lambda_q \left(1 - \frac{1}{(1+x^q)^{1+1/q}}\right) \psi^{(q)}(x)\,. \tag{2.10}$$

For the sake of simplicity of notation, throughout the next sections the $^{(q)}$ will be omitted.

3 – Formal analysis

Proposition 1. *Given the functional equation (2.9), the formal solution of ψ is given by*

$$\begin{aligned}\psi(x) &= \frac{1}{\lambda_q} \sum_j^{*} f_j x^j \sum_{k=0}^{\infty} \lambda_q^{k(j-1)} \frac{1}{(1+kx^q)^{j/q}} = \\ &= \sum_{j=2}^{q} \frac{f_j}{\lambda_q - \lambda_q^j} x^j + O(x^{q+2})\end{aligned} \tag{3.1}$$

and therefore in the limit $q \to \infty$, $\lambda_q \to \lambda_\infty$, one formally recovers the conjugation function of the homologic Siegel problem (2.4)*.*

Proof. In order to write down the formal solution and, in the following sections, study the analytic properties of ψ, we perform the transformation from $\mathbf{C}$ to $\mathbf{C}_q$

$$w = x^{-q} \qquad x \in \mathbf{C} \qquad w \in \mathbf{C}_q \tag{3.2}$$

so that we can exploit the results relative to mappings tangent to the identity [2–5]. We define

$$\begin{aligned}\overline{\psi}(w) &\equiv \psi(w^{-1/q}) = \sum_i^{*} \psi_i w^{-i/q} \\ \overline{f}(w) &\equiv \sum_i^{*} f_i w^{-i/q}\,.\end{aligned} \tag{3.3}$$

The functional equation (2.9) for ψ is transformed to the following form

$$\overline{\psi}(e^{-2\pi i p}(w+1)) - \lambda_q \overline{\psi}(w) + \overline{f}(w) = 0\,. \tag{3.4}$$

We remark that we have to take into account that $e^{-2\pi ip} \neq 1$: in fact, this factor is responsible for the linear dynamics of the map, which on $\mathbf{C}_q$ corresponds to jump $p-1$ sheets each iteration.

We remind the reader of the following definitions and properties of the Borel transform on $\mathbf{C}_q$ [13], [14].

DEFINITION 1. Given a formal power series defined on $\mathbf{C}_q$

$$\hat{h}(w) = \sum_{i=1}^{+\infty} h_i w^{-i/q} \qquad w \in \mathbf{C}_q\,, \tag{3.5}$$

the formal Borel transform of h is given by

$$h_B(t) = \sum_{i=1}^{+\infty} \frac{h_i}{(i/q-1)!} t^{i/q-1} \qquad t \in \mathbf{C}_q\,, \tag{3.6}$$

where the factorial of a rational number is provided by the gamma function

$$(i/q-1)! \equiv \Gamma(i/q-2)\,. \tag{3.7}$$

DEFINITION 2. Let $h_B(t)$ be a power series defined on $\mathbf{C}_q$, analytic on a neighbourhood of the origin except singularities of the type $1/t^{i/q}$, with $i = 1, \dots, q-1$. Then if far from the origin the following estimate holds:

$$|h_B(\rho e^{i\theta})| < A(\theta)\exp(B(\theta)\rho) \qquad \theta \in [0, 2\pi q] \quad A(\theta), B(\theta) \in \mathbf{R}^+\,, \tag{3.8}$$

then we can define the Laplace transform of h_B, carried along the direction θ

$$h_\theta(w) = \int_0^{e^{i\theta}\infty} e^{-tw} h_B(t)dt \qquad w \in \mathbf{C}_q\,. \tag{3.9}$$

PROPERTY 1. *$h_\theta(w)$ is analytic on the domain $\mathrm{Re}(e^{i\theta}w) > B(\theta)$.*

PROPERTY 2. *$\forall\theta \in [0, 2\pi q], h_\theta(w)$ has the same asymptotic expansion of $\hat{h}(w)$ in the neighbourhood of $w = +\infty$.*

Property 3. *If $h_B(t)$ has no singularities in $t=\rho e^{i\theta}$, for $\rho \neq 0$, $\theta \in [\theta_0, \theta_1]$ and $|\theta_0 - \theta_1| < \pi$ then h_{θ_1}, h_{θ_2} define a function that is analytic on the union of the two domains $Re(e^{i\theta_1}w) > B(\theta_1)$, $\mathrm{Re}(e^{i\theta_2}w) > B(\theta_2)$.*

The proofs of these statements are analogous to the case of $\mathbf{C}$. Properties 1-3 show that given an asymptotic expansion $\hat{h}$ whose Borel transform satisfies the conditions of definition 2 (this happens, for instance, if the $\hat{h}$ coefficients are dominated by a factorial), one can reconstruct a function analytic on sectors. Finally, we give the analogous of the property $[h(w+1)]_B = e^{-t}h_B(t)$ for the case of $\mathbf{C}_q$.

Property 4.

$$[h(e^{-2\pi ip}(w+1))]_B = e^{2\pi ip}e^{-t}h_B(e^{2\pi ip}t)\,. \tag{3.10}$$

The proof is straightforward and consists in expanding in Taylor series both sides of (3.10) *and reordering the homogeneous polynomials.*

Using the above properties, we make the Borel transform of the functional equation (3.4), obtaining

$$e^{2\pi ip}e^{-t}\overline{\psi}_B(e^{2\pi ip}t) - \lambda_q\overline{\psi}_B(t) + \overline{f}_B(t) = 0\,. \tag{3.11}$$

In order to explicit $\overline{\psi}_B$ we substitute t with $e^{2\pi ip}t$ and multiply by $e^{2\pi ip}$ e^{-t}/λ_q:

$$\frac{e^{-2t}}{\lambda_q}e^{4\pi ip}\overline{\psi}_B(e^{4\pi ip}t) - e^{-t}e^{2\pi ip}\overline{\psi}_B(e^{2\pi ip}t) + \frac{e^{-t}}{\lambda_q}e^{2\pi ip}\overline{f}_B(e^{2\pi ip}t) = 0\,, \tag{3.12}$$

so that we can substitute in (3.11) and obtain

$$\frac{e^{-2t}}{\lambda_q}e^{4\pi ip}\overline{\psi}_B(e^{4\pi ip}t) - \lambda_q\overline{\psi}_B(t) + \overline{f}_B(t) + \frac{e^{-t}}{\lambda_q}e^{2\pi ip}\overline{f}_B(e^{2\pi ip}t) = 0\,. \tag{3.13}$$

We apply this procedure q times, ending up with

$$\frac{e^{-qt}}{(\lambda_q)^{q-1}}e^{2\pi ipq}\overline{\psi}_B(e^{2\pi ipq}t) - \lambda_q\overline{\psi}_B(t) + \sum_{k=0}^{q-1}\frac{e^{-kt}}{(\lambda_q)^k}e^{2\pi ipk}\overline{f}_B(e^{2\pi ipk}t) = 0 \tag{3.14}$$

and since in $\mathbf{C}_q$ $e^{2\pi ipq}=1$, we can give the following explicit expression for $\overline{\psi}_B$

$$\lambda_q(e^{-qt}-1)\overline{\psi}_B(t)=-\sum_{k=0}^{q-1}\frac{e^{-kt}}{(\lambda_q)^k}e^{2\pi ipk}\overline{f}_B(e^{2\pi ipk}t)\,. \tag{3.15}$$

We expand the r.h.s. of (3.15), using the definition of $\overline{f}_B$ (3.6); since $\overline{f}_B$ is analytic in a neighbourhood of the origin, we can perform the summation over k ending up with

$$\begin{aligned}&\sum_{k=0}^{q-1}\frac{e^{-kt}}{(\lambda_q)^k}\sum_j^{*}\frac{f_j}{(j/q-1)!}(\lambda_q)^{kj}t^{j/q-1}=\\&\quad=(1-e^{-qt})\sum_j^{*}\frac{f_j}{(j/q-1)!}\,\frac{t^{j/q-1}}{1-e^{-t}\lambda_q^{j-1}}\,;\end{aligned} \tag{3.16}$$

therefore $\overline{\psi}_B$ reads

$$\overline{\psi}_B(t)=\frac{1}{\lambda_q}\sum_j^{*}\frac{f_j}{(j/q-1)!}\,\frac{t^{j/q-1}}{1-e^{-t}\lambda_q^{j-1}}\,. \tag{3.17}$$

The expression (3.17) allows the computation of the coefficients of ψ by applying the Laplace transform:

$$\begin{aligned}\overline{\psi}_\theta(w)&=\int_0^{e^{i\theta}\infty}e^{-tw}\overline{\psi}_B(t)dt=\\&=\frac{1}{\lambda_q}\int_0^{e^{i\theta}\infty}e^{-tw}\sum_j^{*}\frac{f_j}{(j/q-1)!}\,\frac{t^{j/q-1}}{1-e^{-t}\lambda_q^{j-1}}dt\,;\end{aligned} \tag{3.18}$$

exchanging the integral with the series and performing the integration we obtain

$$\overline{\psi}_\theta(w)=\frac{1}{\lambda_q}\sum_j^{*}f_j\sum_{k=0}^{\infty}\lambda_q^{k(j-1)}\frac{1}{(w+k)^{j/q}} \tag{3.19}$$

so that on the plane $x=w^{-1/q}$ one recovers the expression (3.1). This completes the proof of Proposition 1.

4 – Convergence proof

PROPOSITION 2. *Let ω satisfy the Cremer condition* (2.2). *We consider the subsequence q_r, p_r that approximate ω according to the algorithm of the continued fraction. For each q_r, the function $\psi^{(q_r)}$ can be split into two parts: $\psi^{(q_r)} = \psi_I^{(q_r)} + \psi_{II}^{(q_r)}$; the first one is analytic on a neighbourhood of the origin and in the limit uniformly converges to the solution of the homologic Siegel problem. The second part $\psi_{II}^{(q_r)}$ can be resummed to $2q$ functions analytic on sectors Σ_{j_r} (where $j = 1, \dots, 2q$) of aperture smaller than 2π; even though each domain has a measure that tends to zero for $r \to \infty$, the union of these domains always include an open neighbourhood of the origin. In the limit $r \to \infty$, each one of the $2q$ resummed functions converges to zero on the domain of analyticity.*

PROOF. We write $\psi(x)$ as the Laplace transform of $\overline{\psi}_B(t)$, separating out the part that in the limit tends to solution of the Siegel homologic problem from the remainder:

$$\begin{aligned}\overline{\psi}_B(t) = \overline{\psi}_{B,I}(t) + \overline{\psi}_{B,II}(t) = \sum_j^* \frac{f_j}{(j/q_r-1)!}\,\frac{t^{j/q_r-1}}{\lambda_{q_r}-(\lambda_{q_r})^j}+ \\ +\sum_j^* \frac{f_j}{(j/q_r-1)!}\,\frac{(\lambda_{q_r})^j t^{j/q_r-1}}{\lambda_{q_r}-(\lambda_{q_r})^j}\,\frac{e^{-t}-1}{\lambda_{q_r}-e^{-t}(\lambda_{q_r})^j}\,;\end{aligned} \tag{4.1}$$

therefore we define

$$\begin{aligned}\psi_{\theta,I}(x) \equiv \int_0^{e^{i\theta}\infty} e^{-t/x^{q_r}} \sum_j^* \frac{f_j}{(j/q_r-1)!}\,\frac{t^{j/q_r-1}}{\lambda_{q_r}-(\lambda_{q_r})^j}dt = \\ = \sum_{j\geq 2}^* \frac{f_j}{\lambda_{q_r}-(\lambda_{q_r})^j}x^j\end{aligned} \tag{4.2}$$

$$\begin{aligned}\psi_{\theta,II}(x) \equiv \int_0^{e^{i\theta}\infty} e^{-t/x^{q_r}} \times \\ \times \sum_j^* \frac{f_j}{(j/q_r-1)!}\,\frac{(\lambda_{q_r})^j t^{j/q_r-1}}{\lambda_{q_r}-(\lambda_{q_r})^j}\,\frac{e^{-t}-1}{\lambda_{q_r}-e^{-t}(\lambda_{q_r})^j}dt\,.\end{aligned} \tag{4.3}$$

We first analyze the behaviour of $\psi_{\theta,I}$, can be rewritten according to

$$\psi_{\theta,I}(x) = S_1(x) + S_2(x) + S_3(x) + S_4(x) =$$

$$= \sum_{j=2}^{\infty} \frac{f_j}{\lambda_\infty - (\lambda_\infty)^j} x^j - \sum_{j \geq q_r+1} \frac{f_j}{\lambda_\infty - (\lambda_\infty)^j} x^j + \tag{4.4}$$
$$+ \sum_{j=2}^{q_r} \frac{(\lambda_{q_r})^j - (\lambda_\infty)^j - \lambda_{q_r} + \lambda_\infty}{[\lambda_\infty - (\lambda_\infty)^j][\lambda_{q_r} - (\lambda_{q_r})^j]} f_j x^j + \sum_{j \geq q_r+1}^{*} \frac{f_j}{\lambda_{q_r} - (\lambda_{q_r})^j} x^j \,.$$

if ω satisfies the Cremer condition

$$\sup_{r \in \mathbf{N}} \frac{\log q_{r+1}}{q_r} < \infty \,, \tag{4.5}$$

then $S_1(x)$ is absolutely convergent on D_R, and therefore $S_2(x)$ converges to zero on the same domain. In order to estimate S_3 we first observe that

$$|\lambda_{q_r} - (\lambda_{q_r})^j|^{-1} = \left| 2 \sin \left(\pi \frac{(j-1)p}{q_r} \right) \right|^{-1} \leq \frac{q_r}{4} \,; \tag{4.6}$$

since the continued fraction expansion of ω satisfy

$$\left| \omega - \frac{p_r}{q_r} \right| \leq \frac{1}{q_r q_{r+1}} \leq \frac{1}{(q_r)^2} \,, \tag{4.7}$$

one has

$$|(\lambda_{q_r})^j - (\lambda_\infty)^j| + |\lambda_{q_r} - \lambda_\infty| \leq 4 \sin \pi j \left| \omega - \frac{p_r}{q_r} \right| \leq \frac{4\pi j}{(q_r)^2} \,. \tag{4.8}$$

If $x \in D_R$ one can prove that

$$|S_3(x)| \leq \frac{\pi}{q_r} \sum_{j=2}^{q_r} \frac{j R^j}{\lambda_\infty - (\lambda_\infty)^j} = O\left(\frac{1}{q_r} \right) \to 0 \,. \tag{4.9}$$

Finally, S_4 is estimated according to

$$(4.10)\qquad |S_4(x)| \le \frac{q_r}{4}\sum_{q_r+1}^{\infty} jR^j \le \frac{q_r}{4}R^{q_r+1}\frac{q_r+2}{(1-R)^2} = O((q_r)^2 R^{q_r}) \to 0\,.$$

This proves that $\psi_{\theta,I}$ uniformly converges to a function analytic on a close neighbourhood of the origin.

In order to complete the proof of proposition 2 one has to show that the remainder $\psi_{\theta,II}$ converges to zero in the limit $r \to \infty$. The function $\psi_{\theta,II}$ is the Laplace transform of $\overline{\psi}_{B,II}$, [see (4.1) and (4.3)], which is defined on $\mathbf{C_q}$, and has $2q$ lines of singularities located on the axes $i\mathbf{R}^+$, $-i\mathbf{R}^-$ in each of the q sheets. This provokes the appearance of $2q$ anti-Stokes lines in the analytic structure of the resummed function $\psi_{\theta,II}$, namely one has that $\{\psi_{\theta,II}\}_{\theta\in[0,2\pi[}$ is a collection of $2q$ functions analytic on sectors of aperture smaller than $2\pi/q$. Each one of the $2q$ functions has the same asymptotic expansion around the origin: in fact they all are formal solutions of the functional equation (2.9). The $2q$ determinations of $\psi_{\theta,II}$ differ by terms that are exponentially small in the distance to the origin, namely proportional to $e^{-1/x}$. We will show that in the limit $q_r \to \infty$ all the $2q$ determination of $\psi_{\theta,II}$ vanish.

Firstly, we have to estimate the denominator $|\lambda_{q_r} - e^{-t}(\lambda_{q_r})^j|$ on a line $t = \rho e^{i\theta}$: one obtains

$$(4.11)\qquad |\lambda_{q_r} - e^{-\rho e^{i\theta}}(\lambda_{q_r})^j|^{-1} \le \frac{c(\theta)}{|1-(\lambda_{q_r})^{j-1}|}\qquad t\in[0,+\infty]$$

where $c(\theta)$ is a positive constant, which diverges for $\theta \to \pm\frac{\pi}{2}$.

For $\theta \in]-\pi/2, \pi/2[$ the numerator of the integrand is limited, and therefore according to property 1 the Laplace integration defines a function $\psi_{\theta,II}(x)$ that is analytic on the plane $\mathrm{Re}x^{-q_r} > 0$ intersected with the convergence domain D_R of $F^{(q_r)}$. For $\theta \in]\pi/2, 3\pi/2[$ the numerator is exponentially growing with constant $B(\theta) = |\cos(\theta)|$ (see 3.8) and therefore the integration defines a function that is analytic on the circle of diameter $1/|\cos\theta|$ which passes through the origin and $x^{q_r} = -1$ intersecated with D_R. All the $\psi_{\theta,II}(x)$ with $\theta \in]-\pi/2, \pi/2[$ are analytically prolongable to the same function, analytic on the domain $D_R/\{x^{q_r} \in \mathbf{R}, x^{q_r} < 0\}$ (see Property 3). Similarly, if $\theta \in]\pi/2, 3\pi/2[$ the $\psi_{\theta,II}(x)$ are prolongable to a function that is analytic on $D_R/\{x^{q_r} \in \mathbf{R}, x^{q_r} > 0 \text{ or } x^{q_r} < -1\}$.

In order to prove the convergence of $\psi^{(q_r)}_{\theta,II}(x)$ in the limit $r \to \infty$ we first explicitly consider the case of $\psi_{0,II}(x)$, which converges to 0 on the domain $\{x/\operatorname{Re} x^{-q_r} \geq R^{-q_r} > 1\}$. Since

$$(4.12)\qquad \begin{aligned} |\psi_{0,II}(x)| \leq \int_0^{+\infty} & |e^{-t/x^{q_r}}| \times \\ & \times \sum_j^{*} \frac{|f_j|}{(j/q_r-1)!} \frac{1-e^{-t}}{|\lambda_{q_r} - e^{-t}(\lambda_{q_r})^j|} \frac{t^{j/q_r-1}}{|\lambda_{q_r} - (\lambda_{q_r})^j|} dt \,, \end{aligned}$$

using (4.11) and exchanging the integral with the series (which is absolutely convergent), one obtains

$$(4.13)\qquad \begin{aligned} |\psi_{0,II}(x)| \leq c(0) \sum_j^{*} & \frac{1}{|1-(\lambda_{q_r})^{j-1}|^2} \frac{1}{(j/q_r-1)!} \times \\ & \times \int_0^{+\infty} e^{-t \operatorname{Re} x^{-q_r}} t^{j/q_r-1} (1-e^{-t}) dt \,. \end{aligned}$$

Using the estimate $1 - e^{-t} \leq t$ and performing the integration, we have

$$(4.14)\qquad |\psi_{0,II}(x)| \leq \frac{c(0) q_r}{16} \sum_j^{*} j (\operatorname{Re} x^{-q_r})^{-j/q_r - 1} \,.$$

Therefore the convergence of $\psi_{0,II}(x)$ to 0 on the domain $\{x/\operatorname{Re} x^{-q_r} \geq R^{-q_r} > 1\}$ can be proved: in fact one has

$$(4.15)\qquad |\psi_{0,II}(x)| \leq \frac{c(0) q_r}{16} R^{q_r} \sum_j^{*} j R^j = O(q_r R^{q_r}) \to 0 \,.$$

The computations for $\theta \in]-\pi/2, \pi/2[$ and $\theta \in]\pi/2, 3\pi/2[$ can be carried out similarily, and lead to the convergence of $\psi^{(q_r)}_{\theta,II}(x) \to 0$ on the domain $\{x/\operatorname{Re}(e^{i\theta} x^{-q_r}) \geq R^{-q_r} > 1\}$. Indeed, considering the union of all the domains of convergence one obtains a closed neighbourhood of the origin, and therefore the proof of Proposition 2 is completed. The difference between the determinations given by $\theta \in]-\pi/2, \pi/2[$ and $\theta \in]\pi/2, 3\pi/2[$ is related to the values of the residues of the Borel transform of ψ.

Finally, we consider the convergence of the preliminary normal form $U^{(q_r)}$; the proof is straightforward since according to the functional equation (2.9) u is proportional to ψ.

PROPOSITION 3. *Using the same subsequence p_r, q_r of Proposition 2, the preliminary normal form $U^{(q_r)}$ converges in a neighbourhood of the origin to the linear part of $F^{(\infty)}$ according to the same properties given for the conjugating function $\psi^{(q_r)}$.*

PROOF. The definition of $U^{(q_r)}$ reads:

$$U^{(q_r)}(y) = \frac{\lambda_{q_r} y}{(1+y^{q_r})^{1/q_r}} + \epsilon \sum_{k=1}^{\infty} f_{kq_r+1} y^{kq_r+1} + \epsilon u(y) \tag{4.16}$$

the resonant standard shift plus the resonant part of f trivially converge to $U^{(\infty)} = \lambda_\infty y$ on D_R; u is given by the second part of (2.9):

$$u(x) = \lambda_{q_r} \left(\frac{1}{(1+x^{q_r})^{1+1/q_r}} - 1 \right) \psi^{(q_r)}(x). \tag{4.17}$$

Since $(1+x^{q_r})^{-1-1/q_r} - 1$ uniformly converges to 0 on D_R, then $U^{(q_r)}$ converges to $U^{(\infty)}$ with the same properties of the convergence of $\psi_{II}^{(q_r)}$ to zero.

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