Rendiconti di Matematica, Serie VII
Volume 17, Roma (1997), 163-186

Nonlinear Dirichlet problems in randomly perforated domains

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Riassunto: *Si studiano successioni di problemi variazionali in regioni perforate in maniera aleatoria, con condizioni di Dirichlet al bordo. Utilizzando un metodo capacitario, si identifica il problema limite. Vengono formulate condizioni necessarie e sufficienti per ottenere un problema limite deterministico.*

Abstract: *Sequences of nonlinear variational problems in random perturbed domains with Dirichlet boundary conditions are investigated. By using a capacitary method, the limit problem is characterized. Necessary and sufficient conditions to have a deterministic limit problem are given.*

1 – Introduction

The aim of this paper is to propose a general framework to study the asymptotic behaviour, as $h \to +\infty$, of sequences of minimum problems in domains with randomly distributed holes and Dirichlet boundary conditions of the form

$$\min_{u\in H_0^{1,p}(D\setminus E_h)} \int_{D\setminus E_h} f(x,Du)\,dx + \int_{D\setminus E_h} gu\,dx \tag{1.1}$$

where (E_h) is a sequence of closed *random* subsets of $D \subseteq \mathbb{R}^n$, $n \geq 2$.

Key Words and Phrases: γ-*convergence – Capacity – Random set.*
A.M.S. Classification: 49A50

Our approach is similar to those ones already provided in [2], [3] and, more recently, in [4] in order to analyze a broad class of linear Dirichlet problems.

The present work, in a sense, may be considered the probabilistic version of the paper of G. Dal Maso and A. Defranceschi [10]; so, following the mentioned authors, we assume that the function $f(x, \xi)$ in (1.1) is measurable in x, convex and p-homogeneous in ξ, and that

$$c_1|\xi|^p \leq f(x,\xi) \leq c_2|\xi|^p$$

for constants $0 < c_1 \leq c_2 < +\infty$, and $1 < p \leq n$.

Next, introducing, for every closed set $B \subseteq D$, the measure given by

$$\infty_E(B) = \begin{cases} 0 & \text{if} \quad C_p(E \cap B) = 0 \\ +\infty & \text{if} \quad C_p(E \cap B) > 0\,. \end{cases} \tag{1.2}$$

where $C_p(A)$ is the p-capacity of A, we can see that the minimum problem (1.1) is equivalent to

$$\min_{u \in H_0^{1,p}(D)} \int_D f(x, Du)\, dx + \int_D |u|^p \, dM_h + \int_D gu \, dx \tag{1.3}$$

for $M_h = \infty_{E_h}$

Therefore, it is convenient to consider problems like (1.3) in the case where (M_h) is a sequence of random measures, *i.e.* measurable functions from a probability space into a set of measures, endowed with a suitable topology.

In order to have at our disposal a good setting for the standard methods of Probability Theory, we consider a peculiar sub-class, denoted by $\mathcal{M}_p^*$, of the set of all non-negative Borel measures on D vanishing on all Borel sets with p-capacity zero. An example of measure belonging to $\mathcal{M}_p^*$ is just the measure in (1.2).

Introducing on $\mathcal{M}_p^*$ the notion of γ_f-convergence (see Definition 2.4), the class $\mathcal{M}_p^*$ becomes a (sequentially) compact metric space.

On this space we define the basic tool in our analisys, that is, the nonlinear variational μ-capacity relative to f defined as

$$C(f, \mu, B) = \min \left\{ \int_D f(x, Du)\, dx + \int_B |u-1|^p \, d\mu;\ u \in H_0^{1,p}(D) \right\} \tag{1.4}$$

for every Borel set $B \subseteq D$ and $\mu \in \mathcal{M}_p^*$.

The probabilistic problem we shall consider can be formulated as follow.

Let (Ω, Σ, P) be a probability space. Consider a sequence (M_h) of random measures, *i.e.* of measurable maps between (Ω, Σ) and $\mathcal{M}_p^*$, endowed with the minimal σ-algebra $\mathcal{B}(\mathcal{M}_p^*)$ for which the maps $C(f, \cdot, K)$ are measurable for every compact subset K of D.

Associated with the measures $M_h(\omega), \omega \in \Omega$, we consider minimum problems like (1.3) and we want to analyze the asymptotic behaviour, as $h \to +\infty$, of the minimum values $m_h(\omega)$ and of the minimum points $U_h(\omega)$ of these problems.

We find necessary and sufficient conditions on (M_h) for the convergence in probability of the sequence of the random minima (m_h) to a constant m_0 given by

$$m_0 = \min_{u \in H_0^{1,p}(D)} \int_D f(x, Du)\, dx + \int_D |u|^p\, d\nu + \int_D gu\, dx \tag{1.5}$$

where ν is a suitable measure of the class $\mathcal{M}_p^*$.

These conditions are given in terms of the asymptotic behaviour of the expectations of the random variables $C(f, M_h(\cdot), B)$ and of the covariances of the random variables $C(f, M_h(\cdot), A)$ and $C(f, M_h(\cdot), B)$ for disjoint subsets A and B of D.

In the case where the functional appearing in (1.3) has a unique minimum point $U_h(\omega)$ for every $h \in \mathbb{N}$, the same conditions guarantee the convergence in probability of the sequence (U_h) to the minimum point U_0 of the problem (1.5).

When these conditions are satisfied, we obtain also a characterization of the limit measure ν. In fact, in this case the expectations of the capacities $C(f, M_h, B)$ converge weakly (in the sense of [14]) to a countably subadditive set function $\alpha(B)$ (which turns to be equal to $C(f, \nu, B)$) and ν is the least measure such that $\nu \geq \alpha$.

Many other authors have been investigated the asymptotic behaviour of nonlinear problems in varying domains (see, for istance, [20], [21], [17], [18], [22], [23], [24], [6], [13]).

Recently, the case of Dirichlet problems in perforated domains with monotone operators has been analyzed in [11], [12], [5].

All the results of the present paper can be extended with some difficult technical changes to the case of monotone operators $A : H_0^{1,p}(D) \to H^{-1,q}(D)$, with $1 < p < +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$. We have planned to give these results in a forthcoming paper.

The present paper can be considered a self-contained exposition in the simple case when A is the subdifferential of the functional $F(u) = \int_D f(x, Du)\,dx$ defined on $H_0^{1,p}$, where $f(x,\xi)$ satisfies the conditions (2.1) and (2.2).

2 – Notation and background material

Let D be a bounded open subset of $\mathbb{R}^n$, $n \geq 2$ and p be a real constant such that $1 < p < \infty$.

We denote by $\mathcal{U}$ the family of all open sets $U \subseteq D$ and by $\mathcal{K}$ the family of all compact sets $K \subseteq D$. Moreover, we indicate by $\mathcal{B}$ the σ-field of all Borel subsets of D.

For every $K \in \mathcal{K}$, we define the p-capacity of K with respect to D by

$$C_p(K) = \inf \left\{ \int_D |D\varphi|^p\,dx \;:\; \varphi \in C_0^\infty(D), \varphi \geq 1 \text{ on } K \right\} .$$

The definition is extended to the sets $U \in \mathcal{U}$ by

$$C_p(U) = \sup \{C_p(K);\, K \subseteq U,\, K \in \mathcal{K}\}$$

and to arbitrary sets $E \subseteq D$ by

$$C_p(E) = \inf \{C_p(U);\, U \supseteq E,\, U \in \mathcal{U}\} .$$

The following proposition summarizes some well-known properties of the p-capacity (see [15]).

Proposition 2.1. *The p-capacity C_p satisfies the following properties:*

(a) $C_p(\emptyset) = 0$*;*

(b) C_p *is increasing, i.e.* $C_p(E_1) \leq C_p(E_2)$ *whenever* $E_1 \subseteq E_2 \subset D$*;*

(c) *if* (E_h) *is an increasing sequence of subsets of* D *and* $E = \bigcup_h E_h$*, then*

$$C_p(E) = \sup_h C_p(E_h)\,;$$

(d) *if* (E_h) *is a sequence of subsets of* D *and* $E \subseteq \bigcup_h E_h$*, then*

$$C_p(E) \leq \sum_h C_p(E_h)\,;$$

(e) C_p *is a strongly subadditive set function, i.e.*

$$C_p(E_1 \cup E_2) + C_p(E_1 \cap E_2) \leq C_p(E_1) + C_p(E_2) \quad \forall\, E_1, E_2 \subseteq D\,.$$

Let E be a subset of D. If a property $P(x)$ holds for all $x \in E$, except for a subset $N \subseteq E$ with $C_p(N) = 0$, then we say that $P(x)$ holds p-quasi everywhere on E (p-q.e.) or for p-quasi every $x \in E$ (for p-q.e. $x \in E$). A set $A \subseteq D$ is said to be *p-quasi open* (respectively *p-quasi closed*, *p-quasi compact*) in D if for every $\varepsilon > 0$ there exists an open (respectively closed, compact) set $U \subseteq D$ such that $C_p(U \triangle A) < \varepsilon$, where $\triangle$ denotes the symmetric difference (the topological notions are in the relative topology of D).

It is well-known that A is p-quasi open if and only if $D \backslash A$ is p-quasi closed and that any countable union or finite intersection of p-quasi open sets is p-quasi open.

In a similar way we give the notion of a p-quasi Borel subset of D and denote by $\mathcal{B}_0$ the σ-field of all p-quasi Borel subsets of D.

A function $f : D \to \overline{\mathbb{R}}$ is said to be p-quasi continuous in D if for every $\varepsilon > 0$ there exists a set $E \subseteq D$ with $C_p(D \backslash E) < \varepsilon$ such that the restriction of f to E is continuous on E.

The notion of p-quasi upper and p-quasi lower semicontinuity are defined in a similar way.

For every set $E \subseteq D$ we denote by 1_E the characteristic function of E, defined by $1_E(x) = 1$ if $x \in E$, and $1_E(x) = 0$ if $x \in D \backslash E$.

It easy to check that a set $E \subseteq D$ is p-quasi open (respectively p-quasi closed) in D if and only if 1_E is p-quasi lower (respectively p-quasi upper) semicontinuous in D.

It can be proved that a function $f : D \to \overline{\mathbb{R}}$ is p-quasi lower (respectively p-quasi upper) semicontinuous if and only if the sets $\{x \in D : f(x) > t\}$ (respectively $\{x \in D : f(x) \geq t\}$), are p-quasi open (respectively p-quasi closed) for every $t \in \mathbb{R}$ (see [16]).

We denote by $H^{1,p}(D)$ the Sobolev space of all functions in $L^p(D)$ with first order distributional derivatives in $L^p(D)$ and by $H_0^{1,p}(D)$ the closure of $C_0^\infty(D)$ in $H^{1,p}(D)$.

For every $x \in \mathbb{R}^n$ and $r > 0$ we set

$$B_r(x) = \{y \in \mathbb{R}^n \, : \, |x - y| < r\} \, ,$$

and for every Borel set $B \subset \mathbb{R}^n$ we denote by $|B|$ its Lebesgue measure.

It is well-known that for every function $u \in H^{1,p}(D)$ the limit

$$\lim_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y) dy$$

exists and is finite p-quasi everywhere in D.

We assume the following convention about the pointwise value of a function $u \in H^{1,p}(D)$: for every $x \in D$ we require that

$$\liminf_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y) dy \leq u(x) \leq \limsup_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y) dy \, .$$

So, the pointwise value $u(x)$ is determinated p-q.e. on D and the function u is p-quasi continuous in D (see [15]).

It can be shown that

$$C_p(E) = \min \left\{ \int_D |Du|^p \, : \, u \in H_0^{1,p}(D) \, , \, u \geq 1 \, \text{p-q.e. on } \, E \right\}$$

for every subset $E \subseteq D$.

A nonnegative countably additive set function μ defined on $\mathcal{B}$ and with value in $[0, +\infty]$ such that $\mu(\emptyset) = 0$ is called a *Borel measure* on D. A Borel measure which assigns finite value to every compact subset of D is called a *Radon measure*.

Definition 2.2. We denote by $\mathcal{M}_p^*$ the class of all Borel measure μ on D such that

(a) $\mu(B) = 0$ for every $B \in \mathcal{B}$ with $C_p(B) = 0$.
(b) $\mu(B) = \inf\{\mu(U) : U$ p-quasi open, $B \subseteq U\}$ for every $B \in \mathcal{B}$.

The set $\mathcal{M}_p^*$ has been introduced in [10]; for p=2 it coincides with the set $\mathcal{M}_0^*$, which has been extensively studied in [8].

We recall that the measures of the class $\mathcal{M}_p^*$ are not required to be regular nor σ-finite. For istance, the measure introduced in the definition below, belong to the class $\mathcal{M}_p^*$.

DEFINITION 2.3. For every p-quasi closed $E \subseteq D$ we denote by ∞_E the Borel measure defined by

$$\infty_E(B) = \begin{cases} 0 & \text{if} \quad C_p(E \cap B) = 0 \\ +\infty & \text{if} \quad C_p(E \cap B) > 0 \end{cases}$$

for every $B \in \mathcal{B}$.

Let us fix a function $f : D \times \mathbb{R}^n \to \mathbb{R}$ and two constants $0 < c_1 \leq c_2 < +\infty$ which satisfy the following conditions:

$$f(x,\xi) \text{ is Lebesgue measurable in } x, \text{ convex and p-homogeneus in } \xi\,; \tag{2.1}$$

$$c_1|\xi|^p \leq f(x,\xi) \leq c_2|\xi|^p \text{ for every } (x,\xi) \in D \times \mathbb{R}^n\,. \tag{2.2}$$

We introduce a variational notion of convergence for sequences (μ_h) in $\mathcal{M}_p^*$.

With every $\mu \in \mathcal{M}_p^*$ we associate the following functional F_μ defined on $L^p(D)$

$$F_\mu(u) = \begin{cases} \displaystyle\int_D f(x, Du(x)) + \int_D |u|^p d\mu & \text{if} \quad u \in H_0^{1,p}(D)\,; \\ +\infty & \text{if} \quad u \in L^p(D),\ u \notin H_0^{1,p}(D)\,. \end{cases}$$

Since $\mu(B) = 0$ for every $B \in \mathcal{B}$ with $C_p(B) = 0$, the functional F_μ is lower semicontinuous in $L^p(D)$.

The following definition of γ_f-convergence for measures (μ_h) in $\mathcal{M}_p^*$ is given in terms of the Γ-convergence (see [7]) of the corresponding functionals F_{μ_h}.

Definition 2.4. Let (μ_h) be a sequence in $\mathcal{M}_p^*$ and let $\mu \in \mathcal{M}_p^*$; we say that (μ_h) γ_f-converges to μ if the following conditions are satisfied:

a) for every $u \in H_0^{1,p}(D)$ and for every sequence (u_h) in $H_0^{1,p}(D)$ converging to u in $L^p(D)$ we have

$$F_\mu(u) \leq \liminf_{h\to\infty} F_{\mu_h}(u_h)\,;$$

b) for every $u \in H_0^{1,p}(D)$, there exists a sequence (u_h) in $H_0^{1,p}(D)$ converging to u in $L^p(D)$ such that:

$$F_\mu(u) \geq \limsup_{h\to\infty} F_{\mu_h}(u_h)\,.$$

Remark 2.5. There exists a metrizable topology on $\mathcal{M}_p^*$ which induces the γ_f-convergence (see [10], Theorem 3.5). It will be called the topology of γ_f-convergence. All topological notions we shall consider on $\mathcal{M}_p^*$ are relative to this topology. The class of measures $\mathcal{M}_p^*$ is sequentially compact with respect to this topology (see [10], Theorem 3.3).

Definition 2.6. For every μ in $\mathcal{M}_p^*$, and $g \in L^q(D)$, $\frac{1}{p} + \frac{1}{q} = 1$, we denote by $m(\mu, g)$ and $U(\mu, g)$ respectively the minimum value and the the set of the minimum points of the problem

$$\min_{u\in H_0^{1,p}(D)} \int_D f(x, Du)\,dx + \int_D |u|^p\,d\mu + \int_D gu\,dx\,.$$

The main motivation of γ_f-convergence is given by the following theorem (see [10], Theorem 4.5).

Theorem 2.7. *Let $1 < p \leq n$. Let (μ_h) be a sequence in $\mathcal{M}_p^*$ which γ_f-converges to $\mu \in \mathcal{M}_p^*$. Then for every $g \in L^q(D)$, $\frac{1}{p} + \frac{1}{q} = 1$, the following properties hold:*

a) $\lim_{h\to\infty} m(\mu_h, g) = m(\mu, g)$;

b) *for every neighbourhood $\mathcal{A}$ of $U(\mu, g)$ in $L^p(D)$ there exists $k \in \mathbb{N}$ such that $U(\mu_h, g) \subseteq \mathcal{A}$ for every $h \geq k$.*

REMARK 2.8. Whenever in Definition 2.6 the function $f(x,\xi)$ in (2.1) and (2.2) is such that the map $U(\cdot,g) : \mathcal{M}_p^* \to L^p(D)$ is single valued, then $U(\cdot,g)$ is continuous, *i.e.* if (μ_h) γ_f-converges to μ in $\mathcal{M}_p^*$ then

$$\lim_{h\to\infty} U(\mu_h, g) = U(\mu, g) \quad \text{in} \quad L^p(D)$$

for every $g \in L^q(D)$, with $\frac{1}{p} + \frac{1}{q} = 1$.

For every set $E \subseteq D$, the capacity of E in D, relative to f, satisfying (2.1), (2.2), is defined by

$$C(f,E) = \min\left\{\int_D f(x,Du)\,dx;\ u \in H_0^{1,p}(D),\ u \geq 1 \text{ p-q.e. on } E\right\}. \tag{2.3}$$

Moreover, for every $\mu \in \mathcal{M}_p^*$ and for every $B \in \mathcal{B}$ the μ-capacity of B in D, relative to f, is defined by

$$C(f,\mu,B) = \min\left\{\int_D f(x,Du)\,dx + \int_B |u-1|^p\,d\mu;\ u \in H_0^{1,p}(D)\right\}. \tag{2.4}$$

The minimum in (2.3) (respectively in (2.4))is attained by the lower semicontinuity of the functional in the weak topology of $H_0^{1,p}(D)$.

REMARK 2.9. If $\mu = \infty_E$ (see Definition 2.3), with E p-quasi closed in D, then

$$C(f,\infty_E,B) = C(f,E\cap B)$$

for every $B \in \mathcal{B}$.

The following proposition collects the main properties of the μ-capacity, relative to f, for an arbitrary $\mu \in \mathcal{M}_p^*$.

The proof of the proposition is analogous to the proofs of Theorem 2.9, Theorem 3.5 and Theorem 3.7 in [8].

PROPOSITION 2.10. *For every $\mu \in \mathcal{M}_p^*$ the set function $C(f,\mu,\cdot)$: $\mathcal{B} \to [0,+\infty]$ satisfies the following properties:*

a) $C(f,\mu,\emptyset) = 0$;

b) $C(f,\mu,B_1) \leq C(f,\mu,B_2)$ *whenever* $B_1, B_2 \in \mathcal{B}$, $B_1 \subseteq B_2$;

c) *if (B_h) is an increasing sequence in $\mathcal{B}$ and $B = \bigcup_h B_h$, then*

$$C(f,\mu,B) = \sup_h C(f,\mu,B_h)\,;$$

d) *if* (B_h) *is a sequence of Borel sets of* B *and* $B \subseteq \bigcup_h B_h$*, then*

$$C(f,\mu,B) \le \sum_h C(f,\mu,B_h)\,;$$

e) $C(f,\mu,B_1 \cup B_2) + C(f,\mu,B_1 \cap B_2) \le C(f,\mu,B_1) + C(f,\mu,B_2)$ *for every* $B_1, B_2 \in \mathcal{B}$*;*

f) $C(f,\mu,B) \le \mu(B)$ *for every* $B \in \mathcal{B}$*;*

g) $C(f,\mu,B) \le C(f,B) \le c_2\, C_p(B)$ *for every* $B \in \mathcal{B}$*;*

h) $C(f,\mu,K) = \inf\{C(f,\mu,U);\ K \subseteq U,\ U \in \mathcal{U}\}$ *for every* $K \in \mathcal{K}$*;*

i) $C(f,\mu,B) = \sup\{C(f,\mu,K);\ K \subseteq B,\ K \in \mathcal{K}\}$ *for every* $B \in \mathcal{B}$*.*

The following theorem states that for an arbitrary $\mu \in \mathcal{M}_p^*$, $1 < p \le n$, the measure μ is the least superadditive set function which is greater than or equal to $C(f,\mu,\cdot)$ on $\mathcal{B}$ (see [9], Theorem 4.2).

THEOREM 2.11. *Suppose that* $1 < p \le n$ *and let* $\mu \in \mathcal{M}_p^*$*. Then for every* $B \in \mathcal{B}$ *we have*

$$\mu(B) = \sup \sum_{i \in I} C(f,\mu,B_i)$$

where the supremum is taken over all finite Borel partitions $(B_i)_{i \in I}$ *of* B*.*

From Theorem 2.11, taking into account Lemma 4.2 in [8], it easy to obtain the following useful formula to reconstruct a measure $\mu \in \mathcal{M}_p^*$ from the corresponding μ- capacity relative to f.

THEOREM 2.12. *Let* $1 < p \le n$ *and let* $\mu \in \mathcal{M}_p^*$*. Then for every* $B \in \mathcal{B}$ *we have*

$$\mu(B) = \lim_{h \to \infty} \sum_{i \in \mathbb{Z}^n} C(f,\mu,B \cap R_h^i)\,,$$

where R_h^i *denotes the cube*

$$R_h^i = \prod_{k=1}^{n} \left] \frac{i_k}{2^h}, \frac{i_k+1}{2^h} \right]$$

for every $h \in \mathbb{N}$ *and for every* $i = (i_1, \dots, i_n)$*.*

The following theorem states a relationship between the γ_f-convergence of a sequence of measures (μ_h) and the behaviour of the corresponding μ-capacities relative to f. The theorem is an easy consequence of Theorem 5.1 and Theorem 5.8 in [10], together with Theorem 6.3 in [8] adapted to our case.

Proposition 2.13. *Let $1 < p \leq n$. Let (μ_h) a sequence in $\mathcal{M}_p^*$ and $\mu \in \mathcal{M}_p^*$.*

Then (μ_h) γ_f converges to μ in $\mathcal{M}_p^*$ if and only if the inequalities

$$\text{a) } C(f,\mu,U) \leq \liminf_{h\to\infty} C(f,\mu_h,U)$$

and

$$\text{b) } C(f,\mu,K) \geq \limsup_{h\to\infty} C(f,\mu_h,K)$$

hold for every $K \in \mathcal{K}$ and for every $U \in \mathcal{U}$.

Remark 2.14. In view of Proposition 2.13 it follows that a sub base for the topology induced by γ_f-convergence on $\mathcal{M}_p^*$, is given by the sets of the form $\{\mu \in \mathcal{M}_p^* : C(f,\mu,U) > t\}$ and $\{\mu \in \mathcal{M}_p^* : C(f,\mu,K) < s\}$ with $t, s \in \mathbb{R}^+$, $U \in \mathcal{U}$ and $K \in \mathcal{K}$.

We denote by $\mathcal{B}(\mathcal{M}_p^*)$ the Borel σ-field of $\mathcal{M}_p^*$ endowed with the topology of γ_f-convergence.

By Proposition 2.13 and Remark 2.14 we can get further measurability properties of the μ-capacity relative to f (see Proposition 2.3 in [2]).

Proposition 2.15. *The family $\mathcal{B}(\mathcal{M}_p^*)$ is the smallest σ-algebra for which the function $C(f,\cdot,U) : \mathcal{M}_p^* \to [0,+\infty]$ is measurable for every $U \in \mathcal{U}$ (respectively the function $C(f,\cdot,K) : \mathcal{M}_p^* \to [0,+\infty]$ is measurable for every $K \in \mathcal{K}$).*

From the previous proposition we have the following consequence.

Corollary 2.16. *Let (Λ,Σ,P) be a measure space and let $m : \Lambda \to \mathcal{M}_p^*$ be a function. The following statements are equivalent:*

a) *m is Σ-$\mathcal{B}(\mathcal{M}_p^*)$ measurable;*

b) *$C(f,m(\cdot),U)$ is Σ-measurable, for every $U \in \mathcal{U}$;*

c) *$C(f,m(\cdot),K)$ is Σ-measurable, for every $K \in \mathcal{K}$.*

We can also to say something about measurability of the function $C(f,\cdot,B)$ with $B \in \mathcal{B}$.

Let us denote by $\hat{\mathcal{B}}(\mathcal{M}_p^*)$ the σ-field of all subsets of $\mathcal{M}_p^*$ which are Q measurable for every probability measure Q on $(\mathcal{M}_p^*, \mathcal{B}(\mathcal{M}_p^*))$.

The following result can be obtained by suitable minor changes in the proof of Proposition 2.4 in [2].

PROPOSITION 2.17. *For every $B \in \mathcal{B}$ the function $C(f,\cdot,B)$ is $\hat{\mathcal{B}}(\mathcal{M}_p^*)$-measurable.*

At the end of this section we recall some probabilistic notions which we shall use in the sequel.

By $\mathcal{P}(\mathcal{M}_p^*)$ we denote the space of all probability measures defined on $\mathcal{B}(\mathcal{M}_p^*)$, *i.e.* an element $Q \in \mathcal{P}(\mathcal{M}_p^*)$ is a non negative countably additive set function defined on $\mathcal{B}(\mathcal{M}_p^*)$ with $Q(\mathcal{M}_p^*) = 1$. On $\mathcal{P}(\mathcal{M}_p^*)$ we consider the following definition of weak convergence.

DEFINITION 2.18. We say that a sequence (Q_h) of measures in $\mathcal{P}(\mathcal{M}_p^*)$ weakly converges to a measure $Q \in \mathcal{P}(\mathcal{M}_p^*)$ if

$$\lim_{h\to\infty} \int_{\mathcal{M}_p^*} g\, dQ_h = \int_{\mathcal{M}_p^*} g\, dQ$$

for every continuous function $g : \mathcal{M}_p^* \to \mathbb{R}$.

Taking into account Remark 2.5, the following proposition holds (see [19]).

PROPOSITION 2.19. *For every sequence (Q_h) of measures in $\mathcal{P}(\mathcal{M}_p^*)$ there exists a subsequence (Q_{h_k}) weakly convergent in $\mathcal{P}(\mathcal{M}_p^*)$.*

Let $Q \in \mathcal{P}(\mathcal{M}_p^*)$. For every $\mathcal{B}(\mathcal{M}_p^*)$-measurable real valued function X we define the expectation of X in the probability space $(\mathcal{M}_p^*, \mathcal{B}(\mathcal{M}_p^*), Q)$ by

$$E_Q[X] = \int_{\mathcal{M}_p^*} X(\mu)\, dQ(\mu)\,.$$

Let X, Y be two real valued functions in $L^2(\mathcal{M}_p^*, Q)$. Then the covariance of X and Y is defined by

$$Cov_Q[X,Y] = E_Q[XY] - E_Q[X]E_Q[Y]\,.$$

The variance of X is defined by

$$\mathrm{Var_Q[X]} = \mathrm{Cov_Q[X,X]}\,.$$

3 – The main result

In this section we give the main result of this paper: sufficient conditions for the convergence of a sequence (Q_h) of measures on $\mathcal{M}_p^*$ of the class $\mathcal{P}(\mathcal{M}_p^*)$ to a measure $\delta_\nu \in \mathcal{P}(\mathcal{M}_p^*)$ of the form

$$\delta_\nu(\mathcal{E}) = \begin{cases} 0 & \text{if} \quad \nu \notin \mathcal{E}\,, \\ 1 & \text{if} \quad \nu \in \mathcal{E}\,, \end{cases} \tag{3.1}$$

for every $\mathcal{E} \in \mathcal{B}(\mathcal{M}_p^*)$, where ν is a finite Borel measure on D of the class $\mathcal{M}_p^*$.

We will show that the conditions are expressed in terms of the asymptotic behaviour, as $h \to \infty$, of the functions $C(f,\cdot,B)$, $B \in \mathcal{B}$, considered as random variables on the probability space $(\mathcal{M}_p^*, \mathcal{B}(\mathcal{M}_p^*), Q_h)$.

Before to state our main result we put some definition.

Let (Q_h) be a sequence in $\mathcal{P}(\mathcal{M}_p^*)$. For every $U \in \mathcal{U}$ we define

$$\alpha'(U) = \liminf_{h\to\infty} E_{Q_h}[C(f,\cdot,U)]$$

$$\alpha''(U) = \limsup_{h\to\infty} E_{Q_h}[C(f,\cdot,U)]$$

where E_{Q_h} denotes the expectation in the probability space $(\mathcal{M}_p^*$, $\mathcal{B}(\mathcal{M}_p^*)$, $Q_h)$.

Next we consider the inner regularizations β' and β'' of the set functions α' and α'', defined for every $U \in \mathcal{U}$ by

$$\begin{cases} \beta'(U) = \sup\{\alpha'(V) \,:\, V \in \mathcal{U},\, V \subset\subset U\}\,, \\ \beta''(U) = \sup\{\alpha''(V) \,:\, V \in \mathcal{U},\, V \subset\subset U\}\,. \end{cases} \tag{3.2}$$

Then we extend the definitions of β' and β'' to the Borel sets $B \in \mathcal{B}$ by:

$$\begin{cases} \beta'(B) = \inf\{\beta'(U) \,:\, U \in \mathcal{U},\, U \supseteq B\}\,, \\ \beta''(B) = \inf\{\beta''(U) \,:\, U \in \mathcal{U},\, U \supseteq B\}\,. \end{cases} \tag{3.3}$$

Finally, we denote by ν' and ν'' the least superadditive set functions on $\mathcal{B}$ greater than or equal to β' and β'', respectively.

Our main result is the following.

THEOREM 3.1. *Let $1 < p \leq n$. Let (Q_h) be a sequence of measures on $\mathcal{M}_p^*$ of the class $\mathcal{P}(\mathcal{M}_p^*)$.*

Assume that

i) *$\nu'(B) = \nu''(B) < +\infty$ for every $B \in \mathcal{B}$ and call $\nu(B)$ the common value of $\nu'(B)$ and $\nu''(B)$;*

ii) *for every $U_1, U_2 \in \mathcal{U}$ with $\overline{U}_1 \cap \overline{U}_2 = \emptyset$*

$$\lim_{h \to \infty} Cov_{Q_h}[C(f, \cdot, U_1), C(f, \cdot, U_2)] = 0\,.$$

Then

a) *ν is finite Borel measure on $\mathcal{B}$ of the class $\mathcal{M}_p^*$;*

b) *(Q_h) converges weakly to the probability measure δ_ν defined in* (3.1)*;*

c) *$\beta'(B) = \beta''(B) = C(f, \nu, B)$.*

REMARK 3.2. Let $\alpha_h : \mathcal{U} \to \mathbb{R}$ be an increasing set function defined by

$$\alpha_h(U) = E_{Q_h}[C(f, \cdot, U)]$$

and let $\alpha : \mathcal{U} \to \mathbb{R}$ be an increasing set function defined by

$$\alpha(U) = C(f, \nu, U)\,.$$

Then the condition c) of Theorem 3.1 is equivalent to say that (α_h) converges weakly to α in the sense of [14] (with respect to the pair $(\mathcal{U}, \mathcal{K})$).

The proof of Theorem 3.1 can be deduced, by means minor changes, from the proof of Theorem 3.1 in [2].

We only recall it is a consequence of Proposition 3.3 below, which provide sufficient conditions in order that a probability measure $Q \in \mathcal{P}(\mathcal{M}_p^*)$ be equal to the Dirac measure defined in (3.1).

PROPOSITION 3.3. *Let $1 < p \leq n$. Let Q be a probability measure on $\mathcal{M}_p^*$ of the class $\mathcal{P}(\mathcal{M}_p^*)$.*

Define

$$\alpha(U) = E_Q[C(f, \cdot, U)]$$

for every $U \in \mathcal{U}$ *and*

$$\alpha(B) = \inf\{\alpha(U)\,;\, U \supseteq B\,,\, U \in \mathcal{U}\}$$

for every $B \in \mathcal{B}$.

Assume that:

i) *there exists a Radon measure* β *on* $\mathcal{B}$ *such that* $\alpha \le \beta$ *on* $\mathcal{B}$*;*

ii) $Cov_Q[C(f,\cdot,U_1), C(f,\cdot,U_2)] = 0$ *for every pair* $U_1, U_2 \in \mathcal{U}$ *such that* $\overline{U}_1 \cap \overline{U}_2 = \emptyset$.

Let ν *be the least superadditive set function on* $\mathcal{B}$ *such that* $\nu \ge \alpha$ *on* $\mathcal{B}$*. Then*

t_1) ν *is a Borel measure on* $\mathcal{B}$ *of the class* $\mathcal{M}_p^*$*;*

t_2) $Q = \delta_\nu$.

Proof. The proof may be obtained by adapting that one of Lemma 3.3 in [2]. For the reader convenience we repeat here the proof in our case.

First of all, we notice that the function α is countably subadditive on $\mathcal{U}$ (hence on $\mathcal{B}$) by the countable subadditivity of $C(f,\mu,\cdot)$ (Proposition 2.10 (d)). Moreover, by Lemma 4.1 of [8], we deduce that ν is a Borel measure. The measure ν is also in $\mathcal{M}_p^*$ because it is a Radon measure and $\nu(B) = 0$ whenever $C_p(B) = 0$ by (h) of Proposition 2.10. This proves (t_1).

Let us denote by $Z(\cdot, B)$ the random variable on the probability space $(\mathcal{M}_p^*, \mathcal{B}(\mathcal{M}_p^*), Q)$ defined by

$$Z(\mu, B) = \mu(B)$$

for every $B \in \mathcal{B}$.

We note that, by Theorem 2.12, for every $\mu \in \mathcal{M}_p^*$ and for every $B \in \mathcal{B}$,

$$Z(\mu, B) = \lim_{h\to\infty} \sum_{i\in\mathbb{Z}^n} C(f,\mu, B \cap R_h^i)$$

where R_h^i denotes the cube defined in Theorem 1.12.

Our aim is to show that the random variable $Z(\cdot, B)$ is constant Q-almost everywhere. In view of Lemma 3.1 in [2], we have only to prove that

$$\lim_{h\to\infty} \mathrm{Var_Q}\Big[\sum_{\mathrm{i}\in\mathbb{Z}^{\mathrm{n}}} \mathrm{C(f,\cdot,B\cap R^i_h)}\Big] = 0\,. \tag{3.4}$$

By properties h) and i) of Proposition 2.10 we can extend the relation ii) to each pair of disjoint sets $A, B \in \mathcal{B}$ and check that

$$\alpha(B) = E_Q[C(f,\cdot,B)]$$

for every $B \in \mathcal{B}$.

Therefore, to get (3.4) it is enough to prove

$$\lim_{h\to\infty} \sum_{i\in\mathbb{Z}^n} \mathrm{Var_Q[C(f,\cdot,B\cap R^i_h]} = 0 \tag{3.5}$$

for every $B \in \mathcal{B}$.

For every $h \in \mathbb{N}$, taking into account property (g) of Proposition 2.10, we have

$$\begin{aligned}&\sum_{i\in\mathbb{Z}^d} \mathrm{Var_Q[C(f,\cdot,B\cap R^i_h)]} = \sum_{\mathrm{i}\in\mathbb{Z}^{\mathrm{d}}} \{\mathrm{E_Q[C(f,\cdot,B\cap R^i_h)^2]}+\\ &-(E_Q[C(f,\cdot,B\cap R^i_h)])^2\} \le \sum_{i\in\mathbb{Z}^d} E_Q[C(f,\cdot,B\cap R^i_h)^2] \le c_2\times\\ &\times \sum_{i\in\mathbb{Z}^d} C_p(B\cap R^i_h)E_Q[C(f,\cdot,B\cap R^i_h)] \le c_2 k_h \beta(B)\end{aligned} \tag{3.6}$$

where we have set

$$k_h = \sup_{i\in\mathbb{Z}^n} C_p(B\cap R^i_h)\,.$$

We observe that $k_h \to 0$ as $h \to \infty$ (because the dimension n is greater than or equal to p). So, taking the limit as $h \to \infty$ in (3.6) we get (3.5) and this proves that $Z(\cdot, B)$ is a constant random variable. Now, let us compute the expectation of $Z(\cdot, B)$. Since the sequence $(\sum_{i\in\mathbb{Z}^d} C(f,\cdot,B\cap R^i_h))_{h\in\mathbb{N}}$ is increasing, we get

$$E_Q[Z(\cdot,B)] = \lim_{h\to\infty} E_Q\Big[\sum_{i\in\mathbb{Z}^d} C(f,\cdot,B\cap R^i_h)\Big] = \lim_{h\to\infty} \sum_{i\in\mathbb{Z}^d} \alpha(B\cap R^i_h) = \nu(B)$$

for every $B \in \mathcal{B}$ where the last equality is proved in [8], Lemma 4.2. So, for every Borel set B in D there exists a subset $\mathcal{M}_B$ of $\mathcal{M}_p^*$ with $Q(\mathcal{M}_B) = 1$ such that $Z(\mu, B) = \nu(B)$ for every $\mu \in \mathcal{M}_B$.

Finally, by means standard density arguments (see for istance the proof of Lemma 3.3 in [2]) we can deduce that there exists a subset $\mathcal{M}$ of $\mathcal{M}_p^*$ such that $Q(\mathcal{M}) = 1$ and $Z(\mu, B) = \nu(B)$ for every $\mu \in \mathcal{M}$ and for every $B \in \mathcal{B}$. This completes the the proof of (t_2).

Remark 3.4. It is not difficult to show that the conditions i) and ii) of Theorem 3.1 are also necessary. The proof of this can be obtained by using the same arguments of Remark 3.2 in [2].

4 – Nonlinear Dirichlet problems in randomly perforated domains

We apply the result obtained in the previous section to analyze the asymptotic behaviour of sequences of nonlinear variational problems in randomly perforated domains of the form

$$\min_{u \in H_0^{1,p}(D \setminus E_h)} \int_{D \setminus E_h} f(x, Du)\, dx + \int_{D \setminus E_h} gu\, dx \tag{4.1}$$

where E_h is a sequence of suitable random closed subsets of D and $g \in L^q(D)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p \leq n$.

Let (Ω, Σ, P) be a probability space.

Definition 4.1. Let $1 < p \leq n$. A random measure of the class $\mathcal{M}_p^*$ is any measurable function $M : \Omega \to \mathcal{M}_p^*$, where $\mathcal{M}_p^*$ is equipped with the Borel σ-field $\mathcal{B}(\mathcal{M}_p^*)$ generated by the topology induced by the γ_f-convergence.

Remark 4.2. We recall that necessary and sufficient conditions for the measurability of a function $M : \Omega \to \mathcal{M}_p^*$ are given in Corollary 2.16.

Let M be a random measure of the class $\mathcal{M}_p^*$ and let Q be the probability measure on $(\mathcal{M}_p^*, \mathcal{B}(\mathcal{M}_p^*))$ defined by

$$Q(\mathcal{E}) = P\{M^{-1}(\mathcal{E})\}$$

for every $\mathcal{E} \in \mathcal{B}(\mathcal{M}_p^*)$. Q is called “the distribution law of M”.

Let (M_h) be a sequence of random measures of the class $\mathcal{M}_p^*$ and M a random measure of the class $\mathcal{M}_p^*$. Let (Q_h) be the sequence of the distribution laws of (M_h) and let Q be the distribution law of M.

DEFINITION 4.3. We say that (M_h) converges in law to the random measure M of the class $\mathcal{M}_p^*$ if and only if the sequence of distribution laws (Q_h) converges weakly in $\mathcal{P}(\mathcal{M}_p^*)$ to the distribution law Q.

We denote by E and by Cov respectively the expectation and the covariance of a random variable on Ω with respect to the measure P.

Let Q be the distribution law of a random measure M of the class $\mathcal{M}_p^*$. It is easy to see that

$$E_Q[C(f,\cdot,U)] = E[C(f,M(\cdot),U)] \tag{4.2}$$

for every $U \in \mathcal{U}$ and

$$Cov_Q[C(f,\cdot,U_1),C(f,\cdot,U_2)] = Cov[C(f,M(\cdot),U_1),C(f,M(\cdot),U_2)] \tag{4.3}$$

for every pair $U_1, U_2 \in \mathcal{U}$.

Let us define the set functions

$$\begin{cases} \alpha'(U) = \liminf\limits_{h\to\infty} E[C(f,M_h(\cdot),U)] \\ \alpha''(U) = \limsup\limits_{h\to\infty} E[C(f,M_h(\cdot),U)] \end{cases} \tag{4.4}$$

for every $U \in \mathcal{U}$.

We shall denote by β' and β'' the inner regularizations of α' and α'' as defined in (3.2) and (3.3), respectively.

The functions ν' and ν'' will be the least superadditive set functions on $\mathcal{B}$ greater than or equal to β' and β'', respectively.

REMARK 4.4. Equalities (4.2), (4.3) and (4.4) allow us to reformulate the hypotheses of Theorem 3.1 in terms of expectations and covariances of the random variables $C(f,M_h(\cdot),U)$. By Defintion 4.3 the thesis of Theorem 3.1 can be restated by saying that the sequence of random measures (M_h) of the class $\mathcal{M}_p^*$, converges in law to a random measure M such that $M(\omega) = \nu$ for P-almost every $\omega \in \Omega$ (*i.e.* to the constant random measure $M = \nu$).

Remark 4.5. Since $\mathcal{M}_p^*$ is a metric space (let $d\gamma_f$ any metric on $\mathcal{M}_p^*$ which induces γ_f-convergence) the convergence in law of the sequence (M_h) toward the constant random measure M is equivalent to the convergence in probability. Thus, by Remark 4.4, we can deduce that if the assumption of Theorem 3.1 hold, then the sequence (M_h) converges in probability to the measure ν in $\mathcal{M}_p^*$, that is, for every $\epsilon > 0$

$$\lim_{h\to\infty} P\{\omega \in \Omega \,:\, d\gamma_f(M_h(\omega), \nu) > \epsilon\} = 0$$

Next theorem states a relationship between the result obtained in the previous section and the convergence of the minimum values of functional associate with the random measure M_h of the class $\mathcal{M}_p^*$.

Theorem 4.6. *Let (M_h) be a sequence of random measures of the class $\mathcal{M}_p^*$, $1 < p \leq n$. Let α' and α'' be the set functions defined in* (4.4) *and let ν' and ν'' be the least superadditive set functions on $\mathcal{B}$ greater than or equal to β' and β'' (respectively the inner regularizations of α' and α'', see* (3.2) *and* (3.3)*).*

Assume that

i) $\nu'(B) = \nu''(B) < \infty$ *for every $B \in \mathcal{B}$*

and denote by $\nu(B)$ the common value of $\nu'(B)$ and $\nu''(B)$ for every $B \in \mathcal{B}$;

ii) *for every $U_1, U_2 \in \mathcal{U}$, such that $\overline{U}_1 \cap \overline{U}_2 = \emptyset$*

$$\lim_{h\to\infty} Cov[C(f, M_h(\cdot), U_1), C(f, M_h(\cdot), U_2)] = 0\,.$$

Let $m_h(\omega) = m(M_h(\omega), g)$ be, for every $\omega \in \Omega$, $h \in \mathbb{N}$ and $g \in L^q(D)$, with $\frac{1}{p} + \frac{1}{q} = 1$, the minimum values defined as

$$m_h(\omega) = \min_{u\in H_0^{1,p}(D)} \int_D f(x, Du)\, dx + \int_D |u|^p\, dM_h + \int_D gu\, dx$$

for any $g \in L^q(D)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Then (m_h) converges in probability, as $h \to \infty$, to m_0 that is, for every $\epsilon > 0$

$$\lim_{h\to\infty} P\{\omega \in \Omega\,,\, |m_h(\omega) - m_0| > \epsilon\} = 0\,,$$

where m_0 is given by

$$m_0 = m(\nu, g) = \min_{u \in H_0^{1,p}(D)} \int_D f(x, Du)\, dx + \int_D |u|^p \, d\nu + \int_D gu \, dx \, .$$

PROOF. By Theorem 2.7 the function $m(\cdot, g) : \mathcal{M}_p^* \to \mathbb{R}$ of Definition 2.6 is continuous. Since $\mathcal{M}_p^*$ is compact (see Remark 2.5), $m(\cdot, g)$ is uniformly continuous too. Hence, for every $\epsilon > 0$ and $h \in \mathbb{N}$ there exists $\delta_\epsilon > 0$ such that

$$P\{\omega \in \Omega \,:\, |m_h(\omega) - m_0| \geq \epsilon\} \leq P\{\omega \in \Omega \,:\, d\gamma_f(M_h(\omega), \nu) \geq \delta_\epsilon\}$$

So, the assertion follows from Theorem 3.1 together with Remark 4.5.

COROLLARY 4.7. *Suppose that the hypotheses of Theorem 4.6 are satisfied. Moreover, assume that the function $f(x, \xi)$ in (2.1) and (2.2) is such that the map $U(\cdot, g) : \mathcal{M}_p^* \to L^p(D)$ of Definition 2.6 is single valued, with $g \in L^q(D)$, $\frac{1}{p} + \frac{1}{q} = 1$.*

Then, if $U_h(\omega) = U(M_h(\omega), g)$, for every $\omega \in \Omega$ denotes the unique minimum point in $H_0^{1,p}(D)$ of the functional

$$\int_D f(x, Du)\, dx + \int_D |u|^p \, dM_h(\omega) + \int_D gu \, dx \, ,$$

we have, for every $\epsilon > 0$,

$$\lim_{h \to \infty} P\left\{\omega \in \Omega \,:\, \|U_h(\omega) - U_0\|_{L^p(D)} > \epsilon\right\} = 0 \, ,$$

where U_0 is the unique minimum point in $H_0^{1,p}(D)$ of the functional

$$\int_D f(x, Du)\, dx + \int_D |u|^p \, d\nu + \int_D gu \, dx \, .$$

Proof. It is enough to notice that in this case, by Remark 2.8, the function $U(\cdot, g) : \mathcal{M}_p^* \to L^p(D)$ is continuous and to apply the same argument used in the previous Theorem 4.6.

In order to consider nonlinear variational problems in domains with random holes like (4.1), we shall see that we have only to choose peculiar sequences (M_h) of random measures of the class $\mathcal{M}_p^*$ in (4.6).

Let $\mathcal{E}(D)$ be the family of all closed sets contained in D.

Definition 4.8. A function $E : \Omega \to \mathcal{E}(D)$ is called a p-random set, $1 < p \leq n$, if the function $M : \Omega \to \mathcal{M}_p^*$ defined by $M(\omega) = \infty_{E(\omega)}$ for each $\omega \in \Omega$ is Σ-measurable, where $\infty_{E(\omega)}$is the measure in $\mathcal{M}_p^*$ as in Definition 2.3.

Remark 4.9. Let $E : \Omega \to \mathcal{E}(D)$ be a function. By Corollary 2.16 and by the equality $C(f, \infty_A, B) = C(f, B \cap A)$ (see Remark 2.9), the following conditions are equivalent:

a) E is a p-random set;
b) $C(f, E(\cdot) \cap U)$ is Σ-measurable for every $U \in \mathcal{U}$;
c) $C(f, E(\cdot) \cap K)$ is Σ-measurable for every $K \in \mathcal{K}$.

Let us take a sequence (E_h) of p-random sets. Let (M_h) be the sequence of random measures of the class $\mathcal{M}_p^*$ so defined: $M_h(\omega) = \infty_{E_h(\omega)}$ for each $\omega \in \Omega$.

We shall analize the asymptotic behaviour of the minimum problems of the form

$$\min_{u \in H_0^{1,p}(D)} \int_D f(x, Du)\, dx + \int_D |u|^p \, d\infty_{E_h} + \int_D gu\, dx \tag{4.5}$$

for any $g \in L^q(D)$, $\frac{1}{p} + \frac{1}{q} = 1$.

For every $h \in \mathbb{N}$ and $\omega \in \Omega$ the minimum problem (4.5) is equivalent to the minimum problem

$$\min_{u \in H_0^{1,p}(D)} \int_{D \setminus E_h(\omega)} f(x, Du)\, dx + \int_{D \setminus E_h(\omega)} gu\, dx$$

in the sense that both problems have the same minimum values and the same minimum points, if we identify each $u \in H_0^{1,p}(D \setminus E_h(\omega))$ with the function of $H_0^{1,p}(D)$ obtained by the usual extension $u = 0$ on $E_h(\omega)$.

In fact, for a function $u \in H_0^{1,p}(D)$ the condition $u = 0$ p-q.e. on B is equivalent to $u \in H_0^{1,p}(D \backslash B)$ for every closed set $B \subseteq D$ (see [1]).

Next results are a straightforward consequence of Theorem 4.6 and Corollary 4.7.

THEOREM 4.10. *Let (E_h) be a sequence of p-random sets, with $1 < p \leq n$. Let α' and α'' be the set functions defined in* (4.4) *where $M_h = \infty_{E_h}$, and let ν' and ν'' be the least superadditive set functions on $\mathcal{B}$ greater than or equal to β' and β'', i.e. the set functions defined in* (3.2) *and* (3.3).

Assume that

i) *$\nu'(B) = \nu''(B) < \infty$ for every $B \in \mathcal{B}$ and denote by $\nu(B)$ the common value of $\nu'(B)$ and $\nu''(B)$ for every $B \in \mathcal{B}$;*

ii) *for every $U_1, U_2 \in \mathcal{U}$ with $\overline{U}_1 \cap \overline{U}_2 = \emptyset$*

$$\lim_{h\to\infty} Cov[C(f, E_h \cap U_1), C(f, E_h \cap U_2)] = 0\,.$$

Let

$$m_h(\omega) = \min_{u\in H_0^{1,p}(D\backslash E_h(\omega))} \int_{D\backslash E_h(\omega)} f(x, Du)\, dx + \int_{D\backslash E_h(\omega)} gu\, dx$$

for any $g \in L^q(D)$, with $\frac{1}{p} + \frac{1}{q} = 1$ and $\omega \in \Omega$.

Then (m_h) converges in probability, as $h \to \infty$, to

$$m_0 = \min_{u\in H_0^{1,p}(D)} \int_D f(x, Du)\, dx + \int_D |u|^p\, d\nu + \int_D gu\, dx\,.$$

COROLLARY 4.11. *Suppose that the hypotheses of Theorem 4.10 are satisfied. Moreover, assume that the function $f(x, \xi)$ in* (2.1) *and* (2.2) *is such that the map $U(\cdot, g) : \mathcal{M}_p^* \to L^p(D)$ of Definition 2.6 is single valued.*

Then, if $U_h(\omega)$ denotes the unique minimum in $H_0^{1,p}(D\backslash E_h(\omega))$ of the functional

$$\int_{D\backslash E_h(\omega)} f(x, Du)\, dx + \int_{D\backslash E_h(\omega)} gu\, dx$$

for every $\omega \in \Omega$*, we have*

$$\lim_{h\to\infty} P\{\omega \in \Omega \,:\, \|U_h(\omega) - U_0\|_{L^p(D)} > \epsilon\} = 0$$

for any $\epsilon > 0$*, where* U_0 *is the minimum point in* $H_0^{1,p}(D)$ *of the functional*

$$\int_D f(x, Du)\, dx + \int_D |u|^p\, d\nu + \int_D gu\, dx\,.$$

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Lavoro pervenuto alla redazione il 1 marzo 1996
ed accettato per la pubblicazione il 4 dicembre 1996.
Bozze licenziate il 16 aprile 1997

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