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A Liouville theorem for radial k-Hessian equations

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Riassunto: *Si considera un'equazione non-lineare che contiene l'operatore k-hessiano. Si dimostra un principio di massimo forte dal quale si deduce un teorema di tipo Liouville.*

Abstract: *We prove a strong maximum principle for the radial solutions of the k-Hessian equation in* $\mathbb{R}^n$ *from which a Liouville theorem is derived.*

1 – Introduction

In a recent paper [1] Caffarelli, Garofalo and Segala consider the following class of quasi-linear equations in $\mathbb{R}^n$

$$\operatorname{div}(\Phi'(|\nabla u|^2)\nabla u) = f(u)\,.$$

Here $\Phi \in C^3(\mathbb{R}^+)$ is such that $\Phi(0) = 0$. In addition, the function Φ satisfies two different sets of conditions. The first one requires existence of numbers $p > 1$, $a \geq 0$ and positive constants c_1, c_2 such that for any

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non-zero σ, $\xi \in \mathbb{R}^n$ the inequalities

$$c_1(a+|\sigma|)^{p-2} \leq \Phi'(|\sigma|^2) \leq c_2(a+|\sigma|)^{p-2}\,,$$
$$c_1(a+|\sigma|)^{p-2}|\xi|^2 \leq \sum_{i,j=1}^{n} (2\phi''(|\sigma|^2)\sigma_i\sigma_j + \Phi'(|\sigma|^2)\delta_{ij})\xi_i\xi_j \leq c_2(a+|\sigma|)^{p-2}|\xi|^2$$

hold. The second condition states that there exist c_1, $c_2 > 0$ such that for any $\sigma \in \mathbb{R}^n$ and any vector $\xi' = (\xi, \xi_{n+1}) \in \mathbb{R}^{n+1}$ orthogonal to $(-\sigma, 1)$

$$c_1(a+|\sigma|)^{-1} \leq \Phi'(|\sigma|^2) \leq c_2(a+|\sigma|)^{-1}\,,$$
$$c_1(a+|\sigma|)^{-1}|\xi'|^2 \leq \sum_{i,j=1}^{n} (2\phi''(|\sigma|^2)\sigma_i\sigma_j + \Phi'(|\sigma|^2)\delta_{ij})\xi_i\xi_j \leq c_2(a+|\sigma|)^{-1}|\xi'|^2.$$

Note that this class contains, in particular, the p-Laplacian equation ($\Phi(s) = 2s^{p/2}/p, p > 1$) as well as the prescribed mean curvature equation ($\Phi(s) = 2(\sqrt{1+s}-1)$). Further in [1] the authors prove a gradient bound for sufficiently smooth solutions of the equations in the above class and derive some consequences, e.g. Liouville theorems.

It is easily seen that the equations of k-Hessian type

$$S_k(\nabla^2 u) = f(u) \tag{1}$$

do not belong to that class if $k > 1$. Recall that the k-Hessian operator S_k is defined by

$$S_k(\nabla^2 u) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \lambda_{i_1} \ldots \lambda_{i_k}\,,$$

where $1 \leq k \leq n$, the function $u \in C^2(\Omega)$ and λ_{i_j} are the eigenvalues of the Hessian of u, i.e. the matrix $\nabla^2 u$, whose elements are the second derivatives of u. Note that S_1 is the Laplace operator and S_n is the Monge-Ampère operator.

In this paper we shall consider the same kind of problem as in [1] for the radial k-Hessian equation ($k > 1$)

$$\frac{1}{r^{n-1}}(r^{n-k}|u'|^{k-1}u')' = f(u) \tag{2}$$

with

$$F(u) = \int_0^u f(t)dt \geq 0\,. \tag{3}$$

One can observe that the cornerstone of [1] is a strong maximum principle for a suitable P-function. The so-called P- function (see [4]) is an expression in the terms of the solution u and its gradient. P assumes its maximum on the boundary of the considered domain or at the critical points of u [4]. Our first aim is therefore to find out an appropriate P-function, which corresponds to the equation (2). In this case it is given by

$$P(u,r) = \frac{2k}{k+1}\,\frac{|u'(r)|^{k+1}}{r^{k-1}} - 2F(u(r))\,. \tag{4}$$

This choice of P enables us to prove in section 2 the following

Theorem 1. *Let $F \geq 0$, $F \in C^2(\mathbb{R})$ and $u \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be a solution of* (2). *Then for every $r > 0$ the inequality*

$$\frac{|u'(r)|^{k+1}}{r^{k-1}} \leq \frac{k+1}{k} F(u(r)) \tag{5}$$

holds.

It is interesting to notice that if we go back to $\mathbb{R}^n$, after an integration, the inequality (5) takes the form

$$\int_{\mathbb{R}^n} T_{k-1}(\nabla^2 u)_{ij} u_i u_j \, dx \leq \frac{k+1}{k} \int_{\mathbb{R}^n} F(u(x))\, dx\,,$$

where $T_{k-1}(\nabla^2 u)_{ij}$ is the Newtonian tensor (see [5] and the references there).

We also note that if $F = 0$, then the inequality (5) implies $u = \text{const}$.

Having at our disposal the Theorem 1, we can derive the next theorem of Liouville type.

Theorem 2. *Let u and F be as in the Theorem 1. Moreover, suppose also that if $F(u_0) = 0$ for some u_0, then $F(u) = 0(|u - u_0|^{k+1})$, when $u \to u_0$.*

If there exists $r_0 > 0$ such that $F(u(r_0)) = 0$, then $u(r) = \text{const}$ in $(0, \infty)$.

Further in section 3 we prove an uniqueness result. Namely,

THEOREM 3. *If u is a radial solution of the k-Hessian equation* (1) *with $F(u) \in L^1(\mathbb{R}^n)$ and $n > 2k$, then $u = \text{const}$.*

The proof is divided in three steps. First we obtain a Pucci-Serrin-Pohozaev type identity [3] for the radial solutions of the k-Hessian equation. This identity is used to show that an appropriate energy functional is monotonically increasing on balls of increasing radii. Then the latter fact and the assumptions in the Theorem 3 imply $u = \text{const}$.

It would be interesting to see if Theorems 1, 2 and 3 remain valid for non-radial solutions.

2 – A strong maximum principle

In this section we prove the Theorem 1. Let us recall that the P-function is given by

$$P(u,r) = \frac{2k}{k+1}\,\frac{|u'(r)|^{k+1}}{r^{k-1}} - 2F(u(r))\,,$$

(see eq. (4)). Our starting point is the following

PROPOSITION 1. *Let $-\infty < a < b < +\infty$ and u be a sufficiently smooth solution of* (2), *such that* $\inf |u'(r)| > 0$, $r \in [a,b]$. *If there is $r_0 \in (a,b)$ for which*

$$P(u,r_0) = \sup_{r\in[a,b]} P(u,r)\,,$$

then $P(u,\,.\,)$ is a constant in (a,b).

PROOF. From the regularity results for the k-Hessian equation [5] it follows that $u \in C^{2,\alpha}((a,b))$. Then it is sufficient to prove an estimate of the type

$$u'^2 \Delta P + B.P' \geq \frac{r^{k-1} P'^2}{2k|u'|^{k-1}}\,, \tag{6}$$

where Δ is the Laplace operator and B is to be determined later. Indeed, the Proposition 1 is a straightforward consequence from the strong maximum principle and the inequality (6).

To begin with, we write the equation (2) in a developed form

$$\frac{k|u'|^{k-1}}{r^{k-1}}\,u'' + \frac{(n-k)}{r^k}\,|u'|^{k-1}u' = f(u)\,. \tag{7}$$

Now we differentiate (4):

$$P' = \frac{2k|u'|^{k-1}u'u''}{r^{k-1}} - 2f(u)u' + \frac{2k(1-k)}{k+1}\,\frac{|u'|^{k+1}}{r^k}\,. \tag{8}$$

Differentiating P twice yields:

$$\begin{aligned} P'' = {} & \frac{2k}{r^{k-1}}\,|u'|^{k-1}(u'')^2 + 2u'\frac{d}{dr}\Big[\frac{k|u'|^{k-1}u''}{r^{k-1}}\Big] - 2f'u'^2 - \\ & - 2fu'' + \frac{2k^2(k-1)}{k+1}\,\frac{|u'|^{k+1}}{r^{k+1}} - 2k(k-1)\frac{|u'|^{k-1}u'u''}{r^k}\,. \end{aligned}$$

Then from the equation (7) we have that

$$\begin{aligned} P'' = {} & \frac{2k}{r^{k-1}}\,|u'|^{k-1}(u'')^2 + 2u'\frac{d}{dr}\Big[f(u) - \frac{(n-k)}{r^k}|u'|^{k-1}u'\Big] - 2f'u'^2 - \\ & - 2f\Big[\frac{fr^{k-1}}{k|u'|^{k-1}} - \frac{(n-k)u'}{kr}\Big] + \frac{2k^2(k-1)}{k+1}\,\frac{|u'|^{k+1}}{r^{k+1}} - \\ & - \frac{(k-1)}{r}\Big[\frac{2k|u'|^{k-1}u'u''}{r^{k-1}}\Big]. \end{aligned}$$

Further, expressing the term $\frac{2k|u'|^{k-1}u'u''}{r^{k-1}}$ from equation (8), we obtain

$$\begin{aligned} P'' = {} & \frac{2k}{r^{k-1}}|u'|^{k-1}(u'')^2 + 2u'\Big[f'u' + \frac{k(n-k)}{r^{k+1}}|u'|^{k-1}u' - \frac{k(n-k)|u'|^{k-1}u''}{r^k}\Big] - \\ & - 2f'u'^2 - \frac{2f^2r^{k-1}}{k|u'|^{k-1}} + \frac{2(n-k)}{kr}fu' + \frac{2k^2(k-1)}{k+1}\,\frac{|u'|^{k+1}}{r^{k+1}} - \\ & - \frac{(k-1)}{r}\Big[P' + 2fu' + \frac{2k(k-1)}{k+1}\,\frac{|u'|^{k+1}}{r^k}\Big]\,. \end{aligned}$$

We now simplify the last expression follows:

$$P''=\frac{2k}{r^{k-1}}|u'|^{k-1}(u'')^2+2f'u'^2+2k(n-k)\frac{|u'|^{k+1}}{r^{k+1}}-\frac{(n-k)}{r}\,\frac{2k|u'|^{k-1}u'u''}{r^{k-1}}-$$

$$-2f'u'^2-\frac{2f^2r^{k-1}}{k|u'|^{k-1}}+\frac{2(n-k)}{kr}fu'+\frac{2k^2(k-1)}{k+1}\,\frac{|u'|^{k+1}}{r^{k+1}}-$$

$$-\frac{(k-1)}{r}P'-\frac{2(k-1)}{r}fu'-\frac{2k(k-1)^2}{k+1}\,\frac{|u'|^{k+1}}{r^{k+1}}=$$

$$=\frac{2k}{r^{k-1}}|u'|^{k-1}(u'')^2+\Big[2k(n-k)-\frac{2k(k-1)^2}{k+1}+\frac{2k^2(k-1)}{k+1}\Big]\frac{|u'|^{k+1}}{r^{k+1}}-$$

$$-\frac{(n-k)}{r}P'-\frac{2(n-k)}{r}fu'-\frac{2k(n-k)(k-1)}{k+1}\,\frac{|u'|^{k+1}}{r^{k+1}}-$$

$$-\frac{2f^2r^{k-1}}{k|u'|^{k-1}}+\frac{2(n-k)}{kr}fu'-\frac{(k-1)}{r}P'-\frac{2(k-1)}{r}fu'\,.$$

Summarizing, we have

$$P''=\frac{2k}{r^{k-1}}|u'|^{k-1}(u'')^2+\frac{2k(2n-k-1)}{k+1}\,\frac{|u'|^{k+1}}{r^{k+1}}-$$

$$-\frac{(n-1)}{r}P'-\frac{2f^2r^{k-1}}{k|u'|^{k-1}}-\frac{2(k-1)n}{k}\,\frac{fu'}{r}\,.$$

Since P is radial the Laplace operator applied to P gives

$$\Delta P=P''+\frac{(n-1)}{r}P'\,.$$

Therefore

$$\Delta P=\frac{2k}{r^{k-1}}|u'|^{k-1}(u'')^2+\frac{2k(2n-k-1)}{k+1}\,\frac{|u'|^{k+1}}{r^{k+1}}-$$
$$-\frac{2f^2r^{k-1}}{k|u'|^{k-1}}-\frac{2(k-1)n}{k}\,\frac{fu'}{r}\,. \tag{9}$$

From (8) we can express u'' as

$$u''=r^{k-1}\Big(P'+2fu'+\frac{2k(k-1)}{k+1}\,\frac{|u'|^{k+1}}{r^k}\Big)\frac{1}{2k|u'|^{k-1}u'}\,. \tag{10}$$

Substituting (10) into (9) we get that

$$\Delta P = \frac{r^{k-1}P'^2}{2k|u'|^{k+1}} + \frac{2r^{k-1}fu'}{k|u'|^{k-1}u'^2}P' + \frac{2k(k-1)^2|u'|^{k+1}}{(k+1)^2r^{k+1}} + $$
$$+ \frac{2(k-1)r^{k-1}}{(k+1)r^k}P' + \frac{4(k-1)}{k+1}\,\frac{fu'}{r} - \frac{2(k-1)n}{k}\,\frac{fu'}{r} + $$
$$+ \frac{2k(2n-k-1)}{k+1}\,\frac{|u'|^{k+1}}{r^{k+1}}\,.$$

Since the third and the last term of the expression above are non-negative for $n \geq k \geq 1$ it follows that

$$\begin{aligned} u'^2\Delta P \geq \frac{r^{k-1}P'^2}{2k|u'|^{k-1}} + 2r^{k-1}\Big(\frac{fu'}{k|u'|^{k-1}} + \frac{(k-1)u'^2}{(k+1)r^k}\Big)P' + \\ + \frac{2(k-1)(2k-n(k+1))}{k(k+1)}\,\frac{fu'^3}{r}\,. \end{aligned} \tag{11}$$

On the other hand, (7) and (8) imply that

$$P' = 2u'\Big[f(u) - \frac{(n-k)}{r^k}|u'|^{k-1}u'\Big] - 2f(u)u' - \frac{2k(k-1)}{k+1}\,\frac{|u'|^{k+1}}{r^k} =$$
$$= -\frac{2(n-k)}{r^k}|u'|^{k+1} - \frac{2k(k-1)}{k+1}\,\frac{|u'|^{k+1}}{r^k} = \frac{2(2k-n(k+1))}{k+1}\,\frac{|u'|^{k+1}}{r^k}\,.$$

That is,

$$P' = \frac{2(2k-n(k+1))}{k+1}\,\frac{|u'|^{k+1}}{r^k}\,.$$

Therefore

$$u'^2\Delta P \geq \frac{r^{k-1}P'^2}{2k|u'|^{k-1}} + r^{k-1}\Big(\frac{(k+1)fu'}{k|u'|^{k-1}} + \frac{2(k-1)u'^2}{(k+1)r^k}\Big)P'\,.$$

Hence we finally obtain that

$$u'^2\Delta P + B.P' \geq \frac{r^{k-1}P'^2}{2k|u'|^{k-1}}\,,$$

where

$$B = -r^{k-1}\Big(\frac{(k+1)fu'}{k|u'|^{k-1}} + \frac{2(k-1)u'^2}{(k+1)r^k}\Big)\,.$$

Now we are ready to prove Theorem 1. Its proof is a consequence of Proposition 1 and the translation invariance of the k-Hessian operator. More precisely, if u is a solution of the k-Hessian equation (2) then by the assumptions of the theorem there exists a constant $C > 0$ such that $\|u\|_{L^\infty(\mathbb{R})} \le C$. This allows us to define the number

$$P_0 = \sup\Big\{P(u,r)|u \text{ is a solution of (2)}, \ \|u\|_{L^\infty(\mathbb{R})} \le C\Big\}\,.$$

Suppose that $P_0 > 0$. Then the translation invariance of (2) and the fact that $u \in C^{1,\mu}$, $0 < \mu < 1$, (see [2]) enable us, as in [1], to apply a diagonalizing procedure, from which we conclude that there exists a solution v of (2) such that $\|v\|_{L^\infty(\mathbb{R})} \le C$ and $P(v, r_0) = P_0$ for some $r_0 \ne 0$. Then we can use the Proposition 1 and the assumption (3) in the same way as in [1] to get a contradiction. Therefore $P_0 \le 0$ which completes the proof.

3 – Liouville results

In this section we present the

PROOF OF THEOREM 2. Let $u(r_0) = u_0$, where $F(u(r_0)) = 0$. Denote

$$A = \{r \in \mathbb{R} | u(r) = u_0\} \ne \emptyset\,.$$

Clearly A is closed. We prove the theorem by a showing that A is also open, which implies that A must be the whole $\mathbb{R}$, that is, $u = u_0 = \text{const}$.

Let $r_1 \in A$. Introduce the function $\varphi(t) = u(r_1 \pm t) - u_0$. Then $|\varphi'(t)| = |u'(r_1 \pm t)|$. Therefore

$$\begin{aligned}|\varphi'(t)|^{k+1} = |u'(r_1 \pm t)|^{k+1} &\le \frac{k+1}{k} F(u(r_1 \pm t))|r_1 \pm t|^{k+1} = \\ &= \frac{k+1}{k}(F(u(r_1 \pm t)) - F(u_0))|r_1 \pm t|^{k+1}\end{aligned}$$

by Theorem 1. By the asymptotic assumptions on F near u_0 we conclude that $|\varphi'(t)| \le \text{const}\,|\varphi(t)|$ for sufficiently small t (note that $k > 1$). Since

$r_1 \in A$ it follows that $\varphi(0) = u(r_1) - u_0 = 0$. Therefore $\varphi = 0$ in a sufficiently small neighbourhood of r_1. In this way we have shown that A is open.

Further we are going to prove the Theorem 3. Before doing this, we need the next

LEMMA 1. *Let $u \in C^2((0,\infty))$ be a solution of* (2). *Then for any $r > 0$ the following identity*

$$F(u(r))r^{n-1} - \frac{k}{k+1} r^{n-k}|u'(r)|^{k+1} = \\ = \frac{n}{r}\int_0^r F(u(t))t^{n-1}dt + \frac{n-2k}{r(k+1)}\int_0^r t^{n-k}|u'(t)|^{k+1}dt$$

holds.

PROOF. We have

$$\begin{aligned} n\int_0^r F(u(t))t^{n-1}dt &= F(u(r))r^n - \int_0^r t^n f(u(t))u'(t)dt = \\ &= F(u(r))r^n - \int_0^r tu'(t^{n-k}|u'|^{k-1}u')'dt = \\ &= F(u(r))r^n - r^{n-k+1}|u'(r)|^{k+1} + \int_0^r t^{n-k}|u'|^{k-1}u'(u' + tu'')dt = \\ &= F(u(r))r^n - r^{n-k+1}|u'(r)|^{k+1} + \int_0^r t^{n-k}|u'|^{k+1}dt + \\ &\quad + \int_0^r t^{n-k+1}|u'|^{k-1}u'u''dt = \\ &= F(u(r))r^n - r^{n-k+1}|u'(r)|^{k+1} + \int_0^r t^{n-k}|u'|^{k+1}dt + \\ &\quad + \int_0^r t^{n-k+1}\frac{1}{k+1}(|u'|^{k+1})'dt = \\ &= F(u(r))r^n - r^{n-k+1}|u'(r)|^{k+1} + \int_0^r t^{n-k}|u'|^{k+1}dt + \\ &\quad + \frac{r^{n-k+1}}{k+1}|u'(r)|^{k+1} - \frac{n-k+1}{k+1}\int_0^r t^{n-k}|u'|^{k+1}dt = \\ &= F(u(r))r^n - \frac{k}{k+1}r^{n-k+1}|u'(r)|^{k+1} - \frac{n-2k}{k+1}\int_0^r t^{n-k}|u'|^{k+1}dt\,. \end{aligned}$$

Dividing by r we obtain the desired identity.

REMARK. We note that the above identity is nothing but the Pucci-Serrin-Pohozaev identity [3] for the radial k-Hessian equation.

The second step is to obtain a monotonicity result for the spherical mean of an appropriate function.

PROPOSITION 2. *Let $n > 2k$. Let a be such that*

$$0 < a < \frac{n-2k}{(k+1)n}\,.$$

For u as in the Theorem 1, we define the function

$$E(r) = \frac{\tau}{r^{n-1}} \int_0^r \Big[F(u(t))t^{n-1} + at^{n-k}|u'(t)|^{k+1}\Big]dt\,,$$

where τ is the measure of the unit sphere in $\mathbb{R}^n$.

Then $E(r)$ is monotone increasing for $r \geq 0$.

PROOF. Differentiating E we obtain:

$$\frac{E'(r)}{\tau} = \frac{1-n}{r^n} \int_0^r \Big[F(u(t))t^{n-1} + at^{n-k}|u'(t)|^{k+1}\Big]dt + \\ + \frac{1}{r^n}\Big[F(u(r))r^{n-1} + ar^{n-k}|u'(r)|^{k+1}\Big]\,.$$

Further, using Lemma 1, we get

$$\frac{E'(r)}{\tau} = \frac{1}{r^n} \int_0^r \Big[F(u(t)) + a\frac{|u'(t)|^{k+1}}{t^{k-1}}\Big]t^{n-1}dt + \\ + \frac{1}{r^n}\Big[\frac{n-2k}{k+1} - na\Big] \int_0^r t^{n-k}|u'(t)|^{k+1}dt + \\ + \frac{1}{r^{k-1}}\Big[a + \frac{k}{k+1}\Big]|u'(r)|^{k+1}\,.$$

The choice of a ensures that all terms above are non-negative. Therefore $E'(r) \geq 0$.

After this preparatory work, we are ready for the

PROOF OF THEOREM 3. Since $F(u) \in L^1(\mathbb{R}^n)$, by Theorem 1 (see (5)) we conclude that the function $|u'|^{k+1}r^{-(k-1)} \in L^1(\mathbb{R}^n)$.

Hence $F(u) + a|u'|^{k+1} r^{-(k-1)} \in L^1(\mathbb{R}^n)$ and

$$0 \leq E(r) \leq \frac{1}{r^{n-1}} \int_{\mathbb{R}^n} \Big[F(u(|x|) + a|u'(|x|)|^{k+1} |x|^{-(k-1)} \Big] dx \, .$$

Thus $\lim_{r \to \infty} E(r) = 0$. By Proposition 2 we conclude that $E(r) = 0$ for any non-negative r. Since $F \geq 0$ and $a \geq 0$, it follows that $u'(r) = 0$ and $u = \text{const}$.

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