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A note on the cardinality of functionally Hausdorff spaces

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Riassunto: *Lo scopo di questa nota è di fornire una nuova stima della cardinalità degli spazi funzionalmente di Hausdorff.*

Abstract: *The aim of this note is to give a new estimation of the cardinality of functionally Hausdorff spaces.*

A space X is said to be functionally Hausdorff if whenever $x \neq y$ in X there is a continuous real valued function f defined on X such that $f(x) = 0$ and $f(y) = 1$.

In this paper we use a new cardinal function related to cellularity in order to obtain a bound on the cardinality of functionally Hausdorff spaces (see [1] and [10] for some cardinal inequalities obtained by means of cellularity-like cardinal functions).

We refer the reader to [2], [5], [9] for notations and terminology not explicitly given.

Definition 1. Let A be a subset of a space X. The τ-closure of A, denoted by $\mathrm{cl}_\tau(A)$, is the set of all points $x \in X$ such that any cozero-set neighbourhood of x intersects A ([7],[8]). A is said to be τ-closed in X

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if $A = \mathrm{cl}_\tau(A)$. Clearly every zero-set in a space X is τ-closed in X. The $\tau\theta$-closure of A, denoted by $\mathrm{cl}_{\tau\theta}(A)$, is the set of all points $x \in X$ such that $\mathrm{cl}_\tau(V) \cap A \neq \emptyset$ for every open neighbourhood V of x [3].

Observe that for every space X and every $A \subset X$ it follows that $\bar{A} \subset \mathrm{cl}_{\tau\theta}(A) \subset \mathrm{cl}_\tau(A)$. Moreover if X is completely regular then $\bar{A} = \mathrm{cl}_{\tau\theta}(A) = \mathrm{cl}_\tau(A)$ for every $A \subset X$.

Definition 2. i) Let X be a topological space and let $p \in X$. We define $\chi(p, X) = \min\{|\mathcal{V}| : \mathcal{V}$ is a local base for $p\}$. The character of X is defined as follows : $\chi(X) = \sup\ \{\chi(p, X) : p \in X\} + \omega$.

(ii) Let X be a functionally Hausdorff space. $H\psi_\tau(X)$ is the smallest infinite cardinal κ such that it is possible to associate to every $x \in X$ a collection $\mathcal{B}_x$ of open neighbourhoods of x, with $|\mathcal{B}_x| \leq \kappa$, in such a way that if $x \neq y$ then there are $U \in \mathcal{B}_x$ and $V \in \mathcal{B}_y$ with $\mathrm{cl}_\tau(U) \cap \mathrm{cl}_\tau(V) = \emptyset$.

Let X be a topological space. A cellular family in X is a collection of pairwise disjoint non-empty open sets in X. The cellularity of X is defined as follows: $c(X) = \sup\{|\mathcal{U}| : \mathcal{U}$ is a cellular family in $X\} + \omega$. Observe that if $c(X) = \kappa$ and $\mathcal{U}$ is an open collection in X then there is a subcollection $\mathcal{V}$ of $\mathcal{U}$ such that $|\mathcal{V}| \leq \kappa$ and $\bigcup\mathcal{U} \subset \overline{\bigcup\mathcal{V}}$. This observation led us to the following

Definition 3. Let X be a topological space. $\tau c(X)$ is the smallest infinite cardinal κ such that for every open collection $\mathcal{U}$ in X there is a $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ such that $\bigcup\mathcal{U} \subset \mathrm{cl}_{\tau\theta}(\bigcup\{\mathrm{cl}_\tau(V) : V \in \mathcal{V}\})$.

Remark 4. i) Note that $H\psi_\tau$ is defined for functionally Hausdorff spaces only. In fact a space X is functionally Hausdorff if and only if for every $x, y \in X$, $x \neq y$, there are open sets U and V in X such that $x \in U$, $y \in V$ and $\mathrm{cl}_\tau(U) \cap \mathrm{cl}_\tau(V) = \emptyset$.

ii) The Hausdorff pseudo-character $H\psi(X)$ of a T_2-space is defined as the smallest infinite cardinal κ such that for every $x \in X$, there is a collection $\mathcal{B}_x$ of open neighbourhoods of x with $|\mathcal{B}_x| \leq \kappa$ such that if $x \neq y$, there exist $U \in \mathcal{B}_x$ and $V \in \mathcal{B}_y$ with $U \cap V = \emptyset$ [6]. It is clear that $H\psi(X) \leq H\psi_\tau(X)$ for every functionally Hausdorff space X and that $H\psi(X) = H\psi_\tau(X)$ for every Tychonoff space X.

iii) Observe that $H\psi_\tau(X) \leq \chi(X)$ for every functionally Hausdorff space X : let $\chi(X) = \kappa$ and for every $x \in X$ let $\mathcal{B}_x$ be a local base for x such that $|\mathcal{B}_x| \leq \kappa$. Let us show that if $x \neq y$ then there are $U \in \mathcal{B}_x$ and

$V \in \mathcal{B}_y$ such that $\mathrm{cl}_\tau(U) \cap \mathrm{cl}_\tau(V) = \emptyset$. Since X is functionally Hausdorff there is a continuous function $f : X \to R$ such that $f(x) = 0$ and $f(x) = 1$. Choose $U \in \mathcal{B}_x$ and $V \in \mathcal{B}_y$ in such a way that $U \subset f^{-1}((-\frac{1}{3}, \frac{1}{3}))$ and $V \subset f^{-1}((\frac{2}{3}, \frac{4}{3}))$. Since $f^{-1}([-\frac{1}{3}, \frac{1}{3}])$ and $f^{-1}([\frac{2}{3}, \frac{4}{3}])$ are zero-sets in X it follows that $\mathrm{cl}_\tau(U) \cap \mathrm{cl}_\tau(V) \subset f^{-1}([-\frac{1}{3}, \frac{1}{3}]) \cap f^{-1}([\frac{2}{3}, \frac{4}{3}]) = \emptyset$. Therefore $H\psi_\tau(X) \le \kappa$.

THEOREM 5. *If X is functionally Hausdorff then* $|X| \le 2^{\tau c(X) H\psi_\tau(X)}$.

PROOF. Let $\tau c(X) H\psi_\tau(X) = \kappa$ and for every $x \in X$ let $\mathcal{B}_x$ be a family of open neighbourhoods of x such that $|\mathcal{B}_x| \le \kappa$ and if $x \ne y$ there are $U \in \mathcal{B}_x$ and $V \in \mathcal{B}_y$ with $\mathrm{cl}_\tau(U) \cap \mathrm{cl}_\tau(V) = \emptyset$.

Construct a sequence $\{S_\alpha : \alpha < \kappa^+\}$ of subsets of X and a sequence $\{\mathcal{G}_\alpha : 0 < \alpha < \kappa^+\}$ of open collections in X such that

(i) $|S_\alpha| \le 2^\kappa$, $\alpha < \kappa^+$;

(ii) $\mathcal{G}_\alpha = \{B : \exists x \in \bigcup_{\beta<\alpha} S_\beta \wedge B \in \mathcal{B}_x\}$, $0 < \alpha < \kappa^+$;

(iii) If $\{U_\gamma : \gamma < \kappa\}$ is a family of subsets of X and each U_γ is the union of τ-closures of $\le \kappa$ elements of $\mathcal{G}_\alpha$ and $\bigcup_{\gamma<\kappa} \mathrm{cl}_{\tau\theta}(U_\gamma) \ne X$ then $S_\alpha \setminus \bigcup_{\gamma<\kappa} \mathrm{cl}_{\tau\theta}(U_\gamma) \ne \emptyset$.

The construction is done by transfinite induction. Set $S_0 = \emptyset$. Let $0 < \alpha < \kappa^+$ and assume that S_β has been constructed for every $\beta < \alpha$. Note that $\mathcal{G}_\alpha$ is defined by (ii) and $|\mathcal{G}_\alpha| \le 2^\kappa$. For every collection $\{U_\gamma : \gamma < \kappa\}$ of subsets of X, each of which is the union of τ-closures of $\le \kappa$ elements of $\mathcal{G}_\alpha$ and $\bigcup\{\mathrm{cl}_{\tau\theta}(U_\gamma) : \gamma < \kappa\} \ne X$, choose a point in $X \setminus \bigcup\{\mathrm{cl}_{\tau\theta}(U_\gamma) : \gamma < \kappa\}$, and let S_α be the set so obtained. Obviously S_α satisfies the above conditions. Now the construction of the sequence $\{S_\alpha : \alpha < \kappa^+\}$ is complete.

Let $S = \bigcup_{\alpha<\kappa^+} S_\alpha$, it is enough to show that $S = X$. Assume not and take a $p \in X \setminus S$. Let $\mathcal{B}_p = \{B_\gamma : \gamma < \kappa\}$ and for every $\gamma < \kappa$ let $\mathcal{W}_\gamma = \{B : B \in \mathcal{B}_x, x \in S, \mathrm{cl}_\tau(B) \cap \mathrm{cl}_\tau(B_\gamma) = \emptyset\}$. Observe that for every $x \in S$ there is a $\gamma < \kappa$ such that $x \in \bigcup \mathcal{W}_\gamma$. Since $\tau c(X) \le \kappa$ it follows that there is a $\mathcal{V}_\gamma \in [\mathcal{W}_\gamma]^{\le\kappa}$ such that $\bigcup \mathcal{W}_\gamma \subset \mathrm{cl}_{\tau\theta}(\bigcup\{\mathrm{cl}_\tau(V) : V \in \mathcal{V}_\gamma\})$ for every $\gamma < \kappa$. Since $\mathrm{cl}_\tau(B_\gamma) \cap \bigcup\{\mathrm{cl}_\tau(V) : V \in \mathcal{V}_\gamma\} = \emptyset$ it follows that $p \notin \mathrm{cl}_{\tau\theta}(\bigcup\{\mathrm{cl}_\tau(V) : V \in \mathcal{V}_\gamma\})$ for every $\gamma < \kappa$.

Now for each $\gamma < \kappa$ let $U_\gamma = \bigcup\{\mathrm{cl}_\tau(V) : V \in \mathcal{V}_\gamma\}$, and choose $\alpha < \kappa^+$ such that $\mathcal{V}_\gamma \subset \mathcal{G}_\alpha$ for every $\gamma < \kappa$. Then $\bigcup_{\gamma<\kappa} \mathrm{cl}_{\tau\theta}(U_\gamma) \ne X$ and $S_\alpha \subset S \subset \bigcup_{\gamma<\kappa} \bigcup \mathcal{W}_\gamma \subset \bigcup_{\gamma<\kappa} \mathrm{cl}_{\tau\theta}(U_\gamma)$, a contradiction.

REMARK 6. A strong cellular family is a family $\mathcal{U}$ of non-empty open subsets of a space X such that $\mathrm{cl}_\tau(U)\cap V=\emptyset$ or $U\cap\mathrm{cl}_\tau(V)=\emptyset$ for any pair U,V of distinct members of $\mathcal{U}$. The strong cellularity of a space X, denoted by $\mathrm{sc}(X)$, is defined as $\sup\{|\mathcal{U}|:\mathcal{U}$ is a strong cellular family of $X\}+\omega$ [4]. Clearly $\mathrm{sc}(X)\leq c(X)$ for any space X. Moreover if X is a space such that $\mathrm{sc}(X)=\kappa$ and $\mathcal{V}$ is a collection of open subsets of X then by prop. 2 in [4] it follows that there is a $\mathcal{W}\subset\mathcal{V}$ such that $|\mathcal{W}|\leq\kappa$ and $\bigcup\mathcal{V}\subset\mathrm{cl}_{\tau\theta}(\bigcup\mathcal{W})$. Therefore $\tau c(X)\leq\mathrm{sc}(X)$ for any space X.

A consequence of theorem 2 is the following

COROLLARY 7. [4] *If X is functionally Hausdorff then* $|X|\leq 2^{\mathrm{sc}(X)\chi(X)}$.

REMARK 8. i) Observe that the inequality in theorem 5 can be sharper than the one given in Corollary 7. Let τ be the euclidean topology on R and let X be R with the topology $\sigma=\{V\setminus A: V\in\tau, A\subset R$ and $|A|\leq\omega\}$. Observe that X is a functionally Hausdorff space with $\tau c(X)H\psi_\tau(X)=\aleph_0$. Moreover $\chi(X)=2^{\aleph_0}$, in fact let $x\in R$ and let $\mathcal{V}_x=\{V_\alpha\setminus A_\alpha\}_{\alpha<\kappa}$ be a collection of open neighbourhoods of x in X such that $\kappa<2^{\aleph_0}$. Let $G_n=(x-\frac{1}{n},x+\frac{1}{n})$ for every positive integer n, and for every $\alpha\in\kappa$ choose $n(\alpha)\in N$ in such a way that $G_{n(\alpha)}\subset V_\alpha$. Now, for every n, take $x_n\in G_n\setminus\bigcup_{\alpha<\kappa}A_\alpha$ such that $x_n\neq x$. $R\setminus\{x_n:n\in N\}$ is an open neighbourhood of x which does not contain any element of $\mathcal{V}_x$ (if there is some α for which $V_\alpha\setminus A_\alpha\subset R\setminus\{x_n:n\in N\}$ then $x_{n(\alpha)}\in R\setminus\{x_n:n\in N\}$, a contradiction). So $\mathcal{V}_x$ is not a local base for x.

Therefore $|X|=2^{\tau c(X)H\psi_\tau(X)}<2^{\mathrm{sc}(X)\chi(X)}$.

ii) In [6] it is shown that $|X|\leq 2^{c(X)H\psi(X)}$ for every Hausdorff space X. It is worth noting that the functionally Hausdorff space X described in [4] is such that $|X|=2^{\aleph_0}$, $H\psi(X)=H\psi_\tau(X)=\chi(X)=\tau c(X)=\mathrm{sc}(X)=\omega$ and $c(X)=2^{\aleph_0}$. So $|X|=2^{\tau c(X)H\psi_\tau(X)}<2^{c(X)H\psi(X)}$.

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