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The initial value problem for a one-dimensional Boltzmann equation with diffusive boundary conditions

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RIASSUNTO: *Si considera un'equazione di evoluzione di tipo Boltzmann unidimensionale, con condizioni diffusive agli estremi di un intervallo, e si dimostra un risultato di esistenza e unicità della soluzione.*

ABSTRACT: *It is solved the initial-boundary value problem for a one-dimensional model of the Boltzmann equation in a slab, with diffusive non constant boundary conditions.*

1 – Introduction

In this paper it is considered the initial value problem for a one-dimensional model of the Boltzmann equation in a slab with diffusive boundary conditions. More precisely one looks for a positive function $f = f(x, v, t)$, $\quad x \in [0, 1]$, $\quad v \in \mathbb{R}$, $\quad t \in [0, T]$, satisfying the following initial boundary value problem (i.b.v.p.):

$$(1.1)\qquad \begin{gathered}\partial_t f(x,v,t) + v\partial_x f(x,v,t) = Q(f,f)(x,v,t) := \\ \int_{vv_1<0} dv_1 S(|v-v_1|)\{f(x,-v,t)f(x,-v_1,t) - f(x,v,t)f(x,v_1,t)\}\end{gathered}$$

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with initial condition:

$$f(x,v,0) = f_0(x,v) \geq 0 \tag{1.2}$$

and with the boundary conditions:

$$f(0,v,t) = -M_0(v) \int_{w<0} w f(0,w,t) dw\,, \qquad v > 0\,, \tag{1.3}$$

$$f(1,v,t) = M_1(v) \int_{w>0} w f(1,w,t) dw\,, \qquad v < 0\,. \tag{1.4}$$

Here $S = S(|v - v_1|)$ is a suitable function such that:

$$S(|v - v_1|) \leq A|v - v_1|^{\alpha}, \qquad \alpha < 1, \quad |v - v_1| > 1 \tag{1.5}$$

$$S(|v - v_1|) \leq B|v - v_1|, \qquad |v - v_1| \leq 1\,, \tag{1.6}$$

while M_0 and M_1, the "maxwellians at the walls", are given by:

$$\begin{aligned} M_0(v) &= 2\beta_0 e^{-\beta_0 v^2}, \qquad v > 0 \\ M_1(v) &= 2\beta_1 e^{-\beta_1 v^2}, \qquad v < 0 \end{aligned}$$

Here A, B, β_0 and β_1 are positive constants.

The above equation can be interpreted as a simplified model for a dilute gas in a kinetic regime in a slab, between two reservoirs in thermal equilibrium at different temperatures. It has been introduced in [7], where the authors exhibit an explicit steady solution.

In any attempt to study the Boltzmann equation and its solutions, there is an evident first order difficulty, which can be sometimes bypassed if one considers systems in one spatial dimension. Indeed, in proving uniqueness results, one has in general to find out some Lipschitz continuity property for the time integral of Q in the L_1-norm, which appears to be the most natural norm in this setup. To do this, the quantities under consideration have to be, at least, dimensionally correct. Roughly speaking, since Q is quadratic in f, four integrals are needed. For the one dimensional systems, this is in principle possible, since the time integral can be turned into a space integral.

We want to mention some of the numerous results in this framework. [3] regards the initial value problem on the whole line, generalized to the three dimensional velocities case in [1] and [4]. In the same context, in [5] and [6] the authors prove some results on existence of solutions, together with some energy and entropy bounds.

The intrinsic difficulty of the problem, induces the authors of all the above mentioned papers to make rather heavy hypotheses on the collision kernel S. The system I consider, having one dimensional velocities, is by no doubt less interesting from a physical point of view. On the other hand, the collision mechanism is fairly simplified, and this makes it possible to obtain a complete result of existence and uniqueness of a strong solution, with plausible assumptions on S.

Problem (1.1)-(1.4) is approached by adapting the method introduced in [1] to this different setup. The equation is splitted into two parts, one ("internal") with bounded velocities and one ("external") with large ones and it is proven an existence and uniqueness result for the former and for the latter separately. Unfortunately, in presence of stochastic boundaries, the usual properties of the energy and the entropy are missing, but the bounds given in [6] on these quantities come very useful to control the density and to handle the large velocities.

The main result is stated in Theorem 2.1 in Section 2, where some notation and preliminary observations are given, together with the above mentioned estimates proven in [6]. The Sections 3 and 4 are devoted to the proof of Theorem 2.1.

2 – The statement of the theorem

The collision operator Q enjoys the following properties, which will be useful in the sequel of the paper:

$$\int_{v>0} Q(f,f)(x,v,t)dv = \int_{v<0} Q(f,f)(x,v,t)dv = 0\,. \tag{2.1}$$

(2.1) imply the local conservation of the density of particles travelling with positive and negative velocity respectively. Moreover it ensures the conservation of the mass. Indeed, defining the flux at the walls:

$$J^{\pm}(\alpha,t) = \int_{v\gtrless 0} dv|v|f(\alpha,v,t) \quad \alpha = 0,1 \tag{2.2}$$

it is, by the definition of the maxwellians M_0 and M_1,

$$J^+(\alpha,t) - J^-(\alpha,t) = 0, \qquad \alpha = 0,1 \tag{2.3}$$

so that, at least at a formal level, defining

$$M(t) = \int f(x,v,t)dx\,dv \tag{2.4}$$

it follows $M(t) = M(0)$.

Equation (1.1) can be written in a mild form:

$$f(x,v,t) = g(x,v,t) + \int_{a(x,v,t)}^{t} Q(f,f)(x - v(t-s), v, s)ds \tag{2.5}$$

where, if $v > 0$:

$$g(x,v,t) = f_0(x - vt, v)\chi\Big(t < \frac{x}{v}\Big) + f\Big(0, v, t - \frac{x}{v}\Big)\chi\Big(t > \frac{x}{v}\Big) \tag{2.6-1}$$

$$a(x,v,t) = \max\Big\{0, t - \frac{x}{v}\Big\} \tag{2.7-1}$$

and if $v < 0$:

$$\begin{aligned} g(x,v,t) = {} & f_0(x - vt, v)\chi\Big(t < \frac{1-x}{|v|}\Big) + \\ & + f\Big(1, v, t - \frac{1-x}{|v|}\Big)\chi\Big(t > \frac{1-x}{|v|}\Big) \end{aligned} \tag{2.6-2}$$

$$a(x,v,t) = \max\Big\{0, t - \frac{1-x}{|v|}\Big\}. \tag{2.7-2}$$

From now on χ will stand for the characteristic function of its argument.

Equation (2.5) has to be complemented with the boundary conditions (1.3) and (1.4).

In what follows it will be convenient to introduce the notation, for any x, v, t such that $0 \le x + vt \le 1$:

$$f^\sharp(x,v,t) = f(x + vt, v, t). \tag{2.8}$$

Then equation (2.5) becomes:

$$f^\sharp(x,v,t) = g^\sharp(x,v,t) + \int_{A(x,v)}^t Q(f,f)^\sharp(x,v,s)ds \tag{2.9}$$

where $A(x,v) = a^\sharp(x,v,t)$, that is, if $v > 0$:

$$A(x,v) = \max\left\{0, -\frac{x}{v}\right\} \tag{2.10}$$

and if $v < 0$:

$$A(x,v) = \max\left\{0, \frac{x-1}{|v|}\right\}. \tag{2.11}$$

We introduce the usual quantities:

$$E(t) = \frac{1}{2}\int f(x,v,t)v^2 dx\, dv\,, \tag{2.12}$$

$$H(t) = \int (f \log f)(x,v,t) dx\, dv\,, \tag{2.13}$$

the energy and the entropy of the system respectively. The result in [6] we want to quote, concerning these quantities, is the following:

Lemma 2.1. *If f is a sufficiently smooth solution of* (1.1)-(1.4) *on $[0,T]$, then under the assumptions 1.5-1.6 there are positive constants γ_i and η_i, $i = 1,2$ depending only on the initial datum such that: $E(t) \le \gamma_1 e^{\gamma_2 t}$ and $H(t) \le \eta_1 e^{\eta_2 t}$ for any $t \in [0,T]$, with $\gamma_1 = \gamma_1(M(0),E(0))$ and $\eta_1 = \eta_1(M(0),E(0),H(0))$.*

The aim of this paper is to prove the following theorem.

Theorem 2.1. *Assume that $M(0), E(0), H(0)$ are finite. Then, for any $T > 0$, there exists a unique solution in $L_\infty([0,T], L_1^+)$ to the problem* (2.5) *with the boundary conditions* (1.3) (1.4)*. Moreover the mass is preserved and the energy and the entropy stay bounded. In addition $Q(f,f)^\sharp(x,v,\cdot) \in L_1([0,T])$ for* a.a. *(x,v).*

REMARK. Notice that in L_1 the boundary conditions have not an obvious meaning and one must give a sense to the fluxes $J(0,t)$ and $J(1,t)$. Indeed, since $Q(f,f)^\sharp \in L_1([0,T])$ for a.a. (x,v), $f^\sharp$ is absolutely continuous in t for a.a. (x,v). As a consequence $f(\alpha,v,t)$, $\alpha = 0,1$, is defined by continuity for a.a. (v,t), which implies that the fluxes are well defined for a.a. $t>0$.

To prove this theorem the i.b.v.p. (1.1) is splitted into two parts as in [1]: a bounded velocities problem and a large velocities one. More precisely, for a fixed $V>0$ which has to be determined later on, I consider the following "internal" problem:

$$\partial_t f_i(x,v) + v\partial_x f_i(x,v) = Q(f_i,f_i) \tag{2.14}$$

with initial condition:

$$f_i(x,v,0) = f_{i,0}(x,v) = f_0(x,v)\chi(|v|\le V) \tag{2.15}$$

and boundary conditions:

$$f_i(0,v,t) = M_0^i(v)J_i^-(0,t)\,, \qquad v>0\,, \tag{2.16}$$

$$f_i(1,v,t) = M_1^i(v)J_i^+(1,t)\,, \qquad v<0\,, \tag{2.17}$$

where:

$$M_\alpha^i(v) = M_\alpha(v)\chi(|v|\le V), \qquad \alpha = 0,1 \tag{2.18}$$

$$J_i^\pm(\alpha,t) = \int_{v\gtrless 0} dv|v|f_i(\alpha,v,t)\,. \tag{2.19}$$

In addition I consider the "external" problem:

$$\partial_t f_e(x,v) + v\partial_x f_e(x,v) = 2Q(f_i,f_e) + Q(f_e,f_e) \tag{2.20}$$

with the initial condition:

$$f_e(x,v,0) = f_{e,0}(x,v) = f_0(x,v)\chi(|v|>V) \tag{2.21}$$

and boundary conditions:

$$f_e(0,v,t) = M_0^e(v)J^-(0,t) + M_0^i(v)J_e^-(0,t) \qquad v>0 \tag{2.22}$$

$$f_e(1,v,t) = M_1^e(v)J^+(1,t) + M_1^i(v)J_e^+(1,t) \qquad v<0 \tag{2.23}$$

where:

$$M_\alpha^e(v) = M_\alpha(v)\chi(|v|>V) \qquad \alpha = 0,1 \tag{2.24}$$

$$J_e^\pm(\alpha,t) = \int_{v\gtrless 0} dv|v|f_e(\alpha,v,t). \tag{2.25}$$

and

$$J^\pm(\alpha,t) = J_i^\pm(\alpha,t) + J_e^\pm(\alpha,t)\,. \tag{2.26}$$

$Q(f,h)$ denotes the symmetrized collision operator, namely:

$$\begin{aligned} Q(f,h)(v) &= \frac{1}{2}\int dv_1\chi(vv_1<0)S(|v-v_1|) \\ &[f(-v)g(-v_1)+g(-v)f(-v_1)-f(v)g(v_1)-f(v_1)g(v)] \end{aligned} \tag{2.27}$$

A solution of system (1.1) is obviously given by $f = f_i + f_e$ so that, to prove Theorem 1.1, one has to prove the existence and uniqueness of f_i and f_e. Notice that f_i is expected to be positive and to satisfy the condition $f_i(x,v,t) = 0$ if $|v|>V$ while f_e is possibly not positive. Moreover the mass is no more preserved for the two systems. Indeed it is

$$J_i^+(0,t) \le J_i^-(0,t) \tag{2.28}$$

$$J_i^-(1,t) \le J_i^+(1,t) \tag{2.29}$$

since:

$$\int_{v\gtrless 0} dv|v|M_\alpha^i(v) \le 1\,. \qquad \alpha = 0,1 \tag{2.30}$$

and analogously, but with opposite inequalities for the external problem. In Sections 2 and 3 the problems 2.14 and 2.20 respectively will be approached.

3 – The bounded velocity problem

The problem (2.14-17) has a mild version analogous to that given in (2.5) for the complete equation, that is:

$$f_i(x,v,t) = g_i(x,v,t) + \int_{a(x,v,t)}^{t} Q(f_i, f_i)(x - v(t-s), v, s)ds \tag{3.1}$$

where g_i is given by equations (2.6) with f and f_0 replaced by f_i and by $f_{0,i}$, and equivalently:

$$f_i^\sharp(x,v,t) = g_i^\sharp(x,v,t) + \int_{A(x,v)}^{t} Q(f_i, f_i)^\sharp(x,v,s)ds. \tag{3.2}$$

Call $M_i(t), E_i(t)$ and $H_i(t)$ the mass, energy and entropy relative to f_i. In this section it will be proven the following theorem:

Theorem 3.1. *Assume that $M_i(0), E_i(0)$ and $H_i(0)$ are finite. Then, for any $V > 0$ and $T > 0$, there exists a unique solution in $L_\infty([0,T]; L_1)$ to the problem* (3.1). *Moreover the mass is preserved and the entropy and the energy stay bounded. In addition*

$$Q(f_i, f_i)^\sharp(x, v, \cdot) \in L_1([0,T])$$

for a.a. (x, v).

Proof. The proof of the above theorem proceeds through the following two steps:

(a) Suppose $\|f_{0,i}\|_{L_1} < \frac{1}{64C_0}$ (where $C_0 = \max\{A, B, 1\}$ with A and B the constants appearing in (1.5), (1.6)) and $T^* \leq \frac{1}{V}$. Then there exists a unique solution to the problem (3.1) over the time interval $[0, T^*]$.

(b) There exists a unique solution to the problem (3.1) for any initial datum and any fixed interval of time $[0, T]$.

Since in this section it is considered only the bounded velocity problem, the subscript "i" will be omitted for the sake of simplicity. Moreover C will denote any constant.

Proof of (a). For any function $f \in L_\infty([0,T];L_1)$ define:

$$F_T(x,v) = \sup_{0\le t\le T} |f^\sharp|(x,v,t) \tag{3.3}$$

and

$$\|F_T\| = \|F_T\|_+ + \|F_T\|_- \tag{3.4}$$

with

$$\|F_T\|_\pm = \int dx \int_{v \gtrless 0} dv\, F_T(x,v) \tag{3.5}$$

By (1.5), (1.6) and a suitable change of variables Q can be estimated as follows:

$$\begin{aligned}
&\int dx \int dv \int_0^T ds |Q(f,f)^\sharp|(x,v,s) \le \\
&\le \int dx \int dv \int_0^T ds \int dv_1 \chi(vv_1 < 0) S(|v-v_1|)\cdot \\
&\quad \cdot [F_T(x+2vs,-v)F_T(x+(v+v_1)s,-v_1)+ \\
&\quad + F_T(x,v)F_T(x+(v-v_1)s,v_1)] = \\
&= 2\int dx \int dv \int_0^T ds \int dv_1 \chi(vv_1<0)\cdot \\
&\quad \cdot S(|v-v_1|)F_T(x,v)F_T(x+(v-v_1)s,v_1) \le \\
&\le 2C_0\|F_T\|^2
\end{aligned} \tag{3.6}$$

Moreover for f and $g \in L_\infty([0,T];L_1)$, proceeding as above:

$$\begin{aligned}
&\int dx \int dv \int_0^T ds |Q(f,f)^\sharp - Q(g,g)^\sharp|(x,v,s) \le \\
&\le \int dx \int dv \int_0^T ds |Q(f+g,f-g)^\sharp|(x,v,s) \le \\
&\le 2C_0(\|F_T\| + \|G_T\|)\|\Delta_T\|
\end{aligned} \tag{3.7}$$

with:

$$\Delta_T(x,v) = \sup_{0\le t\le T} |f^\sharp - g^\sharp|(x,v,t)$$

Following the method in [1], it is considered a sequence of cutoffed equations. Define for any positive integer n:

$$[f,f](x,v,v_1) = \min\left(f(x,v)f(x,v_1),n\right)$$

$$Q^n(f,f) = \int \chi(vv_1<0)S(|v-v_1|)([f,f](x,-v,-v_1) - [f,f](x,v,v_1))dv_1$$

and consider the regularized equation:

$$f^\sharp(x,v,t) = g^\sharp(x,v,t) + \int_{A(x,v)}^t Q^n(f,f)^\sharp(x,v,s)ds\,. \tag{3.8}$$

with g the same as in (2.6).

For fixed n the i.b.v.p. (3.8) admits a unique positive solution, call it f_n. This can be seen by introducing some iterative system as the following:

$$f^{k\sharp}(x,v,t) = g^{k\sharp}(x,v,t) + \int_{A(x,v)}^t Q^n(f^{k-1},f^{k-1})^\sharp(x,v,s)ds \tag{3.9}$$

where for $v>0$

$$g^{k\sharp}(x,v,t) = f_0(x,v)\chi\Big(\frac{x}{v}>0\Big) + f^k\Big(0,v,\frac{-x}{v}\Big)\chi\Big(\frac{x}{v}<0\Big) \tag{3.10-1}$$

and for $v<0$

$$\begin{aligned} g^{k\sharp}(x,v,t) = f_0(x,v)\chi\Big(\frac{1-x}{|v|}>0\Big)&+\\ + f^k\Big(1,v,\frac{x-1}{|v|}\Big)\chi\Big(\frac{1-x}{|v|}<0\Big)& \end{aligned} \tag{3.10-2}$$

for $k=2,3,\dots$ The first term in the iteration is chosen to be $f^1=0$. Since the velocities are bounded, $f^k(t)$ can be explicitely found, once f^{k-1} is known, by performing a finite number of steps back in time, along the characteristics, up to the initial condition. The fact that f^k converges in L_1 to f_n uniformly on $[0, T]$, as k tends to infinity, can be seen with a similar (a fortiori) argument to the one we are going to use for the convergence of f_n to f (solution to (2.5)-(2.7)). Hence, it will be omitted. Notice that for fixed n, $Q^n \in L_\infty([0, T], L_\infty)$; moreover g depends, by definition, upon the value of f at previous time. This, together with the boundedness of the velocities, implies that if f_n and $\partial_x f_n$ are in L_∞ at time zero, they will belong to L_∞ over the time interval $[0, T]$. As a consequence, the boundary conditions make sense on the f_n's and they are regular enough to satisfy the bounds on energy and entropy in Lemma 2.1. As a matter of fact, it is easily seen that Q^n satisfies the same sign properties as Q, when proving the estimate on the entropy, which for this reason does not depend on n. This will turn out to be very useful in a while. In what follows, subscript n will be used for all quantities regarding equation (3.8).

Let T^* be fixed such that $VT^* < 1$ and set $F_{n,T^*} = \sup_{0 \le t \le T^*} f_n(x, v, t)$. For the sake of simplicity, being the whole paragraph (a) devoted to the short time result for $T = T^*$, we will skip the subscript T^*. It is possible to show that $\|F_n\|$ is bounded uniformly in n. Indeed, recalling (2.6) and (2.30) it is:

$$
\begin{aligned}
\|G_n\|_+ &:= \int dx \int_{v>0} dv \sup_{0 \le t \le T^*} |g_n^\sharp|(x, v, t) \le \\
&\le \|f_0\|_+ + \int dx \int_{v>0} dv |f_n| \Big(0, v, \frac{x}{v}\Big) = \\
&= \|f_0\|_+ + \int_0^{T^*} dt \int_{v>0} dvv |f_n|(0, v, t) = \\
&= \|f_0\|_+ + \int_0^{T^*} dt \int_{v>0} dv\, v M_0^i(v) \int_{w<0} dw |w f_n|(0, w, t) \le \\
&\le \|f_0\|_+ + \int_0^{T^*} dt \int_{w<0} dw |w f_n|(0, w, t) = \\
&= \|f_0\|_+ + \int_0^{T^*} dt \int_{w<0} dw |w| |f_n^\sharp|(-wt, w, t)|
\end{aligned}
\tag{3.11}
$$

Notice that, since $VT^* < 1$, $f_n(0, w, t)$ (and hence $f_n^\sharp(-wt, w, t)$) can be expressed again in terms of equation (3.8), without any contribution coming from the boundary, so that by (3.6) it follows:

$$\begin{aligned}\|G_n\|_+ &\le \|f_0\|_+ + \|f_0\|_- + \\ &+ \int_0^{T^*} dt \int_{w<0} dw|w| \int_0^t ds |Q(f_n,f_n)^\sharp|(-wt,w,s) \le \\ &\le \|f_0\| + 2C_0\|F_n\|^2\end{aligned} \tag{3.12}$$

Doing the same estimate for $\|G_n\|_-$ and going back to (3.8), it turns out:

$$\|F_n\| \le 2\|f_0\| + 2C_0\|F_n\|^2 + 4C_0\|F_n\|^2 \,. \tag{3.13}$$

Then, by the assumption $\|f_0\| < \frac{1}{64C_0}$, it is:

$$\|F_n\| \le 4\|f_0\| \le \frac{1}{16C_0} \,. \tag{3.14}$$

The uniform in n estimate (3.14), allows to prove that $\{f_n(t)\}$ is a Cauchy sequence in $L_\infty([0,T^*];L_1)$, converging to the searched solution to the problem (3.1). Indeed, setting $\delta_n(t) = (f_n - f_{n-1})(t)$ and $\Delta_n = \sup_{0\le t\le T^*} |\delta_n^\sharp|$, by (3.8) one has:

$$\begin{aligned}\|\Delta_n\|_+ &\le \int_{x<0} dx \int_{v>0} dv |\delta_n(0,v,-\frac{x}{v})| + \\ &+ \int dx \int_{v>0} dv \int_0^{T^*} ds |Q^n(f_n,f_n)^\sharp - Q^{n-1}(f_{n-1},f_{n-1})^\sharp|(x,v,s) \,.\end{aligned} \tag{3.15}$$

Now it is

$$\begin{aligned}&\int_{x<0} dx \int_{v>0} dv |\delta_n(0,v,-\frac{x}{v})| = \int_0^{T^*} ds \int_{v>0} dv v |\delta_n(0,v,s)| \le \\ &\le \int_0^{T^*} ds \int_{v>0} dv\, v M_0^i(v) \int_{w<0} dw |w\delta_n(0,w,s)| \le \\ &\le \int_0^{T^*} ds \int_{w<0} dw |w\delta_n(0,w,s)| = \int_0^{T^*} ds \int_{w<0} dw |w\delta_n^\sharp(-ws,w,s)| \le \\ &\le \int_0^{VT^*} dx \int_{w<0} dw |\Delta_n(x,w)| \le \int_0^{VT^*} dx \int_{w<0} dw \times \\ &\times \int_0^{T^*} ds |Q^n(f_n,f_n)^\sharp - Q^{n-1}(f_{n-1},f_{n-1})^\sharp|(x,w,s)\end{aligned} \tag{3.16}$$

(since $VT^* < 1$).

Then by (3.15) and (3.16)

$$\|\Delta_n\| \leq 2\int dx \int dv \int_0^{T^*} ds |Q^n(f_n, f_n)^\sharp - Q^{n-1}(f_{n-1}, f_{n-1})^\sharp|(x, v, s) \tag{3.17}$$

Since $Q^n = (Q^n - Q) + Q$, by the Lipschitz property (3.7) and the bound (3.14) it follows:

$$\begin{aligned} \|\Delta_n\| &\leq 2\int dx \int dv \int_0^{T^*} ds |Q(f_n, f_n)^\sharp - Q(f_{n-1}, f_{n-1})^\sharp|(x, v, s) + \\ &+ \int dx \int dv \int_0^{T^*} ds \quad r_n(x, v, s) \leq \\ &\leq \frac{1}{4}\|\Delta_n\| + \int dx \int dv \int_0^{T^*} ds \quad r_n(x, v, s) \end{aligned} \tag{3.18}$$

where

$$r_n(x, v, s) = \sum_{m=n-1}^{n} |Q^m(f_m, f_m)^\sharp - Q(f_m, f_m)^\sharp|(x, v, s) \,.$$

As it has already been proven in [1], the term r_n satisfies:

$$\lim_{n\to\infty} \int dx \int dv \int_0^{T^*} ds \quad r_n(x, v, s) = 0 \,. \tag{3.19}$$

This follows from the definition of Q^m, the uniform bound on $\|F_m\|$ given in (3.14) and the estimate on the entropy given in Lemma 2.1. This achieves the proof of Step (a). Uniqueness and positivity of the solution are standard.

Proof of (b). The proof follows from an argument which has become classical in this context. The main observation is that, due to the boundedness of the velocities and the smallness of the time T^*, the solution $f(x, v, t)$ constructed in the previous step, depends only on the restriction of f_0 in an interval (depending on x) of lenght at most $2VT^*$. Indeed, even if the flux from the boundary influences $f(x, v, t)$, such a flux comes uniquely from the mass in this interval. This fact, together with the finiteness of the entropy, allows to prove estimate (3.14), which is the crucial one in (a), for any initial condition. Given any L_1-function f_0, choose an interval $I \subset [0, 1]$ and put $|I|$ to indicate its amplitude.

After the above observation, $f_i(x,v,t)$ for $x \in I$ and $t \in [0,T^*]$ depends uniquely on $f_{i,0}(x,v)$ for $x \in I_0$ with $|I_0| \leq |I| + 2VT^*$. The interval I has to be chosen in such a way that

$$\int_{I_0} f_{i,0}(x,v)dx\,dv \leq \frac{1}{48C_0}\,,$$

which is possible, since $H(0)$ is finite. Then (3.14) holds for $\|F_{T^*}\|_I$, that is the restriction of the L_1-norm of F_{T^*} on I. The complete estimate (3.14) can be recovered, with a larger constant than 4. Finally, by Lemma 2.1, the entropy can be bounded at later times in terms of initial quantities, so that (3.14) can be iterated in time over $[0,T]$. This implies the result stated in the theorem. ∎

Before going to the next section, it is useful to give a bound on the time integral of the fluxes at the walls, relative to the bounded velocity case. Taking into consideration the time-varying region $[0,V(T-t)]$ with $VT = 1$, it is almost obvious that the mass flowing, in the time interval $[0,T]$, across 0 and $V(T-t)$, is at most the initial mass in $[0,VT] = [0,1]$. Indeed, this is due to the mass conservation property for particle travelling with negative or positive velocity respectively (see (2.1)). More precisely (using the absolute continuity of $f^\sharp$ with respect to t) it is:

$$\begin{aligned}\frac{d}{dt}\int_0^{V(T-t)} dx \int_{v<0} dv f(x,v,t) &= \frac{d}{dt}\int_{v<0} dv \int_{-vt}^{V(T-t)-vt} dx f^\sharp(x,v,t) = \\ &= \int_{v<0} dv[(-v-V)f^\sharp(V(T-t)-vt,v,t) + vf^\sharp(-vt,v,t)]\,.\end{aligned} \tag{3.20}$$

Therefore:

$$\begin{aligned}\int_0^{VT} dx \int_{v<0} dv f(x,v,0) &= -\int_0^T dt \int_{v<0} dv\, v f(0,v,t) + \\ &+ \int_0^T dt \int_{v<0} dv (v+V) f(V(T-t),v,t)\end{aligned} \tag{3.21}$$

The last term is positive so that:

$$\int_0^T dt J^-(0,t) \leq \|f_0\|_-\,. \tag{3.22}$$

The analogous estimate holds for $J^+(1,t)$.

4 – The large velocity problem

The mild form of problem (2.20-23) is:

$$f_e^\sharp(x,v,t) = l^\sharp(x,v,t) + \int_{A(x,v)}^t [2Q(f_i,f_e)^\sharp + Q(f_e,f_e)^\sharp](x,v,s)ds \tag{4.1}$$

where f_i is the solution we have constructed in the previous section and:

$$\text{(4.2-1)}\quad l(x,v,t) = f_{0,e}(x-vt,v)\chi\Big(t<\frac{x}{v}\Big) + f_e\Big(0,v,t-\frac{x}{v}\Big)\chi\Big(t>\frac{x}{v}\Big)$$

$$\text{(4.2-2)}\quad l(x,v,t) = f_{0,e}(x-vt,v)\chi\Big(t<\frac{1-x}{|v|}\Big) + f_e\Big(1,v,t-\frac{1-x}{|v|}\Big)\chi\Big(t>\frac{1-x}{|v|}\Big)$$

for $v>0$ and $v<0$ respectively. f_e on the boundaries with outgoing velocities satisfies the boundary conditions (2.22)and (2.23). Indeed, as before, it will be shown that f_e is absolutely continuous in t for a.a. (x,v), so that the boundary conditions make sense.

In order to establish an a priori estimate for f_e over the time interval $[0,T]$ with T arbitrarily fixed, it is convenient to obtain first this estimate for a short time interval, say $[0,T^*]$. To this extent, call $\beta = \min(\beta_o,\beta_1)$ and choose V, depending on the initial energy and mass, so large to satisfy:

$$\|f_{e,0}\|_{L_1} + e^{-\beta V^2}\|f_0\|_{L_1} \le \frac{1}{512C_0} \tag{4.3}$$

and moreover

$$\frac{1+e^{-\beta V^2}}{1-e^{-\beta V^2}} \le 2 \tag{4.3-1}$$

(this last is an inessential technical assumption which will be used in (4.16)). After the proof of (b) in the preceeding section, estimate (3.14) holds all over the (arbitrarily) fixed time interval $[0,T]$, so that we can choose a positive number D satisfying:

$$D^{\alpha-1}\|F_{i,T}\| \le \frac{1}{256C_0} \tag{4.4}$$

(α is the one introduced in (1.5)) and, consequently, T^* so small to satisfy $VT^* \leq 1$ together with the condition

$$\sup_x \int_x^{x+DT^*} dy \int dv F_{i,T^*}(y,v) \leq \frac{1}{256 C_0} \tag{4.5}$$

It is worthwhile to spend some words on the Assumptions (4.3)-(4.5). V is fixed in (4.3) and its choice depends upon $E(0)$ and $M(0)$. D depends on $V, T, M(0)$ and $H(0)$. Finally, Assumption (4.5) is verified, provided that T^* is small enough, as a function of initial quantities. Indeed, by (3.14), it is sufficient to recall the comments in the proof of Step (b) in Section 3, that clarify the equivalence between looking at F_{i,T^*} on a small set and choosing a small initial condition.

As a conclusion, V, D and T^* depend uniquely upon T and the initial mass, entropy and energy.

The collision part appearing in the right hand side of (4.1) can be controlled, similarly to what has been done in [1]. Also in this proof we will skip the subscript T^* as far as we are concerned with the short time result. As to the linear term, by (2.27) it follows:

$$\begin{aligned}
&\int dx \int dv \int_0^{T^*} ds Q(f_i, f_e)(x,v,s) \leq \\
&\leq \int dx \int dv \int_0^{T^*} ds \int dv_1 S(|v-v_1|)\chi(vv_1<0)+ \\
&\quad + [F_i(x+2vs,-v)F_e(x+(v+v_1)s,-v_1)+ \\
&\quad + F_e(x+2vs,-v)F_i(x+(v+v_1)s,-v_1)+ \\
&\quad + F_i(x,v)F_e(x+(v-v_1)s,v_1) + F_e(x,v)F_i(x+(v-v_1)s,v_1)] = \\
&= 4\int dx \int dv \int_0^{T^*} ds \int dv_1 \times \\
&\quad \times S(|v-v_1|)\chi(vv_1<0)F_i(x+(v-v_1)s,v_1)F_e(x,v) \leq \\
&\leq 4\int dx \int dv \int_0^{T^*} ds \int dv_1 S(|v-v_1|)\chi(|v-v_1|<D)\times \\
&\quad \times F_i(x+(v-v_1)s,v_1)F_e(x,v)+ \\
&\quad + 4\int dx \int dv \int dy \int dv_1 \frac{S(|v-v_1|)}{|v-v_1|}\chi(|v-v_1|\geq D)F_i(y,v_1)F_e(x,v)
\end{aligned} \tag{4.6}$$

being F_e the same as F in (3.3) with f_e in place of f. The second integral, by (1.5) and (4.4), can be bounded by:

$$4C_0 D^{\alpha-1}\|F_i\|\|F_e\| \leq \frac{1}{64}\|F_e\| \tag{4.7}$$

while the first one is controlled by:

$$4C_0 \int dx \int dv \int_x^{x+DT^*} dy \int dv_1 F_i(y,v_1)F_e(x,v) \leq \frac{1}{64}\|F_e\| \tag{4.8}$$

by (4.5). Thus:

$$\int dx \int dv \int_0^{T^*} ds |Q(f_i,f_e)^\sharp|(x,v,s) \leq \frac{1}{32}\|F_e\|\,. \tag{4.9}$$

Recalling (3.6), by (4.1) it follows:

$$\|F_e\| \leq \|L\| + \frac{1}{16}\|F_e\| + 2C_0\|F_e\|^2\,. \tag{4.10}$$

Analogously to (3.11), to bound L one has to control the boundary term. It is:

$$\begin{aligned}
&\int_{x<0} dx \int_{v>0} dv f_e(0,v,-\frac{x}{v}) = \int_0^{T^*} ds J_e^+(0,s) = \\
&= \int_0^{T^*} ds \int_{v>0} dvv[M_0^e(v)J^-(0,s) + M_0^i(v)J_e^-(0,s)] \leq \\
&\leq e^{-\beta_0 V^2}\int_0^{T^*} ds J_i^-(0,s) + 2\int_0^{T^*} ds J_e^-(0,s) \leq \\
&\leq e^{-\beta_0 V^2}\|f_0\|_{L_1} + 2\int_0^{T^*} ds J_e^-(0,s)
\end{aligned} \tag{4.11}$$

The last step is consequence of the smallness of T^* and (3.22). On the other side, applying again (4.1)-(4.2) for $v<0$:

$$\begin{aligned}
&f_e(0,v,s) = \\
&= f_e(-vs,v,0)\chi(-vs<1) + f_e\Big(1,v,s+\frac{1}{v}\Big)\chi(-vs<1) + \\
&+ \int_{A(0,v)}^s [2Q(f_i,f_e)^\sharp + Q(f_e,f_e)^\sharp](-vs,v,r)dr\,.
\end{aligned} \tag{4.12}$$

Thus, recalling (2.22), (2.23), (4.9) and (3.6) it follows:

$$
\begin{aligned}
\int_0^{T^*} ds J_e^-(0,s) \le \|f_{e,0}\|_{L_1} + \frac{1}{32}\|F_e\| + 2C_0\|F_e\|^2 + \\
+ \int_0^{T^*} ds \int_{v<0} dv|v|\Big[M_1^e(v)J^+\Big(1, s+\frac{1}{v}\Big) + \\
+ M_1^i(v)J_e^+\Big(1, s+\frac{1}{v}\Big)\Big]\chi(-vs>1)\,.
\end{aligned}
\tag{4.13}
$$

Since $T^*V \le 1$, then $\chi(-vs>1)\chi(|v|<V) = 0$ for $s \in [0, T^*]$.

Therefore $M_1^i(v)\chi(-vs>1) = 0$ and for the remainder part in the integral in (4.13) one has, again by (3.22):

$$
\begin{aligned}
\int_0^{T^*} ds \int_{v<0} dv|v|M_1^e(v)[J_i^+(1,s) + J_e^+(1,s)]\chi(-vs>1) \le \\
\le e^{-\beta_1 V^2}[\|f_0\|_{L_1} + \int_0^{T^*} ds J_e^+(1,s)]
\end{aligned}
\tag{4.14}
$$

In conclusion:

$$
\begin{aligned}
\int_0^{T^*} ds J_e^-(0,s) \le \|f_{e,0}\|_{L_1} + \frac{1}{32}\|F_e\| + 2C_0\|F_e\|^2 + \\
+ e^{-\beta_1 V^2}[\|f_0\|_{L_1} + \int_0^{T^*} ds J_e^+(1,s)]
\end{aligned}
\tag{4.15}
$$

An analogous estimate can be obviously done for $\int_0^{T^*} ds J_e^+(1,s)$ in terms of $\int_0^{T^*} ds J_e^-(0,s)$, so that, by (4.15),(4.3-1) and (4.10) it is:

$$
\|F_e\| \le 8[\|f_{e,0}\|_{L_1} + e^{-\beta V^2}\|f_0\|_{L_1} + \frac{1}{32}\|F_e\| + 2C_0\|F_e\|^2]
\tag{4.16}
$$

from which, taking into account (4.3), it follows:

$$
\|F_e\| \le \frac{1}{32C_0}
\tag{4.17}
$$

To go to T larger than T^*, one should verify that condition (4.3) still holds at time T^*, that is:

$$
\sup_{0\le t\le T^*} \int dx \int dv|f(x,v,t)|\chi(|v|\ge V) + e^{-\beta V^2}\|f_0\|_{L_1} \le \frac{1}{512C_0}\,.
\tag{4.18}
$$

This is not immediately true, since f could be not regular enough to satisfy the energy estimate in Lemma 2.1, which would guarantee (4.18). What is standard to do in this case is to introduce, as in Section 3, a cutoffed version of equation (4.1) which admits a sufficiently regular solution, satisfying the energy bound. Then, it remains to show that this solution converges to that of the true problem, while removing the cutoff. To this purpose it would be convenient to have a L_∞ right hand side but, since $S(|v-v_1|)$ is unbounded, the regularization chosen in Section 3 is not appropriate. A possible cutoffed system is the one obtained by replacing in (4.1) Q with $\bar{Q}^n$, being this last the same as Q^n introduced in Section 3, but for the collision kernel $\bar{S}$ defined as:

$$\bar{S}(|v-v_1|) = S(|v-v_1|)\chi(|v-v_1| \leq R) \tag{4.19}$$

for some positive R. The fact that the solution to the regularized problem converges in L_1 to the solution to (4.1) can be easily seen. Indeed, taking into account the uniform estimates (3.14) and (4.17) on f_i and f_e, the Lipschitz property of $\int dsQ$ shown in (3.7) and (4.9) and recalling (3.19), the convergence is proven once the following is proven:

$$\begin{aligned} \int dx \int dv \int ds \int dv_1 S(|v-v_1|)\chi(|v-v_1|) \geq R)\times \\ \times f(x,v,s) f(x+(v-v_1)s, v_1, s) \leq \varphi(R) \end{aligned} \tag{4.20}$$

with φ infinitesimal as R goes to infinity. This is actually true, due to the property (1.5) of S. Thus (4.17) can be prolonged up to the arbitrarily fixed time T and this, by the above discussion, implies the existence of a unique solution to problem 4.1. Moreover, as it was to be expected, f_e stays small all over the interval $[0,T]$. This concludes the proof of Theorem 2.1. The positivity of the solution and the boundedness of energy and entropy can be proven by standard arguments.

Final Remark. I want to thank one of the referees for pointing out that the entropy and energy estimates in Lemma 2.1 can be obtained as well by the arguments in [2] (see Lemmas 4.1 and 4.2 in this paper). This approach works directly for mild solutions, making unnecessary the final cutoff argument.

REFERENCES

[1] L. Arkeryd: *Existence theorems for certain kinetic equations and large data*, Arch. for Rat. Mech. and Anal., **103** (1988).

[2] L. Arkeryd – N. Maslova: *On diffuse reflection at the boundary for the Boltzmann equation and related equations*, Jour. of Stat. Phys., **77** (1994).

[3] C. Cercignani: *Recent results in the kinetic theory of gases*, Rend. Sem. Mat. Univ. e Politecnico di Torino, 1988.

[4] C. Cercignani: *Global existence for a model Boltzmann equation*, Jour. of Stat. Phys., **49** (1987).

[5] C. Cercignani: *Weak solutions of the Boltzmann equation and energy conservation and Errata*, Appl. Math. Lett., **8** (1995).

[6] C. Cercignani – R. Illner: *Global weak solutions of the Boltzmann equation in a slab with diffusive boundary conditions*, Arch. for Rat. Mech. and Anal., (to appear).

[7] N. Ianiro – J. L. Lebowitz: *Stationary nonequilibrium solutions of model Boltzmann equation*, Found. of Phys., **15** (1985).

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