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Elliptic operators of divergence type with Hölder coefficients in fractional Sobolev spaces

S. CLAIN

RIASSUNTO: *Il lavoro tratta gli operatori ellittici del tipo della divergenza con coefficienti hölderiani. Si riconosce che, se l'operatore di Laplace definisce un isomorfismo bicontinuo da* $W_p^{1+s}(\Omega)$ *in* $W_p^{s-1}(\Omega)$, $s \in [0,1[$, *allora ciò vale anche per tutta una classe di operatori ellittici con coefficienti hölderiani. Si dà anche una applicazione ad un problema non lineare con ipotesi di regolarità deboli.*

ABSTRACT: *This work concerns linear elliptic operators of divergence type with Hölder coefficients in spaces of type* $W_p^{1+s}(\Omega)$, $s \in [0,1[$ *on a Lipschitz domain* Ω. *We prove that if the Laplace operator* Δ *is a bicontinuous isomorphism from* $W_p^{1+s}(\Omega)$ *onto* $W_p^{s-1}(\Omega)$ *then the result holds for more general elliptic operators with Hölder coefficients. An application to a non linear problem with low regularity in the right-hand side is given.*

– Introduction

This paper is devoted to the elliptic operator study in fractional Sobolev spaces in Lipschitz domain Ω. The main motivation to introduce fractional value for the derivatives comes from the fact that some

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classes of problems cannot have a solution in spaces of type $W_p^2(\Omega)$. Consequently, from a numerical point of view, we cannot give an estimate of the convergence order for approximate solutions (the approximation with finite elements for example) if the solution just lies in a space of type $W_p^1(\Omega)$. The idea consists to see whether the solution belongs to a fractional Sobolev space of type $W_p^{1+s}(\Omega)$. The "fraction" of derivative s will be usefull for numerical estimates in inducing a fractional order of convergence [2].

More precisely, let a be a Hölder function defined on a bounded open set $\Omega \subset \mathbb{R}^N$ of Lipschitz boundary, $N \geq 2$ and let us consider the following elliptic operator $-\nabla(a\nabla u)$, the goal of the paper consists to show that if the Laplace operator $-\Delta$ is a bicontinuous isomorphism from $W_p^{1+s}(\Omega) \cap \overset{\circ}{W}{}_p^1(\Omega)$ onto $W_p^{s-1}(\Omega)$ for a Lipschitz bounded open set Ω then the result holds for general operators when a is regular enough in Hölder spaces. We avoid then the boundary problem assuming that the result holds for the Laplace operator. For an extended study of this point, one can refer to [6].

We use a local pertubation method to express any elliptic operator of divergence form as the Laplace operator on small compacts (rather similar to the method of Korn and Schauder). In the first section, we introduce space definitions and assumptions about the elliptic operators we consider within the paper. The second section is devoted to technical lemmas subsequently used as pertubation arguments. Section three gives theorems about the existence and the uniqueness of solution in $\overset{\circ}{W}{}_p^1(\Omega)$ for the continuous case. We present then a theorem of regularity in $W_p^{s+1}(\Omega) \cap \overset{\circ}{W}{}_p^1(\Omega)$ spaces in the fourth section for the Hölder case. Finally, the last section shows an example where fractional Sobolev spaces give new regularity results.

1 – Spaces and assumptions

In all the sequel, Ω denotes an open bounded set with a Lipschitz boundary. For $p \in]1, \infty[$ and $k = 1, 2$, we denote by $W_p^k(\Omega)$ the classical Sobolev spaces given in [5], p. 16. Let $s \in]0, 1[$, the definition of fractional Sobolev spaces $W_p^s(\Omega)$ and $W_p^{1+s}(\Omega)$ is given in [5], p. 17. For $k = 1, 2$, the spaces $\overset{\circ}{W}{}_p^k(\Omega)$ ($\overset{\circ}{W}{}_p^s(\Omega)$ and $\overset{\circ}{W}{}_p^{1+s}(\Omega)$ respectively) are defined by the closure in $W_p^k(\Omega)$ of the infinitely differential compactly supported functions set (in $W_p^s(\Omega)$ and $W_p^{1+s}(\Omega)$ respectively). The space $W_{p'}^{s-1}(\Omega)$ is the $\overset{\circ}{W}{}_p^{1-s}(\Omega)$ dual space where p' denotes the conjugate number of p. An other definition can be given to caracterise spaces $W_p^{s-1}(\Omega)$ using the truncature on Ω in the distribution sense of all the elements of $W_p^{s-1}(\mathbb{R}^N)$ [5], p. 18:

$$\widehat{W}_p^{s-1}(\Omega) \overset{\text{def}}{=} \{v_{|\Omega};\ v \in W_p^{s-1}(\mathbb{R}^N)\}\,.$$

In the case of Lipschitz regularity for domain Ω, one can prove that most of the time the two definitions are equivalent. Indeed, we have:

$$\widehat{W}_p^{s-1}(\Omega) = W_p^{s-1}(\Omega), \qquad \forall\, s \in]0,1[,\ s \neq \frac{1}{p}\,.$$

It is important to note that spaces $W_p^{\frac{1}{p}-1}(\Omega)$ and $\widehat{W}_p^{\frac{1}{p}-1}(\Omega)$ do not correspond (even if the domain is smooth) [5], p. 31 (we have just $\widehat{W}_p^{\frac{1}{p}-1}(\Omega) \subset W_p^{\frac{1}{p}-1}(\Omega)$ [5], p. 18).

Remark 1.1. We introduce spaces $\widehat{W}_p^{s-1}(\Omega)$ because they are natural spaces for the interpolation theory [9]. In particular we always have with $s \in]0,1[$:

$$(W_p^{-1}(\Omega), L^p(\Omega))_{s,p} = \widehat{W}_p^{s-1}(\Omega) \tag{1.1}$$

even if Ω is not regular. Since we assume the boundary to be Lipschitz, we have with $s \neq \frac{1}{p}$:

$$(W_p^{-1}(\Omega), L^p(\Omega))_{s,p} = W_p^{s-1}(\Omega)\,, \tag{1.2}$$

but the exceptional case $s = \frac{1}{p}$ cannot follow relation (1.2) even if the boundary is smooth. Therefore, we are obliged to introduce spaces $\widehat{W}_p^{s-1}(\Omega)$ to take the case $s = \frac{1}{p}$ into account.

At last, for $\sigma \in [0,1]$, $C^{0,\sigma}(\overline{\Omega})$ denotes the Hölder space endowed with the norm $\|.\|_{C^{0,\sigma}(\overline{\Omega})} = \|.\|_{C^0(\overline{\Omega})} + [.]_\sigma$ where

$$[v]_\sigma \overset{\text{def}}{=} \sup_{x,y\in\overline{\Omega},x\neq y} \frac{|v(x)-v(y)|}{|x-y|^\sigma}.$$

The value $\sigma = 0$ corresponds to the continuous functions space $C^0(\overline{\Omega})$. We present now the notations we use throughout this paper. Let $\sigma \in [0, 1[$, we set the following assumptions:

$$a \in C^{0,\sigma}(\overline{\Omega})\,, \tag{1.3}$$

$$\exists\, E > 0 \text{ such that } \forall\, x \in \overline{\Omega},\ a(x) \geq E\,. \tag{1.4}$$

For every $s \in [0, 1[$ and every $p \in]1, \infty[$, we can define the elliptic operator A from $W^{1+s}_p(\Omega)$ into $W^{s-1}_p(\Omega)$ with $s \neq \frac{1}{p}$ and from $W^{1+s}_p(\Omega)$ into $\widehat{W}^{s-1}_p(\Omega)$ with $s = \frac{1}{p}$ by (see theorem 1.4.1.1 [5], p. 21 and theorem 1.4.4.6 p. 31):

$$A(u) \stackrel{\text{def}}{=} -\nabla \cdot (a\nabla u)\,. \tag{1.5}$$

Let $p \in]1, \infty[$, we say that the operator A has the property H1(p) if and only if

(H1(p)) A is a bicontinuous isomorphism from $\overset{\circ}{W}{}^1_p(\Omega)$ onto $W^{-1}_p(\Omega)$.

Let $p \in]1, \infty[$, $s \in]0, 1[$, we say that the operator A has the property H2(p, s) if and only if A has the property H1(p) and

$$(\mathrm{H2}(p,s)) \qquad \begin{cases} A \text{ is a bicontinuous isomorphism from} \\ W^{1+s}_p(\Omega) \cap \overset{\circ}{W}{}^1_p(\Omega) \quad \text{onto} \quad W^{s-1}_p(\Omega) \quad \text{if} \quad s \neq \dfrac{1}{p}\,, \\ W^{1+s}_p(\Omega) \cap \overset{\circ}{W}{}^1_p(\Omega) \quad \text{onto} \quad \widehat{W}^{s-1}_p(\Omega) \quad \text{if} \quad s = \dfrac{1}{p}\,. \end{cases}$$

The main theorems we prove are the following ones.

Theorem 1.1 (continuous case). *Let* $\sigma = 0$, $p \in]1, \infty[$ *and the coefficient* a *satisfying Assumptions* (1.3)-(1.4). *If* $-\Delta$ *satisfies* H1(p) *then* $\forall\, q \in [\min(p, p'), \max(p, p')]$, A *satisfies* H1(q).

Theorem 1.2 (Hölder case). *Let* $\sigma \in]0, 1[$, $p \in]1, \infty[$ *and the coefficient* a *satisfying Assumptions* (1.3)-(1.4). *If* $-\Delta$ *satisfies* H2(p, s) *for a real number* $s \in]0, \sigma[$, *then* A *satisfies* H2(p, s).

Remark 1.2. As we say in the introduction, boundary regularity does not infer in the theorems. We just need the Lipschitz assumption for the boundary to give the compatibility between the norms $\widehat{W}$ and W [5], p. 25. However, regular assumptions are clearly necessary to prove properties H1(p) and H2(p, s) for the Laplace operator.

2 – Some technical lemmas

This section deals with technical lemmas we shall use in the next sections. The main goal consists to prove that locally the operator A is a pertubation of the Laplace operator in $\overset{\circ}{W}{}^1_p(\Omega)$ or in $W_p^{1+s}(\Omega)$. We build then new operators which approximate A on small domains. To this end, we introduce functions coinciding with a on small compact domains. New operators derived from these functions will help us to obtain local regularity using a pertubation argument. This method is slightly different to the other classical ones since we do not freeze the coefficient a for particular points of Ω but we replace it by a family of functions a_η which coincide with a on small compact domains.

Definition 2.1. Let P be a point of $\mathbb{R}^N$ and η a positive real number. We denote by $B_2(P, \eta)$ ($B_\infty(P, \eta)$ respectively) the closed ball of center P of radius η in the euclidean norm (in the infinity norm respectively).

Remark 2.1. Notice that

$$B_2(P, \eta) \subset B_\infty(P, \eta) \subset B_2(P, 2\eta)\,.$$

We denote by $I = (i_1, \dots, i_N) \in \mathbb{Z}^N$ the points of $\mathbb{R}^N$ with integer coordinates. The balls $B_\infty(I, 1), I \in \mathbb{Z}^N$ cover the whole domain $\mathbb{R}^N$. Let $\mathcal{P}_1$ be the covering, we build now a partition of unit of $\mathbb{R}^N$, invariant by translation of integer value vectors. We first make the construction on $\mathbb{R}$.

Lemma 2.1. *There exist functions $\alpha_i \in \mathcal{D}(\mathbb{R})$, $i \in \mathbb{Z}$ such that:*

$$\text{i) supp } \alpha_i \subset B_\infty\Big(i, \frac{2}{3}\Big)\,, \tag{2.1}$$

$$\text{ii)} \sum_{i \in \mathbb{Z}} \alpha_i = 1 \quad \textit{on} \quad \mathbb{R}\,, \tag{2.2}$$

$$\text{iii)}\, \alpha_i(x) = \alpha_0(x - i)\,. \tag{2.3}$$

PROOF. We first define the function α_0 on $[0,\infty[$. We set:

$$\begin{cases} \alpha_0 = 1 & \text{on } \Big[0, \dfrac{1}{3}\Big], \\ \alpha_0 = 0 & \text{on } \Big[\dfrac{2}{3}, \infty\Big[, \\ \alpha_0 \in C^\infty([0,\infty[)\,. \end{cases}$$

Next, we define the function α_0 on $]-\infty, 0]$ by setting

$$\begin{cases} \alpha_0 = (1-\alpha_0)(1+x) & \text{on} \quad [-1,0]\,, \\ \alpha_0 = 0 & \text{on} \quad]-\infty,-1]\,. \end{cases}$$

We define then α_i by $\alpha_i(x) = \alpha_0(x-i)$. Obviously properties i), iii) are satisfied. Furthermore, let $j \in \mathbb{Z}$ and $x \in [j, j+1]$, we have:

$$\begin{aligned} \sum_{i\in\mathbb{Z}} \alpha_i(x) &= \alpha_j(x) + \alpha_{j+1}(x) = \\ &= \alpha_0(x-j) + \alpha_0(x-1-j) = \\ &= \alpha_0(y) + \alpha_0(y-1) =, \qquad \text{with } y = x-j \\ &= \alpha_0(y) + 1 - \alpha_0(1+y-1) = 1\,. \end{aligned}$$

The lemma is proved. □

LEMMA 2.2. *There exist functions $\alpha_I \in \mathcal{D}(\mathbb{R}^N)$, $I \in \mathbb{Z}^N$ such that:*

(2.4) i) supp $\alpha_I \subset B_\infty\Big(I, \dfrac{2}{3}\Big)$,

(2.5) ii) $\displaystyle\sum_{I\in\mathbb{Z}^N} \alpha_I = 1$ *on* $\mathbb{R}^N$,

(2.6) iii) *if* $I_0 = (0,\dots,0) \in \mathbb{Z}^N$ *then* $\alpha_I(x) = \alpha_{I_0}(x-I)$.

Proof. We define functions α_I, $I = (i_1, \dots, i_N)$ by:

$$\alpha_I = \prod_{j=1}^{N} \alpha_{i_j}(x_j)\,.$$

Obviously, properties i), iii) are satisfied. From relation (2.2) we deduce relation (2.5). ∎

The functions α_I define a partition of unit associated to $\mathcal{P}_1$. Let η be a positive real number and consider $\mathcal{P}_\eta$ the covering of $\mathbb{R}^N$ derived from $\mathcal{P}_1$ by dilation η. Then, if we set

$$\alpha_I^\eta(x) \stackrel{\text{def}}{=} \alpha_I\Big(\frac{x}{\eta}\Big)\,, \tag{2.7}$$

we obtain a partition of unit of $\mathbb{R}^N$ associated to $\mathcal{P}_\eta$. Furthermore, we have the following corollary.

Corollary 2.1. *The functions $\alpha_I^\eta \in \mathcal{D}(\mathbb{R}^N)$ defined by (2.7) satisfy the following properties:*

$$\text{i) supp } \alpha_I^\eta \subset B_\infty\Big(\eta I, \frac{2\eta}{3}\Big)\,, \tag{2.8}$$

$$\text{ii) } \sum_{I \in \mathbb{Z}^N} \alpha_I^\eta = 1 \quad \textit{on} \quad \mathbb{R}^N\,, \tag{2.9}$$

$$\text{iii) } \textit{if } \ I_0 = (0, \dots, 0) \in \mathbb{Z}^N \quad \textit{then} \quad \alpha_I^\eta(x) = \alpha_{I_0}^\eta(x - I\eta)\,, \tag{2.10}$$

$$\text{iv) } \|\alpha_I^\eta\|_{C^r(\mathbb{R}^N)} = \frac{1}{\eta^r}\|\alpha_{I_0}\|_{C^r(\mathbb{R}^N)}, \quad \forall\, r \in \mathbb{N}\,. \tag{2.11}$$

Now, let us denote by Ω' the open set defined by

$$\Omega' = \{x \in \mathbb{R}^N \,|\, \mathrm{dist}(x, \Omega) < 3\}$$

where $\mathrm{dist}(x, \Omega)$ represents the euclidean distance between x and Ω. Obviously, $\Omega \subset \Omega'$. We build a partition of unit on $\overline{\Omega}$ such that the support of this partition is included in $\overline{\Omega'}$.

Proposition 2.1. *Let Ω be an open bounded set and $\eta \in]0, 1[$, there exists a finite number J_η of balls $B_\infty(I\eta, \eta)$ such that we have* dist$(I\eta, \Omega) \leq \eta$. *If* M_k, $k = 1, \dots, J_\eta$ *is a new indexing of the center of the balls satisfying the previous property, we can write:*

$$\text{i)}\, \overline{\Omega} \subset \bigcup_{k=1}^{J_\eta} B_\infty(M_k, \eta) \subset \bigcup_{k=1}^{J_\eta} B_2(M_k, 2\eta) \subset \overline{\Omega'}\,, \tag{2.12}$$

$$\text{ii)} \sum_{k=1}^{J_\eta} \alpha_k^\eta = 1 \quad \textit{on} \quad \overline{\Omega}, \qquad \operatorname{supp} \sum_{k=1}^{J_\eta} \alpha_k^\eta \subset \Omega'\,. \tag{2.13}$$

Using the partition of unit, we build new function a_k^η derived from a. Notice that if a is a function of $C^{0,\sigma}(\overline{\Omega})$ $\sigma \in [0,1]$, we can extend it to a function of $C^{0,\sigma}(\overline{\Omega'})$ – still denoted by a – thanks to Stein's theorem [8], p. 173 and there exists a constant C^* independent of a and σ such that

$$\|a\|_{C^{0,\sigma}(\overline{\Omega})} \leq \|a\|_{C^{0,\sigma}(\overline{\Omega'})} \leq C^* \|a\|_{C^{0,\sigma}(\overline{\Omega})}\,.$$

This extention does not require regularity of the boudary. Thanks to this inequality the norms are equivalent. Hence, we do not make distinction between a or its extention. We first deal with the continuous case and next with the Hölder case. To this end, we consider the following sets:

$$E_1 = \overline{B_2(M_k, 2\eta)}\,,$$

$$E_2 = \{x \in \mathrm{I\!R}^N;\ 2\eta < |x - M_k| \leq 3\eta\}\,,$$

$$E_3 = \{x \in \mathrm{I\!R}^N;\ |x - M_k| > 3\eta\}\,.$$

Remark 2.2. In fact, we should write $E_1^{k,\eta}$, $E_2^{k,\eta}$, $E_2^{k,\eta}$ but for the sake of simplicity, we omit the indexes k and η.

Remark 2.3. In the following lemmas, we adopt two different notations for the same mathematical object. For $x, y \in \mathrm{I\!R}^N$, we denote by $\mathbf{xy}$ or by $x - y$ the vector of origin x and of extremity y. The first notation is more "vectorial" and we shall use it in geometrical considerations. The expressions $|\mathbf{xy}|$ and $|y - x|$ represent the euclidean norm of the vector.

Lemma 2.3. *Let $a \in C^0(\overline{\Omega'})$ and $\eta \in]0,1[$. For every $k = 1, \dots, J_\eta$, there exist functions $a_k^\eta \in C^0(\mathbb{R}^N)$ such that:*

$$\text{i)}\; a_k^\eta = a \quad \text{on} \quad E_1\,, \tag{2.14}$$

$$\text{ii)} \sup_{x \in \mathbb{R}^N} |a_k^\eta(x) - a_k^\eta(M_k)| \le \sup_{z \in E_1} |a(z) - a(M_k)|\,. \tag{2.15}$$

Proof. We build the function a_k^η in the following way. If $x \in E_1$, we set $a_k^\eta(x) = a(x)$. If $x \in E_3$ we set $a_k^\eta(x) = a(M_k)$. If $x \in E_2$ then let y be the unique point on ∂E_1 such that the points M_k, y, x are colinear (fig. 1). We have then

$$\mathbf{M}_k\mathbf{x} = \Big(1 + \frac{\theta}{2}\Big)\mathbf{M}_k\mathbf{y} \tag{2.16}$$

with $\theta\eta = |\mathbf{xy}|$, $\theta \in [0,1]$. We set $a_k^\eta(x) = (1-\theta)a(y) + \theta a(M_k)$. The function $a_k^\eta(x)$ is obviously continuous, it remains to prove (2.15).

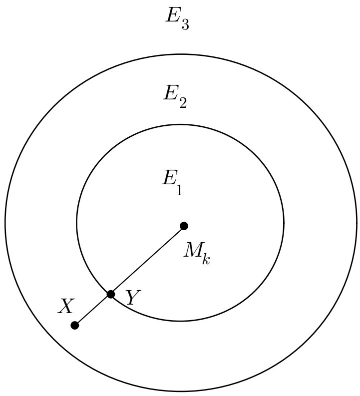

Fig. 1

Let $x \in \mathbb{R}^N$, if $x \in E_1$ we get

$$|a_k^\eta(x) - a_k^\eta(M_k)| = |a(x) - a(M_k)| \le \sup_{z \in E_1} |a(z) - a(M_k)|\,.$$

If $x \in E_3$ we have:

$$|a_k^\eta(x) - a_k^\eta(M_k)| = |a(M_k) - a(M_k)| = 0\,.$$

Now, assume that $x \in E_2$ then we have formula (2.16) for a point $y \in \partial E_1$ and we get:

$$|a_k^\eta(x) - a_k^\eta(M_k)| = (1-\theta)|a(y) - a(M_k)| \leq \sup_{z \in E_1} |a(z) - a(M_k)|\,.$$

So we have proved the lemma. $\square$

COROLLARY 2.2. *Let $a \in C^0(\overline{\Omega'})$, for every $\varepsilon \in]0,1[$, there exists $\eta(\varepsilon) > 0$ such that:*

$$\forall\, k = 1, \ldots, J_{\eta(\varepsilon)}, \qquad \sup_{x \in \mathbb{R}^N} |a_k^{\eta(\varepsilon)}(x) - a_k^{\eta(\varepsilon)}(M_k)| \leq \varepsilon\,. \tag{2.27}$$

PROOF. We use the uniform continuity of a on the compact set $\overline{\Omega'}$ and the relation (2.15). Indeed, we have for $\eta = \eta(\varepsilon)$ small enough

$$|a(x) - a(y)| \leq \varepsilon \qquad \forall\, x, y \in \Omega' \quad \text{with} \quad |x - y| \leq 2\eta\,.$$

Then in particular, we have for every k:

$$\sup_{x \in E_1} |a(x) - a(M_k)| \leq \varepsilon\,.$$

We draw the corollary from relation 2.15. $\square$

LEMMA 2.4. *Let $\sigma \in]0,1[$, $a \in C^{0,\sigma}(\overline{\Omega'})$ and $\eta \in]0,1[$. Then for every $k = 1, \ldots, J_\eta$, the functions a_k^η defined in lemma 2.3 verify the following conditions:*

$$\text{i)}\, a_k^\eta \in C^{0,\sigma}(\mathbb{R}^N)\,, \tag{2.18}$$

$$\text{ii)} \forall\, t \in [0,\sigma],\ [a_k^\eta]_{0,t} \leq 6\eta^{\sigma - t}[a]_{0,\sigma}\,. \tag{2.19}$$

Proof. Let $x_1, x_2 \in \mathbb{R}^N$, $x_1 \neq x_2$. We shall consider two cases according we have $|x_1 - x_2| \leq \eta$ or not. First, let us consider the case when $|x_1 - x_2| > \eta$. We can write:

$$\frac{|a_k^\eta(x_1) - a_k^\eta(x_2)|}{|x_1 - x_2|^t} \leq \frac{|a_k^\eta(x_1) - a_k^\eta(M_k)|}{|x_1 - x_2|^t} + \frac{|a_k^\eta(M_k) - a_k^\eta(x_2)|}{|x_1 - x_2|^t} \leq$$
$$\leq \frac{|a_k^\eta(x_1) - a(M_k)|}{\eta^t} + \frac{|a(M_k) - a_k^\eta(x_2)|}{\eta^t} .$$

We are going to prove that

$$|a_k^\eta(x_1) - a(M_k)| \leq 2\eta^\sigma [a]_{0,\sigma} . \tag{2.20}$$

If $x_1 \in E_1$, we have $a_k^\eta(x_1) = a(x_1)$ so we get

$$\begin{aligned} |a_k^\eta(x_1) - a(M_k)| &\leq |x_1 - M_k|^\sigma [a]_{0,\sigma} \leq \\ &\leq (2\eta)^\sigma [a]_{0,\sigma} \leq \\ &\leq 2\eta^\sigma [a]_{0,\sigma} . \end{aligned}$$

If $x_1 \in E_3$, we have $|a_k^\eta(x_1) - a(M_k)| = 0$. At last, if $x \in E_2$ then thanks to the relation (2.16), there exists a unique $y \in \partial E_1$ such that

$$|a_k^\eta(x_1) - a(M_k)| = (1-\theta)|a(y) - a(M_k)| .$$

Since y and M_k belong to E_1 we draw the same estimate (2.20). We have also this estimate for x_2. To sum up, in the case when $|x_1 - x_2| > \eta$ we have prove relation (2.19).

Now we deal with the case when $|x_1 - x_2| \leq \eta$. These leads us to consider five subcases:

-1) $x_1 \in E_1$ **and** $x_2 \in E_1$**.**

By definition of a_k^η we have:

$$\begin{aligned} \frac{|a_k^\eta(x_1) - a_k^\eta(x_2)|}{|x_1 - x_2|^t} &= \frac{|a(x_1) - a(x_2)|}{|x_1 - x_2|^t} \leq \\ &\leq |x_1 - x_2|^{\sigma - t} [a]_{0,\sigma} \leq \\ &\leq \eta^{\sigma - t} [a]_{0,\sigma} . \end{aligned}$$

-2) $x_1 \in E_3$ **and** $x_2 \in E_3$**.**

Obviously we have $|a_k^\eta(x_1) - a_k^\eta(x_2)| = |a(M_k) - a(M_k)| = 0$.

-3) $x_1 \in E_2$ **and** $x_2 \in E_2$**.**

It is the most difficult case. Let (y_i, θ_i) given by formula (2.16) associated to x_i, $i = 1, 2$. We can write:

$$\mathbf{x}_1\mathbf{x}_2 = \Big(1 + \frac{\theta_1}{2}\Big)\mathbf{y}_1\mathbf{y}_2 + \frac{\theta_2 - \theta_1}{2}\mathbf{M}_k\mathbf{y}_2 \,. \tag{2.21}$$

On the other hand, the scalar product formula gives:

$$|\mathbf{M}_k\mathbf{y}_1|^2 = |\mathbf{M}_k\mathbf{y}_2|^2 + |\mathbf{y}_2\mathbf{y}_1|^2 + 2\langle \mathbf{M}_k\mathbf{y}_2, \mathbf{y}_2\mathbf{y}_1\rangle \,.$$

Since $|\mathbf{M}_k\mathbf{y}_1|^2 = |\mathbf{M}_k\mathbf{y}_2|^2 = (2\eta)^2$, this yields:

$$\langle \mathbf{M}_k\mathbf{y}_2, \mathbf{y}_1\mathbf{y}_2\rangle = \frac{|\mathbf{y}_1\mathbf{y}_2|^2}{2} \,.$$

From (2.21), we deduce:

$$\begin{aligned}\langle \mathbf{x}_1\mathbf{x}_2, \mathbf{y}_1\mathbf{y}_2\rangle &= \Big(1 + \frac{\theta_1}{2}\Big)|\mathbf{y}_1\mathbf{y}_2|^2 + \frac{\theta_2 - \theta_1}{4}|\mathbf{y}_1\mathbf{y}_2|^2 = \\ &= \Big(1 + \frac{\theta_1 + \theta_2}{4}\Big)|\mathbf{y}_1\mathbf{y}_2|^2 \,.\end{aligned}$$

Using the Cauchy-Schwartz inequality, we get:

$$|\mathbf{x}_1\mathbf{x}_2| \geq \Big(1 + \frac{\theta_1 + \theta_2}{4}\Big)|\mathbf{y}_1\mathbf{y}_2| \tag{2.22}$$

Since $\theta_1, \theta_2 \in [0, 1]$, we have:

$$|\mathbf{x}_1\mathbf{x}_2| \geq |\mathbf{y}_1\mathbf{y}_2| \,. \tag{2.23}$$

Furthermore, we can write that:

$$\begin{aligned}|\theta_1 - \theta_2|\eta &= \big||\mathbf{x}_1\mathbf{y}_1| - |\mathbf{y}_2\mathbf{x}_2|\big| \leq \\ &\leq |\mathbf{x}_1\mathbf{y}_1 - \mathbf{y}_2\mathbf{x}_2| = \\ &= |\mathbf{x}_1\mathbf{x}_2 + \mathbf{y}_2\mathbf{y}_1| \leq \\ &\leq |\mathbf{x}_1\mathbf{x}_2| + |\mathbf{y}_2\mathbf{y}_1| \leq \\ &\leq 2|\mathbf{x}_1\mathbf{x}_2| \,.\end{aligned}$$

and we obtain:

$$|\theta_1 - \theta_2| \leq \frac{2}{\eta}|x_1 - x_2|\,. \tag{2.24}$$

The quantity $a_k^\eta(x_1) - a_k^\eta(x_2)$ can be written in the following way:

$$a_k^\eta(x_1) - a_k^\eta(x_2) = (1-\theta_1)(a(y_1) - a(y_2)) + (\theta_2 - \theta_1)(a(y_2) - a(M_k))\,.$$

Then, we obtain using (2.23) and (2.24):

$$\begin{aligned}\frac{|a_k^\eta(x_1) - a_k^\eta(x_2)|}{|x_1 - x_2|^t} &\leq |1-\theta_1|\frac{|a(y_1) - a(y_2)|}{|x_1 - x_2|^t} + |\theta_2 - \theta_1|\frac{|a(y_2) - a(M_k)|}{|x_1 - x_2|^t} \leq \\ &\leq \frac{|y_1 - y_2|^\sigma}{|x_1 - x_2|^t}[a]_{0,\sigma} + 2\frac{|x_1 - x_2|^{1-t}}{\eta}|y_2 - M_k|^\sigma [a]_{0,\sigma} \leq \\ &\leq |x_1 - x_2|^{\sigma - t}[a]_{0,\sigma} + 2|x_1 - x_2|^{1-t}2^\sigma \eta^{\sigma-1}[a]_{0,\sigma}\,.\end{aligned}$$

Since $|x_1 - x_2| \leq \eta$, we draw:

$$\frac{|a_k^\eta(x_1) - a_k^\eta(x_2)|}{|x_1 - x_2|^t} \leq 5\eta^{\sigma - t}[a]_{0,\sigma}\,.$$

-4) $x_1 \in E_1$ **and** $x_2 \in E_2$**.**

Let $x_0 \in \partial E_1$ such that the points x_1, x_0, x_2 are colinear, we have then:

$$|\mathbf{x}_1\mathbf{x}_2| = |\mathbf{x}_1\mathbf{x}_0| + |\mathbf{x}_0\mathbf{x}_2|\,.$$

We can then write:

$$\begin{aligned}\frac{|a_k^\eta(x_1) - a_k^\eta(x_2)|}{|x_1 - x_2|^t} &\leq \frac{|a(x_1) - a(x_0)|}{|x_1 - x_2|^t} + \frac{|a_k^\eta(x_0) - a_k^\eta(x_2)|}{|x_1 - x_2|^t} \leq \\ &\leq \frac{|a(x_1) - a(x_0)|}{|x_1 - x_0|^t} + \frac{|a_k^\eta(x_0) - a_k^\eta(x_2)|}{|x_0 - x_2|^t}\,.\end{aligned}$$

Applying estimates obtained in Subsections 1 and 3, we get immediately:

$$\frac{|a_k^\eta(x_1) - a_k^\eta(x_2)|}{|x_1 - x_2|^t} \leq 6\eta^{\sigma - t}[a]_{0,\sigma}\,.$$

-5) $x_1 \in E_2$ **and** $x_2 \in E_3$**.**

Let $x_0 \in \partial B_2(M_k, 3\eta)$ such that x_1, x_0, x_2 are colinear. Since $|x_1 - x_2| \leq |x_1 - x_0|$, we get:

$$\frac{|a_k^\eta(x_1) - a_k^\eta(x_2)|}{|x_1 - x_2|^t} = \frac{|a_k^\eta(x_1) - a_k^\eta(x_0)|}{|x_1 - x_2|^t} \leq$$

$$\leq \frac{|a_k^\eta(x_1) - a_k^\eta(x_0)|}{|x_1 - x_0|^t} .$$

Since points x_1 and x_0 are in E_2, Subcase 3 gives the estimate. □

3 – The continuous case

We consider first the continuous case for an elliptic operator of divergence form A in spaces $\overset{\circ}{W}{}_p^1(\Omega)$. This means that we assume $a \in C^0(\overline{\Omega})$. The main idea of this section consists to prove an inequality of type Agmon-Douglis-Nirenberg (also called generalized GÅRDING'S inequality [7]) for operator A assuming that the Laplace operator satisfies H1(p) for a $p \in]1, \infty[$. This inequality associated with a duality argument gives the first theorem. A similar result has been obtained by C. G. SIMADER [7] assuming the boundary of Ω to be C^1. The case of a polygonal domain has been also studied by M. DAUGE [4]. Nevertheless, the method is different, the main result states in the following way: if, for a Lipschitz open bounded domain $\Omega \subset \mathbb{R}^N$, the Laplace operator is a bicontinuous isomorphism from $\overset{\circ}{W}{}_p^1(\Omega)$ onto $W_p^{-1}(\Omega)$ with $p \in]1, \infty[$ then the result holds for an elliptic operator with continuous coefficients. In the sequel, we always consider that the space $\overset{\circ}{W}{}_p^1(\Omega)$ is endowed with its seminorm $\|\nabla v\|_{L^p(\Omega)}$ and the dual space $W_{p'}^{-1}(\Omega)$ is endowed with the dual norm [1], p. 3 associated to the seminorm of $\overset{\circ}{W}{}_p^1(\Omega)$.

PROPOSITION 3.1. *Let $p \in]1, \infty[$ such that operator A satisfies* H1(p). *Then for all $q \in [\min(p, p'), \max(p, p')]$, A satisfies* H1(q).

PROOF. If A satisfies H1(p), by duality A satisfy H1(p'). Indeed, for every $v \in \overset{\circ}{W}{}^1_p(\Omega)$, we have

$$\|\nabla v\|_{L^p(\Omega)} \leq C_p \|Av\|_{W_p^{-1}(\Omega)},$$

where C_p does not depend of v. This inequality yields that the operator A is one-to-one on $\overset{\circ}{W}{}^1_p(\Omega)$ with a closed range in $W_p^{-1}(\Omega)$ [1], p. 30. This implies the adjoint operator $A^* = A$ is surjective from $\overset{\circ}{W}{}^1_{p'}(\Omega)$ onto $W_{p'}^{-1}(\Omega)$. On the other hand, thanks to the HANN-BANACH theorem [1], p. 4, we can write for every $v \in \overset{\circ}{W}{}^1_{p'}(\Omega)$:

$$\begin{aligned}\|\nabla v\|_{L^{p'}(\Omega)} &= \sup_{f \in W_p^{-1}(\Omega)} \frac{\langle v, f\rangle}{\|f\|_{W_p^{-1}(\Omega)}} = \\ &= \sup_{w \in \overset{\circ}{W}{}^1_p(\Omega)} \frac{\langle v, Aw\rangle}{\|Aw\|_{W_p^{-1}(\Omega)}} \leq \\ &\leq C_p \sup_{w \in \overset{\circ}{W}{}^1_p(\Omega)} \frac{\langle Av, w\rangle}{\|\nabla w\|_{L^p(\Omega)}}.\end{aligned}$$

We get then the estimate:

$$\|\nabla v\|_{L^{p'}(\Omega)} \leq C_p \|Av\|_{W_{p'}^{-1}(\Omega)}. \tag{3.1}$$

Relation (3.1) yields that A is one-to-one in $\overset{\circ}{W}{}^1_{p'}(\Omega)$ and A^{-1} is continuous from $W_{p'}^{-1}(\Omega)$ onto $\overset{\circ}{W}{}^1_{p'}(\Omega)$, the continuity being implied by relation (3.1). Operator A satisfies H1(p) and H1(p') hence, by interpolation [9], p. 185, we deduce the result for all $q \in [\min(p, p'), \max(p, p')]$. $\square$

REMARK 3.2. A particular case of proposition 3.1 is $A = -\Delta$, the constant C_p has then the value:

$$C_p = \| - \Delta^{-1}\|_{\mathcal{L}(W_p^{-1}(\Omega), \overset{\circ}{W}{}^1_p(\Omega))}. \tag{3.2}$$

PROPOSITION 3.2. *Let $a \in C^0(\overline{\Omega})$ satisfying Assumption* (1.4) *then the operator A is a bicontinuous isomorphism from $\overset{\circ}{W}{}^1_2(\Omega)$ onto $W_2^{-1}(\Omega)$*

with the estimate

$$\|\nabla v\|_{L^2(\Omega)} \leq \frac{1}{E}\|Av\|_{W_2^{-1}(\Omega)} \tag{3.3}$$

PROOF. An elementary proof can be found in [1]. □

PROPOSITION 3.3. *Let $p \in]1,\infty[$ such that $-\Delta$ satisfies* H1(p). *Then there exists $\eta_1 \in]0,1[$ such that for $k = 1,\dots, J_{\eta_1}$, the operator $A_k^{\eta_1}$ defined by*

$$A_k^{\eta_1} v \stackrel{\text{def}}{=} -\nabla \cdot (a_k^{\eta_1} \nabla v) \tag{3.4}$$

is a bicontinuous isomorphism from $\overset{\circ}{W}{}_p^1(\Omega)$ onto $W_p^{-1}(\Omega)$. Furthermore, we get the estimate

$$\|\nabla v\|_{L^p(\Omega)} \leq \frac{2}{E} C_p \|A_k^{\eta_1} v\|_{W_p^{-1}(\Omega)} \tag{3.5}$$

PROOF. Let $v \in \overset{\circ}{W}{}_p^1(\Omega)$ and $\eta \in]0,1[$, we can write:

$$A_k^\eta v \stackrel{\text{def}}{=} -\nabla.(a_k^\eta \nabla v) = -a(M_k)\Delta v + \nabla \cdot ([a(M_k) - a_k^\eta]\nabla v)\,.$$

If G_p denotes the inverse operator of $-\Delta$ in $W_p^{-1}(\Omega)$, we have:

$$\frac{1}{a(M_k)} G_p A_k^\eta(v) = \Big\{ v + G_p \circ \nabla \cdot \Big(\frac{a(M_k) - a_k^\eta}{a(M_k)} \nabla v \Big) \Big\}\,.$$

If we prove for a small enough real number η_1 that the norm of the operator

$$v \mapsto G_p \circ \nabla \cdot \Big(\frac{a(M_k) - a_k^{\eta_1}}{a(M_k)} \nabla v \Big)$$

is small (less than $1/2$ for example) then we have proved that $G_p A_k^{\eta_1}$ is an isomorphism on $\overset{\circ}{W}{}_p^1(\Omega)$, hence $A_k^{\eta_1}$ is an isomorphism from $\overset{\circ}{W}{}_p^1(\Omega)$ onto $W_p^{-1}(\Omega)$.

To this end, let ε a positive real number, Corollary 2.2 says that we can find $\eta \in]0,1[$ such that for every $k = 1, \dots, J_\eta$ we have then:

$$\|\nabla \cdot \Big(\frac{a(M_k) - a_k^\eta}{a(M_k)} \nabla v\Big)\|_{W_p^{-1}(\Omega)} \leq \frac{\varepsilon}{E} \|\nabla v\|_{L^p(\Omega)} .$$

We choose now $\varepsilon = \varepsilon_1(p)$ such that

$$\varepsilon_1 = \frac{E}{2C_p} . \tag{3.6}$$

This gives the value of $\eta_1(p)$ thanks to the uniform continuity of a. Therefore, we get the estimate [10], p. 151:

$$\| \left[G_p A_k^{\eta_1} / a(M_k)\right]^{-1} \|_{\mathcal{L}(\overset{\circ}{W}{}_p^1(\Omega), \overset{\circ}{W}{}_p^1(\Omega))} \leq 2$$

thus we have

$$\begin{aligned}
\|\nabla v\|_{L^p(\Omega)} &= \| \left[G_p A_k^{\eta_1} / a(M_k)\right]^{-1} G_p A_k^{\eta_1} v / a(M_k)\|_{\overset{\circ}{W}{}_p^1(\Omega)} \leq \\
&\leq \frac{2}{E} \|G_p A_k^{\eta_1} v\|_{\overset{\circ}{W}{}_p^1(\Omega)} \leq \\
&\leq \frac{2C_p}{E} \|A_k^{\eta_1} v\|_{W_p^{-1}(\Omega)}
\end{aligned}$$

and the proposition is proved. □

Remark 3.2. In all the section, we can just consider the case where $p \geq 2$. Indeed, proposition 3.1 shows that we can always obtain this case by duality.

Remark 3.3. The value of η_1 depends on p.

Proposition 3.4. *Let $p \in [2, 2^*[$ with $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{N}$ and let us assume that $-\Delta$ satisfies* H1(p)*. For every $f \in W_p^{-1}(\Omega)$, if v is the unique solution in $\overset{\circ}{W}{}_2^1(\Omega)$ of $Av = f$ then $\alpha_k^{\eta_1} v \in \overset{\circ}{W}{}_p^1(\Omega)$ for every $k = 1, \dots, J_{\eta_1}$. Furthermore, we have the estimate*

$$\|\nabla(\alpha_k^{\eta_1} v)\|_{L^p(\Omega)} \leq C(\eta_1, p, \Omega, a) \|Av\|_{W_p^{-1}(\Omega)} . \tag{3.7}$$

PROOF. We can write $A_k^{\eta_1}(\alpha_k^{\eta_1} v) = A(\alpha_k^{\eta_1} v)$ because $a_k^{\eta_1} = a$ on the ball $B_\infty(M_k, \eta_1)$. Proposition 3.3 says that the operator $A_k^{\eta_1}$ is a bicontinuous isomorphism from $\overset{\circ}{W}{}_p^1(\Omega)$ into $W_p^1(\Omega)$. It remains to prove that $A(\alpha_k^{\eta_1} v) \in W_p^{-1}(\Omega)$.

Using the fact that $Av = f$, we obtain:

$$\begin{aligned} A(\alpha_k^{\eta_1} v) &= \alpha_k^{\eta_1} Av + av\Delta\alpha_k^{\eta_1} + a\nabla\alpha_k^{\eta_1} \cdot \nabla v + \nabla\alpha_k^{\eta_1} \cdot \nabla(av) = \\ &= \alpha_k^{\eta_1} f + av\Delta\alpha_k^{\eta_1} + a\nabla\alpha_k^{\eta_1} \cdot \nabla v + \nabla\alpha_k^{\eta_1} \cdot \nabla(av)\,. \end{aligned}$$

Since $W_p^{-1}(\Omega) \subset W_2^{-1}(\Omega)$ proposition 3.2 says that $v \in \overset{\circ}{W}{}_2^1(\Omega)$ with the estimate (3.3). Consequently, the functions av, $a\nabla v$ and $\nabla(av)$ belong to $W_p^{-1}(\Omega)$ since $p < 2^*$ and $A(\alpha_k^{\eta_1} v) \in W_p^{-1}(\Omega)$. On the other hand, Corollary 2.1 gives the estimates

$$|\nabla\alpha_k^{\eta_1}| \le \frac{C}{\eta_1} \qquad |\Delta\alpha_k^{\eta_1}| \le \frac{C}{\eta_1^2}$$

where C does not depend on k and η_1. We draw then the estimate

$$\begin{aligned} \|\nabla(\alpha_k^{\eta_1} v)\|_{L^p(\Omega)} &\le \frac{2C_p}{E}\|A_k^{\eta_1}(\alpha_k^{\eta_1} v)\|_{W_p^{-1}(\Omega)} = \\ &= \frac{2C_p}{E}\|A(\alpha_k^{\eta_1} v)\|_{W_p^{-1}(\Omega)} \le \\ &\le \frac{C}{\eta_1}\|Av\|_{W_p^{-1}(\Omega)} + C\Big(\frac{1}{\eta_1^2} + \frac{1}{\eta_1}\Big)\|a\|_{L^\infty(\Omega)}\|\nabla v\|_{L^2(\Omega)} \le \\ &\le C_1\|Av\|_{W_p^{-1}(\Omega)} + C_2\|Av\|_{W_2^{-1}(\Omega)} \le \\ &\le C(p, \eta_1, \Omega, a)\|Av\|_{W_p^{-1}(\Omega)} \end{aligned}$$

and the proposition is proved. □

From the previous porposition, we deduce the following corollary.

COROLLARY 3.1. *Let $p \in [2, 2^*[$ with $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{N}$ such that $-\Delta$ satisfies* H1(p) *and $a \in C^0(\overline{\Omega})$ satisfying* (1.4). *Then for every $r \in [p', p]$, operator A given by* (1.5) *is a bicontinuous isomorphism from $\overset{\circ}{W}{}_r^1(\Omega)$ onto $W_r^{-1}(\Omega)$ with the estimate*

$$\|\nabla v\|_{L^r(\Omega)} \le C(p, \Omega, a)\|Av\|_{W_r^{-1}(\Omega)}. \tag{3.8}$$

Proof. Let $f \in W_p^{-1}(\Omega)$, proposition 3.4 says that there exists a unique solution $u \in \overset{\circ}{W}{}_p^1(\Omega)$ such that $Au = f$. Furthermore, the following estimate hold

$$\begin{aligned}\|\nabla u\|_{L^p(\Omega)} &\le \sum_{k=1}^{J_{\eta_1}} \|\nabla(\alpha_k^{\eta_1} u)\|_{L^p(\Omega)} \le \\ &\le \sum_{k=1}^{J_{\eta_1}} C(p,\Omega,a,\eta_1)\|Av\|_{W_p^{-1}(\Omega)} \le \\ &\le J_{\eta_1} C(p,\Omega,a,\eta_1)\|Av\|_{W_p^{-1}(\Omega)}\,.\end{aligned}$$

Proposition 3.1 gives the conclusion for every $r \in [p', p]$. ∎

Remark 3.4. The proposition 3.4 and the Corollary 3.1 contain the case $p = 2^*$ if $2^* < \infty$.

Using a bootstrapping method, we conclude with the main theorem.

Theorem 3.1 (continuous case). *Let $p \in]1, \infty[$ such that $-\Delta$ satisfies* H1(p) *and $a \in C^0(\overline{\Omega})$ satisfying* (1.4). *Then for every $r \in [\min(p,p'), \max(p,p')]$, operator A given by* (1.5) *is a bicontinuous isomorphism from $\overset{\circ}{W}{}_r^1(\Omega)$ onto $W_r^{-1}(\Omega)$ with the estimate*

$$\|\nabla v\|_{L^r(\Omega)} \le C(p,\Omega,a)\|Av\|_{W_r^{-1}(\Omega)}\,.$$

Proof. Thanks to the proposition 3.1, we can assume $p \ge 2$. If $p < 2^*$ with $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{N}$ the theorem is proved. If $2^* < \infty$, we repeat the same technique as we do in proposition 3.4 for the intervalle $[2^*, 2^{**}[$ with $\frac{1}{2^{**}} = \frac{1}{2^*} - \frac{1}{N}$. For every $p \in [2^*, 2^{**}[$ there exists $q \in [2, 2^*[$ such that we have the continuous embedding $L^p(\Omega) \subset W_q^{-1}(\Omega)$. We get then that the functions $\alpha_k^{\eta_1} v$ belong to $\overset{\circ}{W}{}_p^1(\Omega)$ and we obtain the estimate with the technique developed in the Corollary 3.1. If $p \ge 2^{**}$, we shall consider the intervalle $[2^{**}, 2^{***}[$ and repeat the process as far as we can put p in a similar intervalle —remark that for a fixed dimension N we always have a finite number of such intervals. ∎

4 – The Hölder case

We consider now an elliptic operator A in spaces $W_p^{1+s}(\Omega)$ with Hölder coefficients. Let $p \in]1,\infty[$, $s,\sigma \in]0,1[$ with $\sigma > s$ and a a real function satisfying Assumptions (1.3)-(1.4). We assume that $-\Delta$ realises property H2(p,s). By definition, $-\Delta$ realises also property H1(p) and theorem 3.1 says that the operator A is a bicontinuous isomorphism from $\overset{\circ}{W}{}^1_p(\Omega)$ onto $W_p^{-1}(\Omega)$. Hence, A is one-to-one from $W_p^{1+s}(\Omega)\cap\overset{\circ}{W}{}^1_p(\Omega)$ into $W_p^{s-1}(\Omega)$ for $s \neq \frac{1}{p}$ and from $W_p^{1+s}(\Omega)\cap\overset{\circ}{W}{}^1_p(\Omega)$ into $\widehat{W}{}_p^{s-1}(\Omega)$ for $s = \frac{1}{p}$. It remains to prove the surjectivity. It is now a problem of regularity.

Following the technique used in section three, we shall show that there exists $\eta_2 \in]0,1[$ such that for every $k = 1,\dots,J_{\eta_2}$, $\alpha_k^{\eta_2} u \in W_p^{1+s}(\Omega)$. In all this section, we do the proof only in the case $s \neq \frac{1}{p}$ for the sake of simplicity but we have also the result for $s = \frac{1}{p}$.

Lemma 4.1. *Let $t \in]0,1[$, then there exists $C>0$ independent of t such that:*

$$\forall\, y \in \overline{\Omega}, \qquad \int_\Omega |x-y|^{t-N}\,dx \leq \frac{C}{t}\,.$$

Proof. Denote by Ω_y the domain derived from Ω by translation of vector y, we have:

$$\int_\Omega |x-y|^{t-N} dx = \int_{\Omega_y} |x|^{t-N} dx\,.$$

For all $x \in \Omega$, we have dist$(x,y) \leq$diam(Ω). Hence, if we set

$$R = \text{diam }(\Omega),$$

we have using the spheric coordinates:

$$\begin{aligned}\int_\Omega |x-y|^{t-N} dx &\leq \int_{B_2(0,R)} |x|^{t-N}\,dx \leq \\ &\leq N|B_2(0,1)|\int_0^R r^{t-1}\,dr \leq \\ &\leq N|B_2(0,1)|\frac{R^t}{t} \leq \\ &\leq N|B_2(0,1)|\frac{\sup(R,1)}{t}\,. \end{aligned}$$

□

Lemma 4.2. *Let $s, \sigma \in]0,1[$ with $\sigma > s$, $p \in]1,\infty[$ and $a \in C^{0,\sigma}(\overline{\Omega'})$. There exists a constant C_2 such that for every $w \in W^s_p(\Omega)$, for every $\eta \in]0,1[$ and $k = 1, \dots, J_\eta$:*

$$\|(a^\eta_k - a(M_k))w\|_{W^s_p(\Omega)} \le \frac{C_2}{\sigma - s}\eta^{\frac{\sigma-s}{2}}[a]_{C^{0,\sigma}(\overline{\Omega'})}\|w\|_{W^s_p(\Omega)}\,. \tag{4.1}$$

The constant C_2 does not depend on η, k, w, s, σ and a.

Proof. We can write:

$$\|(a^\eta_k - a(M_k))w\|_{W^s_p(\Omega)} \le A + B$$

with

$$A = \left(\int_\Omega\int_\Omega \frac{|a^\eta_k(x) - a^\eta_k(y)|^p|w(y)|^p}{|x-y|^{sp}}\,\frac{dx\,dy}{|x-y|^N}\right)^{\frac{1}{p}}$$

and

$$B = \left(\int_\Omega\int_\Omega \frac{|a^\eta_k(x) - a(M_k)|^p|w(x) - w(y)|^p}{|x-y|^{sp}}\,\frac{dx\,dy}{|x-y|^N}\right)^{\frac{1}{p}}\,.$$

We study first expression A. Lemma 2.4 yields that:

$$\frac{|a^\eta_k(x) - a^\eta_k(y)|^p}{|x-y|^{sp}} = \frac{|a^\eta_k(x) - a^\eta_k(y)|^p}{|x-y|^{sp}}\frac{|x-y|^{p\frac{\sigma-s}{2}}}{|x-y|^{p\frac{\sigma-s}{2}}} \le$$

$$\le 6^p\eta^{p\frac{\sigma-s}{2}}[a]^p_{0,\sigma}|x-y|^{p\frac{\sigma-s}{2}}\,.$$

Then, we get:

$$A \le 6\eta^{\frac{\sigma-s}{2}}[a]_{0,\sigma}\left(\int_\Omega\int_\Omega |w(y)|^p|x-y|^{p\frac{\sigma-s}{2}-N}\,dx\,dy\right)^{\frac{1}{p}}\,.$$

Using lemma 4.1 and noticing that $(p\sigma - ps)^{\frac{1}{p}} \ge (\sigma - s)$, we have a constant c such that:

$$A \le \frac{c}{\sigma - s}\eta^{\frac{\sigma-s}{2}}[a]_{0,\sigma}\|w\|_{L^p(\Omega)}\,.$$

We study now expression B. Lemma 2.4 yields that:

$$|a_k^\eta(x) - a(M_k)|^p \leq \sup_{x,y\in\mathbb{R}^N} |a_k^\eta(x) - a_k^\eta(y)|^p \leq$$

$$\leq 6^p \eta^{p\sigma} [a]_{0,\sigma}^p .$$

Hence we obtain immediatly:

$$B \leq 6\eta^\sigma [a]_{0,\sigma} \|w\|_{W_p^s(\Omega)} .$$

Since η and $\sigma - s \in]0,1[$, we have:

$$\eta^\sigma \leq \frac{1}{\sigma - s} \eta^{\frac{\sigma - s}{2}}$$

and the lemma is proved. ∎

Thanks to the previous result, we can prove now that for η small enough operator A_k^η defined by relation (3.4) is a bicontinuous isomorphisms from $W_p^{1+s}(\Omega) \cap \overset{\circ}{W}{}_p^1(\Omega)$ onto $W_p^{s-1}(\Omega)$.

PROPOSITION 4.1. *Let $p \in]1,\infty[$, $\sigma, s \in]0,1[$ with $\sigma > s$ and a real function satisfying Assumptions* (1.3)-(1.4). *We assume that $-\Delta$ realise property* H2(p,s). *We denote by $G_{s,p}$ the inverse operator of $-\Delta$ and by $C_{s,p}$ the norm of $G_{s,p}$ in $\mathcal{L}(W_p^{s-1}(\Omega), W_p^{1+s}(\Omega) \cap \overset{\circ}{W}{}_p^1(\Omega))$. Then there exists $\eta_2 \in]0,1[$ such that for every $k = 1, \dots, J_{\eta_2}$, operator $A_k^{\eta_2}$ is a bicontinuous isomorphism operator from $W_p^{1+s}(\Omega) \cap \overset{\circ}{W}{}_p^1(\Omega)$ onto $W_p^{s-1}(\Omega)$ for $s \neq \frac{1}{p}$ and from $W_p^{1+s}(\Omega) \cap \overset{\circ}{W}{}_p^1(\Omega)$ onto $\widehat{W}_p^{s-1}(\Omega)$ for $s = \frac{1}{p}$ with the estimate:*

$$\|v\|_{W_p^{1+s}(\Omega)} \leq \frac{2C_{s,p}}{E} \|A_k^{\eta_2} v\|_{W_p^{s-1}(\Omega)} \quad s \neq \frac{1}{p} , \tag{4.2}$$

$$\|v\|_{W_p^{1+s}(\Omega)} \leq \frac{2C_{s,p}}{E} \|A_k^{\eta_2} v\|_{\widehat{W}_p^{s-1}(\Omega)} \quad s = \frac{1}{p} . \tag{4.3}$$

Proof. In the proof, we assume $s \neq \frac{1}{p}$ for the sake of simplicity. Let $v \in W_p^{1+s}(\Omega)$ and $\eta \in]0,1[$, we can write:

$$A_k^\eta v = -a(M_k)\Delta v + \nabla \cdot ([a(M_k) - a_k^\eta]\nabla v)\,.$$

We have then:

$$\frac{1}{a(M_k)} G_{s,p} A_k^\eta v = v + G_{s,p}\nabla \cdot \Big(\frac{a(M_k) - a_k^\eta}{a(M_k)}\nabla v\Big)\,.$$

If we prove that the linear operator $v \mapsto G_{s,p}\nabla \cdot \Big(\frac{a(M_k)-a_k^\eta}{a(M_k)}\nabla v\Big)$ has a small norm (for example lower than $1/2$), we have proved that operator $G_{s,p}A_k^\eta$ is an isomorphism from $W_p^{1+s}(\Omega) \cap \overset{\circ}{W}{}_p^1(\Omega)$ onto itself, hence A_k^η is an isomorphism form $W_p^{1+s}(\Omega) \cap \overset{\circ}{W}{}_p^1(\Omega)$ onto $W_p^{s-1}(\Omega)$.

Lemma 4.2 yields that:

$$\begin{aligned}\|\nabla \cdot \Big(\frac{a(M_k) - a_k^\eta}{a(M_k)}\nabla v\Big)\|_{W_p^{s-1}(\Omega)} &\leq \sum_{i=1}^N \|\frac{a(M_k) - a_k^\eta}{a(M_k)}\partial_{x_i} v\|_{W_p^s(\Omega)} \leq \\ &\leq N\frac{C_2}{E(\sigma - s)}\eta^{\frac{\sigma-s}{2}}[a]_{0,\sigma}\|v\|_{W_p^{1+s}(\Omega)}\,.\end{aligned}$$

We now choose η_2 such that:

$$N\frac{C_2}{E(\sigma - s)}\eta_2^{\frac{\sigma-s}{2}}[a]_{0,\sigma}C_{s,p} = \frac{1}{2}\,. \tag{4.4}$$

Under this condition, operator $A_k^{\eta_2}$ is an isomorphism. Furthermore, we have the estimate [10], p. 151:

$$\|\,[G_{s,p}A_k^{\eta_2}/a(M_k)]^{-1}\,\|_{\mathcal{L}(W_p^{1+s}(\Omega), W_p^{1+s}(\Omega))} \leq 2$$

thus we have like in proposition 3.3

$$\begin{aligned}\|v\|_{W_p^{1+s}(\Omega)} &\leq \frac{2}{E}\|G_{s,p}A_k^{\eta_2}v\|_{W_p^{1+s}(\Omega)} \leq \\ &\leq \frac{2C_{s,p}}{E}\|A_k^{\eta_2}v\|_{W_p^{s-1}(\Omega)}\end{aligned}$$

and we get the estimate (4.2). $\square$

PROPOSITION 4.2. *Under the assumptions of proposition 4.1, let $f \in W_p^{s-1}(\Omega)$ if $s \neq \frac{1}{p}$ and $f \in \widehat{W}_p^{s-1}(\Omega)$ if $s = \frac{1}{p'}$. Let v be the unique solution in $\overset{\circ}{W}{}_p^1(\Omega)$ of $Av = f$. Then, for all $k = 1, \dots, J_{\eta_2}$, $\alpha_k^{\eta 2} v \in W_p^{1+s}(\Omega)$.*

Furthermore, we have the following estimates:

$$\|\alpha_k^{\eta_2} v\|_{W_p^{1+s}(\Omega)} \leq C_3 \|Av\|_{W_p^{s-1}(\Omega)} \qquad s \neq \frac{1}{p}\,, \tag{4.5}$$

$$\|\alpha_k^{\eta_2} v\|_{W_p^{1+s}(\Omega)} \leq C_3 \|Av\|_{\widehat{W}_p^{s-1}(\Omega)} \qquad s = \frac{1}{p}\,. \tag{4.6}$$

PROOF. We do the proof only in the case $s \neq \frac{1}{p}$. We can also prove the same result for the exceptional case $s = \frac{1}{p}$. We can write $A_k^{\eta_2}(\alpha_k^{\eta_2} v) = A(\alpha_k^{\eta_2} v)$ because $a_k^{\eta_2} = a$ on ball $B_\infty(M_k, \eta_2)$. Operator $A_k^{\eta_2}$ is a bicontinuous isomorphism from $W_p^{1+s}(\Omega) \cap \overset{\circ}{W}{}_p^1(\Omega)$ onto $W_p^{s-1}(\Omega)$. It remains to prove that $A(\alpha_k^{\eta_2} v) \in W_p^{s-1}(\Omega)$.

We can rewrite $A(\alpha_k^{\eta_2} v)$ in the following way:

$$A(\alpha_k^{\eta_2} v) = \alpha_k^{\eta_2} Av + av\Delta\alpha_k^{\eta_2} + a\nabla\alpha_k^{\eta_2} \cdot \nabla v + \nabla\alpha_k^{\eta_2} \cdot \nabla(av)\,.$$

We show that each term in the right hand side member belongs to $W_p^{s-1}(\Omega)$ since $Av \in W_p^{s-1}(\Omega)$.

-1) the term $\alpha_k^{\eta_2} Av$.

Relation (2.11) says that:

$$\|\alpha_k^{\eta_2}\|_{C^1} \leq \frac{C'}{\eta_2}\,, \qquad \|\alpha_k^{\eta_2}\|_{C^2} \leq \frac{C'}{{\eta_2}^2}$$

where C' does not depend of k and η_2. Thus we have:

$$\begin{aligned} \|\alpha_k^{\eta_2} Av\|_{W_p^{s-1}(\Omega)} &\leq \|\alpha_k^{\eta_2}\|_{C^1} \|Av\|_{W_p^{s-1}(\Omega)} \leq \\ &\leq \frac{C'}{\eta_2} \|Av\|_{W_p^{s-1}(\Omega)}\,. \end{aligned} \tag{4.7}$$

-2) the term $av\Delta\alpha_k^{\eta_2}$.

Thanks to theorem 3.1, we know that $v \in \overset{\circ}{W}{}_p^1(\Omega)$. Hence we can write:

$$
\begin{aligned}
\|av\Delta\alpha_k^{\eta_2}\|_{W_p^{s-1}(\Omega)} &\le \|av\Delta\alpha_k^{\eta_2}\|_{L^p(\Omega)} \le \\
&\le \|\alpha_k^{\eta_2}\|_{C^2}\|av\|_{L^p(\Omega)} \le \\
&\le \frac{C'}{{\eta_2}^2}\|av\|_{L^p(\Omega)} \le \\
&\le \frac{C'}{{\eta_2}^2}\|a\|_{L^\infty(\Omega)}\|v\|_{L^p(\Omega)} \le \\
&\le \frac{C'}{{\eta_2}^2}\|a\|_{C^{0,\sigma}(\overline{\Omega})}\|\nabla v\|_{L^p(\Omega)} \le \\
&\le \frac{C'C_1}{{\eta_2}^2}\|a\|_{C^{0,\sigma}(\overline{\Omega})}\|Av\|_{W_p^{-1}(\Omega)} \le \\
&\le \frac{C''C_1}{{\eta_2}^2}\|a\|_{C^{0,\sigma}(\overline{\Omega})}\|Av\|_{W_p^{s-1}(\Omega)}\,.
\end{aligned} \tag{4.8}
$$

-3) the term $a\nabla\alpha_k^{\eta_2}.\nabla v$.

Like Subsection 2, we write:

$$
\begin{aligned}
\|a\nabla\alpha_k^{\eta_2}.\nabla v\|_{W_p^{s-1}(\Omega)} &\le \|a\nabla\alpha_k^{\eta_2}.\nabla v\|_{L^p(\Omega)} \le \\
&\le \|\alpha_k^{\eta_2}\|_{C^1}\|a\nabla v\|_{L^p(\Omega)} \le \\
&\le \frac{C'}{\eta_2}\|a\nabla v\|_{L^p(\Omega)} \le \\
&\le \frac{C'}{{\eta_2}^2}\|a\|_{L^\infty(\Omega)}\|\nabla v\|_{L^p(\Omega)} \le \\
&\le \frac{C'}{{\eta_2}^2}\|a\|_{C^{0,\sigma}(\overline{\Omega})}\|\nabla v\|_{L^p(\Omega)} \le \\
&\le \frac{C'C_1}{{\eta_2}^2}\|a\|_{C^{0,\sigma}(\overline{\Omega})}\|Av\|_{W_p^{-1}(\Omega)} \le \\
&\le \frac{C''C_1}{{\eta_2}^2}\|a\|_{C^{0,\sigma}(\overline{\Omega})}\|Av\|_{W_p^{s-1}(\Omega)}\,.
\end{aligned} \tag{4.9}
$$

-4) the term $\nabla\alpha_k^{\eta_2}\cdot\nabla(av)$.

Since $v \in \overset{\circ}{W}{}_p^1(\Omega)$ and $a \in C^{0,\sigma}(\overline{\Omega})$, $av \in \overset{\circ}{W}{}_p^s(\Omega)$, hence $\nabla(av) \in$

$W_p^{s-1}(\Omega)$ with the estimate [5], p. 31:

$$\begin{aligned}\|\nabla(av)\|_{W_p^{s-1}(\Omega)} &\le N\|av\|_{W_p^s(\Omega)} \le \\ &\le N\|a\|_{C^{0,\sigma}(\overline{\Omega})}\|v\|_{W_p^s(\Omega)} \le \\ &\le NC\|a\|_{C^{0,\sigma}(\overline{\Omega})}\|v\|_{W_p^1(\Omega)} \le \\ &\le NCC_1\|a\|_{C^{0,\sigma}(\overline{\Omega})}\|Av\|_{W_p^{-1}(\Omega)} \le \\ &\le NC'C_1\|a\|_{C^{0,\sigma}(\overline{\Omega})}\|Av\|_{W_p^{s-1}(\Omega)}\,.\end{aligned}$$

We obtain then:

$$\begin{aligned}\|\nabla\alpha_k^{\eta_2}.\nabla(av)\|_{W_p^{s-1}(\Omega)} &\le \|\alpha_k^{\eta_2}\|_{C^2}\|\nabla(av)\|_{W_p^{s-1}(\Omega)} \le \\ &\le \frac{C'}{{\eta_2}^2}\|\nabla(av)\|_{W_p^{s-1}(\Omega)} \le \\ &\le N\frac{C''C_1}{{\eta_2}^2}\|a\|_{C^{0,\sigma}(\overline{\Omega})}\|Av\|_{W_p^{s-1}(\Omega)}\,.\end{aligned}\tag{4.10}$$

Summing up relations (4.7)-(4.10), we have $A(\alpha_k^{\eta_2}v) \in W_p^{s-1}(\Omega)$ with the following estimate:

$$\begin{aligned}\|A_k^{\eta_2}(\alpha_k^{\eta_2}v)\|_{W_p^{s-1}(\Omega)} &= \|A(\alpha_k^{\eta_2}v)\|_{W_p^{s-1}(\Omega)} \le \\ &\le \frac{C''}{{\eta_2}^2}\|a\|_{C^{0,\sigma}(\overline{\Omega})}C_1\|Av\|_{W_p^{s-1}(\Omega)}\,,\end{aligned}$$

where the constants C', C'', C_1 do not depend on k,η_2, a. We can now write:

$$\begin{aligned}\|\alpha_k^{\eta_2}v\|_{W_p^{s+1}(\Omega)} &\le \frac{2C_{s,p}}{E}\|A_k^{\eta_2}(\alpha_k^{\eta_2}v)\|_{W_p^{s-1}(\Omega)} \le \\ &\le \frac{2C_{s,p}}{E}\|A(\alpha_k^{\eta_2}v)\|_{W_p^{s-1}(\Omega)} \le \\ &\le C''\frac{2C_1}{{\eta_2}^2}\|a\|^2_{C^{0,\sigma}(\overline{\Omega})}\|Av\|_{W_p^{s-1}(\Omega)}\,.\end{aligned}$$

□

To sum up all the results of this section, we have the theorem.

THEOREM 4.1 (Hölder case). *Let* $p \in]1, \infty[$, $\sigma, s \in]0,1[$ *with* $\sigma > s$ *and a real function satisfying Assumptions* (1.3)-(1.4). *We assume that* $-\Delta$ *satisfies assumption* H2(p,s). *Then* A *satisfies Assumption* H2(p,s) *with the estimate:*

$$\|v\|_{W_p^{1+s}(\Omega)} \leq C_4 \|Av\|_{W_p^{s-1}(\Omega)} \qquad s \neq \frac{1}{p}, \tag{4.11}$$

$$\|v\|_{W_p^{1+s}(\Omega)} \leq C_4 \|Av\|_{\widehat{W}_p^{s-1}(\Omega)} \qquad s = \frac{1}{p}. \tag{4.12}$$

PROOF. We do the proof in the case $s \neq \frac{1}{p}$. Proposition 4.2 gives that $v \in W_p^{1+s}(\Omega)$. Using estimate (4.5), we can write:

$$\begin{aligned} \|v\|_{W_p^{1+s}(\Omega)} &= \| \sum_{k=1}^{J_{\eta_2}} \alpha_k^{\eta_2} v\|_{W_p^{1+s}(\Omega)} \leq \\ &\leq \sum_{k=1}^{J_{\eta_2}} \|\alpha_k^{\eta_2} v\|_{W_p^{1+s}(\Omega)} \leq \\ &\leq \sum_{k=1}^{J_{\eta_2}} C_3 \|Av\|_{W_p^{s-1}(\Omega)} \leq \\ &\leq J_{\eta_2} C_3 \|Av\|_{W_p^{s-1}(\Omega)}. \end{aligned}$$

Taking for example $C_4 = J_{\eta_2} C_3$, we obtain the estimate. □

REMARK 4.1. In fact, we can take a weaker property $H_2(s,p)$ assuming the $H_2(s,p)$ assumption but a weakened assumption for $H_1(p)$. We can prove the embeddings (4.7)-(4.10) are satisfied since A has the property $H_1(\max(p_s^*, 2))$ for $p \geq 2$ with $\frac{1}{p_s^*} = \frac{1}{p} + \frac{1-s}{N}$ and $H_1(\min(p_s, 2))$ for $p \leq 2$ with $\frac{1}{p_s} = \frac{1}{p} - \frac{s}{N}$. In particular, H_1 assumption is not necessary when $p_s^* \leq 2$ if $p \geq 2$ or when $p_s \geq 2$ if $p \leq 2$ since H1(2) is automatically satisfied thanks to the energy inequality.

5 – Application to a non linear problem

We present an application of the theorems we have established in the previous sections. We first recall the following result.

Proposition 5.1. *Let Ω be an open bounded set of $\mathbb{R}^N$ with a $C^{1,1}$ boundary. For every $p \in]1,\infty[$, the Laplace operator $-\Delta$ is a bicontinuous isomorphism from $\overset{\circ}{W}{}^1_p(\Omega)$ onto $W_p^{-1}(\Omega)$. For every $p \in]1,\infty[$, the Laplace operator $-\Delta$ is a bicontinuous isomorphism from $W_p^2(\Omega) \cap \overset{\circ}{W}{}^1_p(\Omega)$ onto $L^p(\Omega)$.*

Proof. This first result is given in [7], p. 123, theorem 7.2. The second result is given in [5], p. 124, theorem 2.4.2.5. □

We deduce then the following lemma.

Corollary 5.1. *Let $s \in]0,1[$ and $p \in]1,\infty[$. Under the assumptions of proposition 5.1, the Laplace operator is a bicontinuous isomorphism from $W_p^{1+s}(\Omega) \cap \overset{\circ}{W}{}^1_p(\Omega)$ onto $W_p^{s-1}(\Omega)$ if $s \neq \frac{1}{p}$ and from $W_p^{1+s}(\Omega) \cap \overset{\circ}{W}{}^1_p(\Omega)$ onto $\widehat{W}_p^{s-1}(\Omega)$ if $s = \frac{1}{p}$.*

Proof. Let $p \in]1,\infty[$, proposition 5.1 says that $-\Delta$ is a bicontinuous isomorphism in the following spaces:

$$\overset{\circ}{W}{}^1_p(\Omega) \mapsto W_p^{-1}(\Omega),$$
$$W_p^2(\Omega) \cap \overset{\circ}{W}{}^1_p(\Omega) \mapsto L^p(\Omega)\,.$$

Using interpolation of operators $-\Delta$ and $-\Delta^{-1}$ as it is done in [9], p. 401, we get that $-\Delta$ is a bicontinuous isomorphism in interpolated spaces:

$$(\overset{\circ}{W}{}^1_p(\Omega), W_p^2(\Omega) \cap \overset{\circ}{W}{}^1_p(\Omega))_{s,p} \mapsto (W_p^{-1}(\Omega), L^p(\Omega))_{s,p}\,.$$

Moreover, we have [9], p. 320:

$$(W_p^{-1}(\Omega), L^p(\Omega))_{s,p} = W_p^{s-1}(\Omega) \text{ if } s \neq \frac{1}{p}\,,$$
$$(W_p^{-1}(\Omega), L^p(\Omega))_{\frac{1}{p},p} = \widehat{W}_p^{\frac{1}{p}-1}(\Omega)\,,$$
$$(\overset{\circ}{W}{}^1_p(\Omega), W_p^2(\Omega) \cap \overset{\circ}{W}{}^1_p(\Omega))_{s,p} = W_p^{1+s}(\Omega) \cap \overset{\circ}{W}{}^1_p(\Omega)\,.$$

The corollary is proved. □

Corollary 5.2. *Let $s, \sigma \in]0,1[$ with $s < \sigma$ and $a \in C^{0,\sigma}(\overline{\Omega})$ satisfying Assumptions* (1.3)-(1.4)*. Then for every $p \in]1, \infty[$, under assumptions of proposition 5.1, operator A is a bicontinuous isomorphism from $W_p^{1+s}(\Omega) \cap \overset{\circ}{W}{}_p^1(\Omega)$ onto $W_p^{s-1}(\Omega)$ if $s \neq \frac{1}{p}$ and from $W_p^{1+s}(\Omega) \cap \overset{\circ}{W}{}_p^1(\Omega)$ onto $\widehat{W}_p^{s-1}(\Omega)$ if $s = \frac{1}{p}$.*

Proof. We apply theorem 4.1. ∎

We present a short example where fractional Sobolev spaces give regularity result. We consider a bounded open set Ω of $\mathbb{R}^2$ with a $C^{1,1}$ boundary. Assume $p \in]1, 2[$ and let f be a function of $L^p(\Omega)$. We consider the following problem: find (u, v) such that:

$$
\begin{cases}
-\nabla.(k(u)\nabla v) = f & \text{in} \quad \Omega\,, \\
v = 0 & \text{on} \quad \partial\Omega\,, \\
-\Delta u = k(u)|\nabla v|^2 & \text{in} \quad \Omega\,, \\
u = 0 & \text{on} \quad \partial\Omega\,,
\end{cases}
\tag{5.1}
$$

where k is a bounded Lipschitz function of $\mathbb{R}$ satisfying $k(\xi) \geq E > 0$, for all $\xi \in \mathbb{R}$. It is a classical system in thermistor or in induction heating problems.

Using a fixed point theorem and the Meyers lemma, we can prove that there exists a solution (u, v) for the problem (5.1) in $\overset{\circ}{W}{}^1_{\frac{p^*}{2}}(\Omega) \cap W^2_{\frac{p^*}{2}}(\Omega) \times \overset{\circ}{W}{}^1_{p^*}(\Omega)$ with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{2}$ [3]. If $p > \frac{4}{3}$, the Morey theorem says that $u \in C^{0,1}(\overline{\Omega})$, hence using a classical regularity result [5], p. 125, we get that $v \in W_p^2(\Omega)$. If $p \in]1, \frac{4}{3}[$, we only have $u \in C^{0,\alpha}(\overline{\Omega})$ with $\alpha = 4(1 - \frac{1}{p})$. In this case, let $s \in]0, \alpha[$ and set

$$
q = \frac{2p}{2 + (s-1)p}
$$

then using the embedding $L^p(\Omega) \subset W_q^{s-1}(\Omega)$, we obtain thanks to the Corollary 5.2 a solution $v \in W_q^{1+s}(\Omega)$. This "$s-$regularity" could be very usefull to get numerical estimates.

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