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Fuzzy topologies and a type of their decomposition

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Riassunto: *In questa nota si considerano alcune operazioni sugli insiemi fuzzy. Queste operazioni producono decomposizioni degli spazi topologici fuzzy. Si dimostra una condizione necessaria e sufficiente per l'esistenza di tali decomposizioni; si definiscono decomposizioni identiche e si danno condizioni necessarie e sufficienti affinchè due decomposizioni lo siano. Si costruiscono esempi di esistenza e di non esistenza di decomposizioni.*

Abstract: *In this paper we start with certain operations on fuzzy sets. These operations provide a decomposition of fuzzy topologies. We establish necessary and sufficient condition for the existence of decomposition. We also define identical decomposition and prove necessary and sufficient condition for two decomposition to be identical. We illustrate different situations with interesting examples of decompositions.*

1 – Introduction

The notion of fuzzy topological spaces was introduced by Chang ([5]) in the year 1969. Since then research is going on rigouresly in this area of fuzzy mathematics. During this period, two prominent directions emerged. In one direction the research was carried out keeping as a model the concepts and results of general topological spaces and extending them to the framework of fuzzy setting (see [1], [6], [7], [10]). Although such

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type of study has an advantage that when succesfully carriedout, reduces the results of general topology to a particular case, it allowed the purist to raise a question that: what new fuzzy topology is doing to mathematics? In the other direction, the work is more of a philosophical nature. The studies are situated in a categorical framework. The workers in this area are in search of new categories. Attempts are being made to define categories of fuzzy topological spaces which behave as nicely as the category TOP of general topological spaces and also accomodate it as a proper subcategory (see [2], [3], [4], [9], [11]).

This paper is written to explore a new dimension in the studies of fuzzy topological spaces. Keeping in view, the requirements of both the above mentioned directions, in this preliminary paper we try to unveil certain peculiarities and specialities of fuzzy setting without providing their justification from categorical point of view. Firstly we introduce certain closure and interior operators. These operators, when allowed to act on a given fuzzy topological space can split it into two fuzzy topological spaces. This splitting is called a decomposition; since this decomposition can be obtained by every real number α in the closed unit interval, we call it an α-decomposition. We investigate conditions under which a fuzzy topology admits an α-decomposition. We also illustrate our view point with interesting examples. Then we define identical and strongly identical α-decompositions and show that they are equivalent. We also investigate necessary and sufficient conditions for a fuzzy topological space to have identical α -decomposition.

In a forthcoming paper we shall pursue these studies from categorical point view. We shall also demonstrate that these decompositions can really help us in locating certain important subcategories of Chang's category of fuzzy topological spaces.

2 – Basic concepts and results

Let X be any non empty set. We define fuzzy sets with respect to evaluation lattice the closed unit interval $I = [0, 1]$. We denote by I^X the family of all fuzzy sets in X. For $A \in I^X$, the complement $C(A)$ is defined by $C(A)(x) = 1 - A(x) \quad \forall\, x \in X$. For $A, B \in I^X$, we say A is contained in B (or B includes A) if $A(x) \leq B(x) \quad \forall\, x \in X$, and denote it by $A \subseteq B$.

For any subfamily $\{A_i\}_{i\in\Lambda}$, of I^X, the union $\bigcup_{i\in\Lambda} A_i$ and intersection $\bigcap_{i\in\Lambda} A_i$ are defined respectively by:

$$\Big(\bigcup_{i\in\Lambda} A_i\Big)(x) = \sup_{i\in\Lambda} A_i(x) \quad \forall\, x \in X\,,$$

and

$$\Big(\bigcap_{i\in\Lambda} A_i\Big)(x) = \inf_{i\in\Lambda} A_i(x) \quad \forall\, x \in X\,.$$

For the definition of a fuzzy topological space (briefly f.t.s.) we follow Chang and denote it by a pair $\langle X, \tau\rangle$ where τ is a family of fuzzy sets in X closed under arbitrary suprema and finite infima.

The constant fuzzy sets which take each member of X to zero and 1 respectively are denoted by 0_X and 1_X.

For a fixed $\alpha \in [0,1]$, we define two operations $\bar{\alpha}$ and $\underline{\alpha}$ on I^X, as follows:

$$A \longrightarrow A^{\alpha} \in I^X\,, \tag{2.1}$$

$$A \longrightarrow A_{\alpha} \in I^X\,, \tag{2.2}$$

where the fuzzy sets A^{α} and A_{α} in X are defined by:

$$A^{\alpha}(x) = \begin{cases} 1 & \text{if} \quad A(x) > \alpha \\ A(x) & \text{if} \quad A(x) \le \alpha\,; \end{cases}$$

and

$$A_{\alpha}(x) = \begin{cases} A(x) & \text{if} \quad A(x) \ge \alpha \\ 0 & \text{if} \quad A(x) < \alpha\,; \end{cases}$$

We call the first operation "upper agreement at α level" (shortly: uppag(α)) and the second "lower agreement at α level" (shortly: lowag(α)) (see [12]).

We observe that A^0 and A_1 are the characteristic functions of ordinary subsets; moreover, for ordinary subsets, we have: $A^1 = A = A_0$.

The following proposition is easy to establish:

Proposition 2.1. *Let A, B and A_i $(i \in \Lambda)$ be fuzzy sets of X. Then the following hold,* $\forall\, \alpha \in [0,1]$:

a) $(0_X)^\alpha = 0_X = (0_X)_\alpha$ *and* $(1_X)^\alpha = 1_X = (1_X)_\alpha$*;*

b) $A_\alpha \subseteq A \subseteq A^\alpha$*;*

c) *if* $A \subseteq B$ *then* $A^\alpha \subseteq B^\alpha$ *and* $A_\alpha \subseteq B_\alpha$*;*

d) $(A^\alpha)^\alpha = A^\alpha$*;* $(A_\alpha)_\alpha = A_\alpha$*;*

e) $C(A^\alpha) = (C(A))_{1-\alpha}$ *or, equivalenty,* $C(A_\alpha) = (C(A))^{1-\alpha}$*;*

f) *if* $\alpha \leq \beta$ *then* $A^\beta \subseteq A^\alpha$ *and* $(A^\alpha)^\beta = A^\alpha$*;*

g) *if* $\alpha \leq \beta$ *then* $A_\beta \subseteq A_\alpha$ *and* $(A_\beta)_\alpha = A_\beta$*;*

h) $(\bigcup_{i\in\Lambda} A_i)^\alpha = \bigcup_{i\in\Lambda}(A_i)^\alpha$ *and* $(\bigcap_{i\in\Lambda} A_i)_\alpha = \bigcap_{i\in\Lambda}(A_i)_\alpha$*;*

i) *If* Λ *is finite, then* $(\bigcap_{i\in\Lambda} A_i)^\alpha = \bigcap_{i\in\Lambda}(A_i)^\alpha$ *and* $(\bigcup_{i\in\Lambda} A_i)_\alpha = \bigcup_{i\in\Lambda}(A_i)_\alpha$;

l) $(\bigcap_{i\in\Lambda} A_i)^\alpha \subseteq \bigcap_{i\in\Lambda}(A_i)^\alpha$ *and* $\bigcup_{i\in\Lambda}(A_i)_\alpha \subseteq (\bigcup_{i\in\Lambda} A_i)_\alpha$.

Next let $\mathcal{M}$ be a family of fuzzy sets in X. Then we define the following:

$$\mathcal{M}^\alpha = \{L \in I^X \mid \exists \quad M \in \mathcal{M} \quad \text{such} \quad \text{that} \quad L = M^\alpha\};$$

$$\mathcal{M}_\alpha = \{L \in I^X \mid \exists \quad M \in \mathcal{M} \quad \text{such} \quad \text{that} \quad L = M_\alpha\}.$$

In view of proposition 2.1, we have

PROPOSITION 2.2. *Let* $\langle X, \tau\rangle$ *be the discrete f.t.s., then* $\forall\, \alpha \in [0,1]$, *the pair* $\langle X, \tau^\alpha\rangle$ *is a* f.t.s. (*with closed sets given by the family* $\tau_{1-\alpha}$).

PROOF. It follows in view of a), h), i) and e) of proposition 2.1.

Let us recall that in a f.t.s. $\langle X, \tau\rangle$, the closure and interior of a fuzzy set $A \in I^X$ are defined by

$$\bar{A} = \bigcap\{B | A \subset B, C(B) \in \tau\}, \tag{2.3}$$

and

$$A^0 = \bigcup\{B | B \subset A, B \in \tau\}. \tag{2.4}$$

Moreover, we have

Definition 2.3. An operator $\psi : I^X \longrightarrow I^X$ is a fuzzy closure operator if the following conditions are satisfied:

(i) $\psi(0_X) = 0_X$;

(ii) $A \subset \psi(A), \quad \forall A \in I^X$;

(iii) $\psi(A \cup B) = \psi(A) \cup \psi(B), \quad \forall A, B \in I^X$;

(iv) $\psi(\psi A)) = \psi(A), \quad \forall A \in I^X$.

Now we define a fuzzy set $A \in I^X$ to be closed if and only if $\psi(A) = A$. Thus each closure operation ψ on I^X determines a unique fuzzy topology $\tau(\psi)$, given by

$$\tau(\psi) = \{C(A)|\psi(A) = A\} .$$

It can be easily seen that the operator $^-$ defined by (2.3) is a closure operator. So that with a given fuzzy topology, we can associate a closure operator and vice-versa with a closure operator we can associate a fuzzy topology. Moreover, these associations are reflexive in the sense that the associated fuzzy topology of the closure operator in some fuzzy topology τ is τ itself; and the associated closure operator associated with the fuzzy topology of some closure operator ψ is ψ itself.

Dually, we define interior operator.

Definition 2.4. An operator $\theta : I^X \longrightarrow I^X$ is a fuzzy interior operator if the following conditions are satisfied:

(i) $\theta(1_X) = 1_X$;

(ii) $\theta(A) \subset A, \quad \forall A \in I^X$;

(iii) $\theta(A \cap B) = \theta(A) \cap \theta(B), \quad \forall A, B \in I^X$;

(iv) $\theta(\theta(A)) = \theta(A), \quad \forall A \in I^X$.

Clearly the operator ^{0} defined by (2.4) is a fuzzy interior operator. Also, with a fuzzy interior operator θ on I^X, we can associate a unique fuzzy topology $\tau(\theta)$, given by

$$\tau(\theta) = \{A|\psi(A) = A\} .$$

The association of fuzzy interior operator is also reflexive. To conclude, we have that for a given fuzzy topology, there is a unique associated closure operator and an interior operator and they mutually determine each other. On the other way round (for more details see [9], [10]), given an interior operator (closure operator) there is a unique associated fuzzy topology, which in turn gives back the interior operator (closure operator).

Keeping in view the above discussion, we try to investigate the behavior of operations uppag(α) and lowag(α). We have the following:

PROPOSITION 2.5. *Let* $\alpha \in [0,1]$ *be a fixed real number. Then the map* $\psi^\alpha : I^X \longrightarrow I^X$, *defined by*

$$A \longrightarrow \psi^\alpha(A) = A^\alpha$$

is a fuzzy closure operator.

PROOF. Conditions (i) and (ii) of definition 2.3, follow from proposition 2.1 (a) and (b) respectively. On the other hand, conditions (iii) and (iv) of fuzzy closure operator can be established by using (h) and (d) of the same proposition.

The fuzzy topology $\tau(\psi^\alpha)$, associated with the fuzzy closure operator ψ^α of proposition 2.5, is given by

$$\tau(\psi^\alpha) = \{C(A) | \psi^\alpha(A) = A\},$$

that is,

$$\tau(\psi^\alpha) = \{C(A^\alpha) | A \in I^X\}.$$

But $C(A^\alpha) = (C(A))_{1-\alpha}$, and since A varies over whole of I^X, $C(A)$ also varies over whole of I^X. Hence we can write the associated fuzzy topology

$$\tau(\psi^\alpha) = \{A_{1-\alpha} | A \in I^X\}.$$

Also, we have

PROPOSITION 2.6. *Let* $\alpha \in [0,1]$ *be a fixed real number. Then the map* $\theta_\alpha : I^X \longrightarrow I^X$, *defined by*

$$A \longrightarrow \theta_\alpha(A) = A_\alpha$$

is a fuzzy interior operator.

Proof. It is immediate in view of proposition 2.1.

The fuzzy topology $\tau(\theta_\alpha)$ associated with interior operator θ_α is given

$$\tau(\theta_\alpha) = \{A | \theta_\alpha(A) = A\},$$

that is

$$\tau(\theta_\alpha) = \{A_\alpha | A \in I^X\}.$$

Therefore

$$\tau(\psi^\alpha) = \tau(\theta_{1-\alpha}).$$

We denote the above fuzzy topology by $d\tau_{1-\alpha}$. Moreover, proposition 2.1 also yields the fact that the collection

$$d\tau^\alpha = \{A^\alpha | A \in I^X\}$$

is a fuzzy topology (with closed sets given by $A_{1-\alpha}$). We shall call the f.t.s. $d\tau_{1-\alpha}$ to be the $\alpha-$dual of $d\tau^\alpha$ in the discrete fuzzy topology on X.

The pair $(d\tau^\alpha, d\tau_{1-\alpha})$ constitutes a decomposition of the discrete fuzzy topology .

We illustrate our view point with a trivial example.

Let $X = \{a, b\}$ and $\alpha = \frac{1}{2}$. Define fuzzy sets A_i, B_i, $C_{i,j}$ by

$$A_i(a) = 1 \quad A_i(b) = i \qquad \forall\, i \in \left[0, \frac{1}{2}\right]$$

$$B_j(b) = 1 \quad B_j(a) = j \qquad \forall\, j \in \left[0, \frac{1}{2}\right]$$

$$C_{i,j}(a) = i \quad C_{i,j}(b) = j \qquad \forall\, i, j \in \left[0, \frac{1}{2}\right].$$

Then $d\tau^\alpha = \{A_i, B_j, C_{i,j} |\, \forall\, i, j \in [0, \frac{1}{2}]\}$. On the other hand, we define:

$$A'_i(a) = 0 \qquad A'_i = i \qquad \forall\, i \in \left[\frac{1}{2}, 1\right],$$

$$B'_j(b) = 0 \qquad B'_j = j \qquad \forall\, j \in \left[\frac{1}{2}, 1\right],$$

$$C'_{i,j}(a) = i \quad C'_{i,j}(b) = j \qquad \forall\, i, j \in \left[\frac{1}{2}, 1\right].$$

Then $\quad d\tau_{1-\alpha} = \{A'_i, B'_j, C'_{i,j} |\, \forall\, i, j \in [\frac{1}{2}, 1]\}$.

In case of arbitrary fuzzy topological spaces the situation turns out to be very interesting and it suggests a new direction in the study of fuzzy topologies. Let $\langle X, \tau \rangle$ be a f.t.s. where the closed sets are given by the family σ. Let $\alpha \in [0,1]$ be fixed. Then, in view of proposition 2.1, we immediately have the following:

PROPOSITION 2.7. *The system* $\langle X, \tau^\alpha \rangle$ *is a* f.t.s. (*where closed sets are given by the family* $\sigma_{1-\alpha}$).

It seems to be a rewarding exercise to determine the closure (interior) operator associated with the f.t.s. $\langle X, \tau^\alpha \rangle$ and its relationship with closure (interior) operator of $\langle X, \tau \rangle$.

Let $A \in I^X$, and let us denote by $A^{(0)}$ the interior of A in the given f.t.s. $\langle X, \tau \rangle$ and by $A^{(o),\alpha}$ the interior of A in the f.t.s. $\langle X, \tau^\alpha \rangle$. Then the following facts are immediate and are shown by the diagram

$$\begin{array}{ccc} A^{(o)} & \xrightarrow{} & A \\ & (1) & \\ \Big\downarrow & (4) \qquad (2) & \Big\downarrow \\ & (3) & \\ (A^{(o)})^\alpha & \xrightarrow{} & A^\alpha \end{array}$$

where the arrows denote inclusion relations.

The arrow (2) yields that, $A^{(0)} \subseteq (A^\alpha)^{(0)}$; this alongwith arrow (4) imply that

$$A^{(0)} \subseteq (A^{(0)})^\alpha \cap (A^\alpha)^{(0)}\,.$$

Next, consider the families $\{A_i\}_{i \in \Lambda}$ and $\{A_j\}_{j \in \Gamma}$ of open sets of τ, given by

$$\{A_i\}_{i \in \Lambda} = \{A_i \in \tau | A_i \subset A\}, \quad \{A_j\}_{j \in \Gamma} = \{A_j \in \tau | A_j^\alpha \subset A\}\,.$$

Then $\{A_j\}_{j \in \Gamma} \subset \{A_i\}_{i \in \Lambda}$, since we have $A_j \subset A_j^\alpha \ \forall\, j \in \Gamma$. But

$$A^{(0),\alpha} = \bigcup_{j \in \Gamma} A_j^\alpha\,, \quad A^{(0)} = \bigcup_{i \in \Lambda} A_i\,.$$

Now, since: $\bigcup_{j \in \Gamma} A_j^\alpha \subseteq \bigcup_{i \in \Lambda} A_i^\alpha = (\bigcup_{i \in \Lambda} A_i)^\alpha$, we have

$$A^{(0),\alpha} \subseteq (A^{(0)})^\alpha\,.$$

The following example shows that the above inclusion is proper.

Example. Let $X = [0,1]$. Consider the fuzzy topology τ, given by

$$\tau = \{1_X, 0_X\} \cup \{A \in I^X | A(x) \leq x \quad \forall\, x \in X\}\,.$$

Now choose an arbitrary $\alpha \in]0,1[$ and fix it, then choose $\beta \in]0,\alpha[$. Define a fuzzy set η in X by

$$\eta(x) = \frac{\alpha - \beta}{\alpha} x + \beta \quad \forall\, x \in X\,.$$

Then $(\eta^\alpha)^{(0)} = C$, where C is the fuzzy set given by $C(x) = x \quad \forall\, x \in X$, whereas

$$\eta^{(0),\alpha} = \cup\{A_i^\alpha \in \tau^\alpha | A_i^\alpha \subset \eta\}\,.$$

Since the upper bound of η is $\frac{\alpha-\beta}{\alpha} + \beta \neq 1$, then

$$\eta^{(0),\alpha}(x) \leq \frac{\alpha - \beta}{\alpha} + \beta \quad \forall\, x \in X\,.$$

But we have $\quad \frac{\alpha-\beta}{\alpha} + \beta < C(x) \quad \forall\, x \in]\frac{\alpha-\beta}{\alpha} + \beta, 1]\,.$

3 – A decomposition of fuzzy topological Spaces

In section 2, we witnessed a decomposition of the discrete fuzzy topological spaces on a given set X. We also noticed that this decomposition can be obtained for every real number α in the closed unit interval. The decomposed fuzzy topological spaces of a f.t.s. $\langle X, \tau\rangle$ are denoted by $d\tau^\alpha$, $d\tau_{1-\alpha}$ respectively. In this section, we intend to carry out this process of decomposition to arbitrary fuzzy topological spaces. These studies are certainly going to throw more light on the inner structure of a given fuzzy topology. Moreover, the investigation of such type of properties points towards the peculiarities of fuzzy setting which is, so far, a neglected area of research work.

Before we venture for such type of investigations, let us first fix concretly the meaning of decompositions.

Definition 3.1. Let $\langle X,\tau\rangle$ be a f.t.s. and let $\langle X,\delta\rangle$, $\langle X,\Delta\rangle$ be fuzzy topological spaces, such that

$$\delta=\tau^{\alpha}\,,\quad \Delta=\tau_{1-\alpha}\,,$$

for some $\alpha\in[0,1]$. Then we say that the pair (δ,Δ) is an *α-decomposition* of the f.t.s. $\langle X,\tau\rangle$.

We observe that any f.t.s. $\langle X,\tau\rangle$ has 1-decomposition. In fact, for $\alpha=1$ we have

$$\tau^{1}=\tau=\tau_{0}\,.$$

We call an α-decomposition (δ,Δ) to be trivial if we have

$$\delta=\tau=\Delta\,.$$

With this view of decomposition, we claim that those fuzzy topologies are in abundance which admit $\alpha-$decomposition for each $\alpha\in[0,1]$. We have already seen that a discrete fuzzy topological space is always of this kind. In order to further substantiate our argument we let Im A denote the range set of a fuzzy set $A\in I^{X}$. In the sequel we also need the following notation

$$\text{Im }\tau=\bigcup_{A\in\tau}\text{Im }A\,.$$

Now as a straightforward consequence of our proposition 2.1, we present the following:

Proposition 3.2. *Let* $< X,\tau >$ *be a* f.t.s. *with finitely many open sets or* Card Im $\tau<+\infty$. *Then,* $\forall\,\alpha\in[0,1]$, *the* f.t.s. $\langle X,\tau\rangle$ *has an* $\alpha-$*decomposition.*

Now before proceeding furhter in our investigations, we provide an example of α-decomposition which demonstrates that the class of α-decomposable fuzzy topologies is not limited to the class discussed in proposition 3.2.

Example. Let $X=N$. Let us choose arbitrary numbers α, $\beta\in]0,1]$ such that $\alpha<\beta$. Then consider the family $\{A_i\}_i$ of all those sequences whose subsequences of even integers has upper bound α and that of odd integers has lower bound β . Now, it is immediate that the family τ given by

$$\tau = \{1_X,\, 0_X\} \cup \{A_i\}_i$$

is a fuzzy topology. We show here that the system $\langle X, \tau_\alpha \rangle$ is a f.t.s. So, let $\{A_i\}_{i\in\Lambda}$ be any family of open sets of τ . Then, we know that $\forall\, i \in \Lambda$

$$A_i = (x_n^i)_{n\in N} \text{ where } x_{2n}^i \le \alpha\ ,\ x_{2n+1}^i \ge \beta \quad \forall\, n \in N\,.$$

Now, if

$$(*) \qquad \bigcup_{i\in\Lambda} (A_i)_\alpha = (\bigcup_{i\in\Lambda} A_i)_\alpha\,,$$

then there is nothing to prove since the union $\bigcup_{i\in\Lambda} A_i$ is an open set in τ and hence $(\bigcup_{i\in\Lambda} A_i)_\alpha$ is a member of the class τ_α by the very nature of operation lowag(α) .

On the other hand, if the equality $(*)$ does not hold then in view of the fact that

$$\bigcup_i (A_i)_\alpha \subset (\bigcup_i A_i)_\alpha\,,$$

we have

$$(**) \qquad (\cup(A_i)_\alpha)(x) < (\bigcup_i A_i)_\alpha(x) \quad \text{for some} \quad x \in N\,.$$

The integer x in this case has to be even. For if x is an odd integer then the values of both the fuzzy sets $\bigcup_i (A_i)_\alpha$ and $\cup A_i$ will certainly exceed α, since it exceed β and hence the equality of $(*)$ will hold. Let us denote by

$$N^* = \{x = 2m | (\bigcup_i (A_i)_\alpha)(x) < (\bigcup_i A_i)_\alpha(x)\}\,.$$

Now, for a given $x \in N^*$, we write

$$(\bigcup_i A_i)_\alpha(x) = k_x\,.$$

Then

$$(***) \qquad \sup_i \{(A_i)_\alpha(x)\} < k_x\,.$$

Therefore $(A_i)_\alpha(x) < k_x \quad \forall\, i \in \Lambda$. Also, since the integer x is even, $(\bigcup_i A_i)(x) \le \alpha$. Hence, we have

$$(\bigcup_i A_i)_\alpha(x) = 0 \text{ or } \alpha\,.$$

So by $(***)$ we obtain that $k_x = \alpha$, and thus $(A_i)_\alpha < \alpha \quad \forall\, i \in \Lambda$. This implies $A_i(x) < \alpha$ and hence $(A_i)_\alpha = 0 \quad \forall\, i \in \Lambda$. Define a fuzzy set $\hat{A}$ in N by

$$\hat{A}(x) = (\bigcup_i A_i)(x) \qquad \forall\, x \in N - N^*$$

$$\hat{A}(x) = \alpha_x\,,\ \alpha_x \in \left]0, \frac{1}{2}\right[\quad \forall\, x \in N^*\,.$$

It is obvious that $\hat{A}$ is an open set in τ . Also, it can be easily seen that

$$\bigcup_{i\in\Lambda} (A_i)_\alpha = \hat{A}_\alpha\,.$$

Therefore $\bigcup_i (A_i)_\alpha$ is a member of τ_α and hence τ_α is closed under arbitrary unions. Also by proposition 2.1, τ_α is closed under finite intersections.

Our next pursuit is to remove the strong conditions imposed on fuzzy topological spaces in proposition 3.2. We know, by proposition 2.2, that for an arbitrary f.t.s. $\langle X, \tau\rangle$, the system $\langle X, \tau^\alpha\rangle$ is always a f.t.s for each $\alpha \in [0,1]$. On the other hand, the exact nature of the system $\langle X, \tau_{1-\alpha}\rangle$ demands our closer attention.

Before, we put forward a characterization for the system $\langle X, \tau_{1-\alpha}\rangle$ to be a f.t.s., let us formulate the following definitions:

Definition 3.4. A family $\{A_i\}_{i\in\Lambda} \subset I^X$ is said be *α-reachable*, for $\alpha \in]0,1]$, if for some $x \in X$

$$A_i(x) < \alpha \quad \forall\, i \in \Lambda \quad \text{and} \quad \sup_i A_i(x) = \alpha\,.$$

Definition 3.5 Let $\{A_i\}_{i\in\Lambda} \subset I^X$ be an α-reachable family for some $\alpha \in]0,1]$. Then the set X^* of *α-terminal points* of $\{A_i\}_{i\in\Lambda}$ we define by

$$X^* = \{x \in X |\ A_i(x) < \alpha \quad \forall\, i \in \Lambda\,, \qquad \sup_i A_i(x) = \alpha\}\,.$$

Moreover, we call a fuzzy set $\hat{A} \in I^X$ a *cofinal set* for $\{A_i\}_{i \in \Lambda}$ if

$$\hat{A}(x) = \begin{cases} \beta_x \in [0, \alpha[& \text{if} \quad x \in X^* \\ (\bigcup_i A_i)(x) & \text{if} \quad x \in X - X^*\,. \end{cases}$$

Now we are in a position to present the following

THEOREM 3.6. *Let* $\langle X, \tau \rangle$ *be any* f.t.s. *and* $\alpha \in]0, 1]$. *Then* $< X, \tau_\alpha >$ *is a* f.t.s. *if and only if every* α*−reachable family* $\{A_i\}_{i \in \Lambda}$ *of open sets in* τ *has an cofinal open set in* τ.

PROOF. Suppose $\langle X, \tau_\alpha \rangle$ is a f.t.s. for a fixed $\alpha \in]0, 1]$. Let $\{A_i\}_{i \in \Lambda}$ be any $\alpha-$reachable family of open sets in τ. Then by definition of $< X, \tau_\alpha >$, the family $\{(A_i)_\alpha\}_i$ is of open sets in τ_α. Therefore, the union $\bigcup_i (A_i)_\alpha$ is an open set in τ_α. Hence, there exists $\hat{A} \in \tau$ such that

$$(*) \qquad \bigcup_i (A_i)_\alpha = \hat{A}_\alpha\,.$$

We show that $\hat{A}$ is a cofinal set for the family $\{A_i\}_{i \in \Lambda}$. That is

$$\hat{A}(x) = \beta_x \in [0, \alpha[\quad \forall\, x \in X^* \quad \text{and} \quad \hat{A}(x) = (\bigcup_i A_i)(x) \quad \forall\, x \in X - X^*\,,$$

where X^* is the set of $\alpha-$terminal points of the family $\{A_i\}_{i \in \Lambda}$. Since the family $\{A_i\}_{i \in \Lambda}$ is $\alpha-$reachable, $\forall\, x \in X^*$ and $\forall\, i \in \Lambda$ we have

$$(**) \qquad A_i(x) < \alpha \quad \text{and} \quad \sup_i A_i(x) = \alpha\,.$$

Therefore $(A_i)_\alpha(x) = 0 \quad \forall\, x \in X^*$. So that $(\bigcup_i (A_i)_\alpha)(x) = 0 \quad \forall\, x \in X^*$. Consequently, $(*)$ implies that

$$\hat{A}_\alpha(x) = \beta_x \quad \forall\, x \in X^* \quad \text{where}\, \beta_x \in [0, \alpha[\,.$$

Also, for a given $x \in X - X^*$, in view of definition of X^* we have, either

$$\sup_i A_i(x) < \alpha \quad \text{or} \quad A_i(x) \geq \alpha \quad \text{for some } i \in \Lambda\,.$$

In case there exists $i \in \Lambda$ such that $A_i(x) \geq \alpha$ then $(A_i)_\alpha(x) \geq \alpha$. Therefore

$$(\bigcup_i (A_i)_\alpha)(x) = \sup_i (A_i)_\alpha(x) = \sup_i A_i(x) = (\bigcup_i A_i)(x)\,.$$

Again, by $(*)$ we have

$$(\bigcup_i (A_i)_\alpha)(x) = \hat{A}_\alpha(x) = (\bigcup_i A_i)(x)\,.$$

And in case $\sup_i A_i(x) < \alpha$, we have $A_i(x) < \alpha \; \forall\, i \in \Lambda$. Therefore

$$(\bigcup_i A_i)(x) = 0 = (\bigcup_i (A_i)_\alpha)(x)\,.$$

So in both cases we have

$$\hat{A}_\alpha(x) = (\bigcup_i A_i)(x) \quad \forall\, X \in X - X^*\,.$$

Therefore $\hat{A}$ is a cofinal set for the family $\{A_i\}_{i\in\Lambda}$.

Conversely suppose that every $\alpha-$reachable family $\{A_i\}$ of open sets in τ has an open cofinal set in τ. We have to show that $\langle X, \tau_\alpha\rangle$ is a f.t.s. Now τ_α is closed under finite intersections follows as a consequence of proposition 2.1 . In order to show that τ_α is closed under arbitrary unions, let $\{(A_i)_\alpha\}_{i\in\Lambda}$ be any family of fuzzy sets in τ_α. Therefore $\{A_i\}_{i\in\Lambda}$ is a family of open sets in τ. Now, if $\{A_i\}_{i\in\Lambda}$ is an $\alpha-$reachable family then there exists an open cofinal set $\hat{A} \in \tau$. Therefore

$$\begin{aligned} \hat{A}(x) &= \beta_x \quad \forall\, x \in X^*\,, \quad \text{where } \beta_x \in [0, \alpha[\\ &= (\bigcup_i A_i)(x) \quad \forall\, x \in X - X^*\,, \end{aligned}$$

where X^* is the set of $\alpha-$terminal points of the family $\{A_i\}_{i\in\Lambda}$. Now, we show that

$$\bigcup_{i\in\Lambda} (A_i)_\alpha = \hat{A}_\alpha\,,$$

and thus the union of $\{(A_i)_\alpha)_i\}$ is member of τ_α. To this end, let us first choose $x \in X^*$, then $A_i(x) < \alpha \; \forall\, i \in \Lambda$ and hence $(A_i)_\alpha(x) = 0 \quad \forall\, i \in \Lambda$. This implies $(\bigcup_i (A_i)_\alpha)(x) = 0$. Also $\hat{A}_\alpha(x) = 0$, since $\beta_x < \alpha$. Therefore

$$(\bigcup_i (A_i)_\alpha(x) = \hat{A}_\alpha(x) \quad \forall\, x \in X^* .$$

On the other hand, if we choose $x \in X - X^*$, then either

$$\sup_i A_i(x) < \alpha \,\text{or}\, A_i(x) \geq \alpha \,\text{for some}\, i \in \Lambda .$$

In case $\sup_i A_i(x) < \alpha$, we have $A_i(x) < \alpha \; \forall\, i \in \Lambda$ and thus $(A_i)_\alpha(x) = 0$ $\forall\, i \in \Lambda$. So that $\sup_i (A_i)_\alpha(x) = 0$. This implies

$$(\bigcup_i (A_i)_\alpha)(x) = 0 = (\bigcup_i A_i)(x) .$$

In case, for some $i \in \Lambda$, $A_i(x) \geq \alpha$, we have then $(A_i)_\alpha(x) \geq \alpha$. Therefore, it follows that

$$(\bigcup_i (A_i)_\alpha)(x) = (\bigcup_i A_i)(x) .$$

Consequently in both cases, we have

$$(\bigcup_i (A_i)_\alpha)(x) = (\bigcup_i A_i)(x) \quad \forall\, x \in X - X^* .$$

Therefore we proved that $\bigcup_i (A_i)_\alpha = \hat{A}_\alpha$. The proof of the sufficiency part will be complete if we show that, also in case when $\{A_i\}_i$ is not an $\alpha-$reachable family, the union of $\{(A_i)_\alpha\}$ is in τ_α. So, suppose $\{A_i\}_i$ is not $\alpha-$reachable, then $X^* = \emptyset$. Therefore for every $x \in X$ either $\sup_i A_i(x) < \alpha$ or for some $i \in \Lambda \quad A_i(x) \geq \alpha$. In both cases, we have as in earlier part of the proof that

$$(\bigcup_i (A_i)_\alpha)(x) = (\bigcup_i A_i)_\alpha(x) .$$

Hence $\bigcup_i (A_i)_\alpha = (\bigcup_i A_i)_\alpha$. But $(\bigcup_i A_i)_\alpha$ is a member of the family τ_α, since $\bigcup_i A_i$ is an open set in τ. Thus τ_α is closed under arbitrary unions. This completes the proof of the theorem.

Remark. After going through the proof of the above theorem, it follows immediatly that no finite family of fuzzy sets can be $\alpha-$reachable for any α in the closed unit interval. This allows us to look at proposition 3.2 as a corollary of theorem 3.5.

The characterization presented in the above theorem gives us the impression that for a fixed α the class of α-decomposable fuzzy topological spaces is a proper subclass of the class of all fuzzy topological spaces. Indeed, there are situations, where for some given α and a given f.t.s. $\langle X, \tau \rangle$, the system $\langle X, \tau_\alpha \rangle$ is not a topological space; consequently there is no α- decomposition for $\langle X, \tau \rangle$.

Counterexample. Let $X = R^+$. Now choose any $\alpha \in]0,1]$ and fix it. Then $\forall \beta \in]\frac{1}{2}, 1]$, define a fuzzy set μ_β in X by

$$\mu_\beta(x) = 1 \qquad \forall x \geq 2,$$

$$\mu_\beta(x) = 0 \qquad \forall x \in \Big[\frac{1}{2}, \beta\Big[\cup \Big[2-\beta, \frac{3}{2}\Big[,$$

$$\mu_\beta(x) = 1 - 2x \quad \forall x \in \Big[0, \frac{1}{2}\Big[\text{ and } \mu_\beta(x) = 2x - 3 \quad \forall x \in \Big[\frac{3}{2}, 2\Big[.$$

For the remaining part of R^+, define

$$\begin{aligned} \mu_\beta(x) &= 2\alpha(x-\beta) & \forall x \in [\beta, 1[\\ \mu_\beta(x) &= 2\alpha(2-\beta-x) & \forall x \in [1, 2-\beta[. \end{aligned}$$

Thus the functions μ_β are defined for whole R^+. Now, let τ be the fuzzy topology with subbase

$$\langle\, \{1_X, O_X\} \cup \Big\{\mu_\beta \,|\, \beta \in \Big]\frac{1}{2}, 1\Big]\Big\}\,\rangle.$$

Consider the family $\mathcal{P}$ of open sets of τ, given by

$$\mathcal{P} = \Big\{\mu_\beta \,|\, \beta \in \Big]\frac{1}{2}, 1\Big]\Big\}.$$

Then the union of the family $\mathcal{P}$ is an open set of τ . Write

$$U = \bigcup_{\mu_\beta \in \mathcal{P}} \mu_\beta \,,$$

where the fuzzy set U in R^+ is given by

$$U(x) = 1 - 2x \qquad \forall\, x \in \Big[0, \frac{1}{2}\Big[, \quad U(x) = \alpha(2x-1) \quad \forall\, x \in \Big[\frac{1}{2}, 1\Big[$$

$$U(x) = \alpha(3-2x) \quad \forall\, x \in \Big[1, \frac{3}{2}\Big[, \quad U(x) = 2x - 3 \qquad \forall\, x \in \Big[\frac{3}{2}, 2\Big[$$

and $U(x) = 1 \quad \forall\, x \geq 2$.

Now, it can be easily verified that the family $\mathcal{P}$ is α-reachable for our choosen α . Moreover the set X^* of α-reachable points of $\mathcal{P}$ is $\{1\}$, the singleton. Also, it is routine to verify that the family $\mathcal{P}$ does not have any open cofinal set in the fuzzy topology τ . Thus by theorem 3.5, the system $\langle X, \tau_\alpha \rangle$ is not a f.t.s.

Remark. In the above counterexample, if we let $r = 1 - \alpha$ then the fuzzy topological space $< X, \tau >$ discussed therein does not have a $r-$decomposition, since τ_{1-r} is not a fuzzy topology.

Example 3.3, provides us a view of the fact that for different real numbers α and β , a fuzzy topological space may have same decomposition. Therefore, we can formulate

Definition 3.7. Let $\langle X, \tau \rangle$ be a f.t.s. having decompositions for distinct real numbers α , $\beta \in [0, 1]$ such that

$$\tau^\alpha = \tau^\beta \quad \text{and} \quad \tau_{1-\alpha} = \tau_{1-\beta} \,.$$

Then, we say that the α-decomposition of $< X, \tau >$ is *identical* with its β-decomposition. Moreover, if in addition $\forall\, A \in \tau$

$$A^\alpha = A^\beta \quad \text{and} \quad A_{1-\alpha} = A_{1-\beta} \,,$$

then the decompositions are called *strongly identical.*

The following proposition show the previuos concepts are equivalent.

PROPOSITION 3.8. *Let $\langle X, \tau \rangle$ be a f.t.s. having decompositions for distinct real numbers α, $\beta \in [0,1]$. Then, the α-decomposition of $\langle X, \tau \rangle$ is identical with its β-decomposition if and only if they are strongly identical.*

PROOF. Sufficiency is obvious. To show necessity, let α, $\beta \in [0,1]$ such that $0 \leq \alpha \leq \beta \leq 1$. Since the α-decomposition of $\langle X, \tau \rangle$ is identical with β-decomposition, we have

$$(*) \qquad \tau^\alpha = \tau^\beta \quad \text{and} \quad \tau_{1-\alpha} = \tau_{1-\beta}\,.$$

In order to show that the decompositions are strongly identical, we show that $\forall\, B \in \tau$

$$B^\alpha = B^\beta \quad \text{and} \quad B_{1-\alpha} = B_{1-\beta}\,.$$

So, let $B \in \tau$. Then by $(*)$, we have for some $A \in \tau$

$$A^\alpha = B^\beta\,.$$

Consider the following cases.

Case (i) $0 < \alpha < \beta < 1$. Now, let $x \in X$. Then, either $B(x) \leq \alpha$ or $B(x) > \alpha$. If $B(x) \leq \alpha$ then $B(x) \leq \alpha < \beta < 1$ imply that

$$B(x) = B^\beta(x) = A^\alpha(x) = A(x)\,.$$

So that $B^\alpha(x) = A^\alpha(x)$ and thus $B^\alpha(x) = B^\beta(x)$. Again, if $B(x) > \alpha$ then either

$$B(x) > \beta \quad \text{or} \quad \alpha < B(x) \leq \beta\,.$$

Now, if $B(x) > \beta$ then $B(x) > \alpha$ and thus $B^\alpha = 1$, and also $B^\beta(x) = 1$.

On the other hand, if $\alpha < B(x) \leq \beta < 1$ then $B(x) \leq \beta$ imply that $B(x) = B^\beta(x)$. But $B^\beta(x) = A^\alpha(x)$. Therefore $A^\alpha(x) \neq 1$ and hence $A^\alpha(x) = A(x)$. So we have

$$B(x) = B^\beta(x) = A(x) \leq \alpha < B(x)\,.$$

This is a contradiction and hence in all possible cases, we have $B^\alpha(x) = B^\beta(x) \quad \forall\, x \in X$. Therefore $B^\alpha = B^\beta$. A similar argument yeild that $B_{1-\alpha} = B_{1-\beta} \quad \forall\, B \in \tau$.

Case (ii) $0 = \alpha < \beta < 1$. Then by $(*)$, we have $A^0 = B^\beta$. This leads to the fact that $B(x) > \beta$ if $A(x) > 0$ and $B(x) = 0$ if $A(x) = 0$. Consequently we have $B^0 = B^\beta$.

Case (iii) $0 < \alpha < \beta = 1$. Again by $(*)$, we have $A^\alpha = B^1$. But $B^1 = B$ and thus $B(x) = 1$ if $A(x) > \alpha$ and $B(x) = A(x)$ if $A(x) \leq \alpha$. Therefore, $B^\alpha = B^1$.

Case (iv) $0 = \alpha < \beta = 1$. Then by definition, we have $B^0 = B^1$.

A dual argument can be used in the four cases to show that $B_{1-\alpha} = B_{1-\beta}$. This establishes the necessity part of the proposition.

For a fixed $\alpha \in [0,1]$, we define upper $\alpha - cut$ $\mathrm{Im}_{\alpha^+}\tau$ and lower $\alpha - cut$ $\mathrm{Im}_{\alpha^-}\tau$ of $\mathrm{Im}\tau$, as follows

$$\mathrm{Im}_{\alpha^+}\tau = \{t \in \mathrm{Im}\,\tau \,|\, \alpha < t\},$$

$$\mathrm{Im}_{\alpha^-}\tau = \{t \in \mathrm{Im}\,\tau \,|\, t < \alpha\}.$$

Then we write

$$\varsigma_\alpha = \inf_{t \in \mathrm{Im}_{\alpha^+}\tau} \{t\},$$

$$\xi_\alpha = \sup_{t \in \mathrm{Im}_{\alpha^-}\tau} \{t\}.$$

So that $\alpha \in [\xi_\alpha, \varsigma_\alpha]$.

Finally, we are in a position to establish necessary and sufficient conditions for a decomposition to be identical with a given α-decomposition.

THEOREM 3.9. *Let* $< X, \tau >$ *be a f.t.s. with an* $\alpha-$*decomposition* $(\tau^\alpha, \tau_{1-\alpha})$ *for some* $\alpha \in [0,1]$. *Then* $\forall\, \alpha_i \in [\xi_\alpha, \varsigma_\alpha[$ *and* $\forall\, \beta_i \in]\xi_\beta, \varsigma_\beta]$, *the pair* $(\tau^{\alpha_i}, \tau_{\beta_i})$ *constitues a decomposition which is identical with* α*-decomposition* $(\tau^\alpha, \tau_{1-\alpha})$, *where* $\beta = 1 - \alpha$, and $\beta \neq \xi_\beta$, $\alpha \neq \varsigma_\alpha$.

PROOF. Firstly, we show that $\forall\, \alpha_i \in [\xi_\alpha, \varsigma_\alpha[$ the fuzzy topologies τ^{α_i} and τ^α are identical. So, let us choose $\alpha_i \in [0,1]$, such that $\xi_\alpha \leq \alpha_i < \varsigma_\alpha$. Then

$$A^{\alpha_i} \subseteq A^{\xi_\alpha} \qquad \forall\, A \in \tau.$$

Infact, we now prove that $A^{\alpha_i} = A^{\xi_\alpha}$ $\forall\, A \in \tau$. Suppose, on the contrary, for some $A \in \tau$, $A^{\alpha_i} \neq A^{\xi_\alpha}$. Then, there exists $x \in X$, such that

$$A^{\alpha_i}(x) < A^{\xi_\alpha}(x).$$

So that $A(x)$ cannot exceed α_i , and thus $A(x) \leq \alpha_i$. Therefore $A^{\alpha_i}(x) = A(x)$. This implies $A^{\xi_\alpha}(x) = 1$ and hence $A(x) > \xi_\alpha$. Consequentely

$$\xi_\alpha < A(x) \leq \alpha_i < \varsigma_\alpha \,.$$

But this contradicts the fact that the open interval $]\xi_\alpha, \varsigma_\alpha[$ intersects with Imτ trivially .

Thus we have established that $A^{\alpha_i} = A^{\xi_\alpha} \quad \forall\, A \in \tau$. Hence the topologies τ^α and τ^{ξ_α} are identical. But this is true for each α_i in the interval $[\xi_\alpha, \varsigma_\alpha[$, in particular for $\alpha_i = \alpha$. Consequently, the topologies τ^{α_i} and τ^α are identical.

Next, we show that $\forall\, \beta_i \in]\xi_\beta, \varsigma_\beta]$, where $\beta = 1 - \alpha$, the fuzzy topologies τ_{β_i} and τ_β are also identical. So, choose any real number $\beta_i \in [0,1]$ and satisfying $\xi_\beta < \beta_i \leq \varsigma_\beta$.

Then $\forall\, A \in \tau$, we have

$$A_{\varsigma_\beta} \subseteq A_{\beta_i} \,.$$

Now, we prove that $A_{\varsigma_\beta} = A_{\beta_i} \quad \forall\, A \in \tau$. Suppose, if possible for some $A \in \tau$, $A_{\varsigma_\beta} \neq A_{\beta_i}$. Then, there exists $x \in X$ satisfiying

$$A_{\varsigma_\beta}(x) < A_{\beta_i}(x) \,.$$

So that $A_{\beta_i}(x)$ is non zero and hence $A_{\beta_i}(x) \geq \beta_i$, thus $A_{\beta_i}(x) = A(x)$. But $A_{\varsigma_\beta}(x) = 0$ or $A(x)$. This implies that the only possibility is $A(x) < \varsigma_\beta$. Consequentely, we have the following inequality

$$\xi_\beta < \beta_i \leq A(x) < \varsigma_\beta(x) \,.$$

Again since the open interval $]\xi_\alpha, \varsigma_\beta[$ intersect Im τ trivially, we are assure of a contradiction. Therefore the topologies τ_{β_i} and τ_{ς_β} contains the same open sets and hence are identical. Again, since this argument is valid for each β_i in the interval $]\xi_\beta, \varsigma_\beta]$, it is also valid for $\beta_i = \beta$. Therefore, we have

$$\tau_{\beta_i} = \tau_\beta = \tau_{\varsigma_\beta} \quad \forall\, \beta_i \in]\xi_\beta, \varsigma_\beta] \,.$$

In the earlier part, it is shown that

$$\tau^{\alpha_i} = \tau^{\alpha} = \tau^{\xi_\alpha} \quad \forall\, \alpha_i \in [\xi_\alpha, \varsigma_\alpha[\,.$$

Combining the above two equations, we get that $(\tau^{\alpha_i}, \tau_{\beta_i})$ is the $\alpha-$decomposition $(\tau^{\alpha}, \tau_{1-\alpha})$.

THEOREM 3.10. *Let* $\langle X, \tau\rangle$ *be a* f.t.s. *with the property that* Im τ *is lower well ordered. Then*

$$\alpha_i \in [\xi_\alpha, \varsigma_\alpha[\quad \textit{and} \quad \beta_i \in [\xi_\beta, \varsigma_\beta]\,,$$

if the pair $(\tau^{\alpha_i}, \tau_{\beta_i})$ *constitues a decomposition which is identical with* $\alpha-$*decomposition* $(\tau^{\alpha}, \tau_{\beta})$, *where* $\beta = 1-\alpha$, *and* $\alpha \neq \varsigma_\alpha$ *and* $\beta \neq \xi_\beta$.

PROOF. Suppose that for given α_i, $\beta_i \in [0,1]$, the pair $(\tau^{\alpha_i}, \tau_{\beta_i})$ is a decomposition of $\langle X, \tau\rangle$, identical with $(\tau^{\alpha}, \tau_{\beta})$. Then

$$(*) \qquad A^{\alpha_i} = A^{\alpha} \quad \forall\, A \in \tau\,.$$

Now, suppose if possible $\alpha_i \notin [\xi_\alpha, \varsigma_\alpha[$. Then, either

$$\alpha_i < \xi_\alpha \quad \text{or} \quad \varsigma_\alpha \leq \alpha_i\,.$$

If $\varsigma_\alpha \leq \alpha_i$ then $\forall\, A \in \tau$, we have

$$A^{\alpha_i} \subseteq A^{\varsigma_\alpha}\,.$$

By the hypothesis, Im τ is well ordered, therefore the upper α-cut $\mathrm{Im}_{\alpha^+}\tau$ being a subset of Im τ has a least element, say t_1. Now, since $t_1 \in \mathrm{Im}_{\alpha^+}\tau$, for some $\hat{A} \in \tau$ and $x_1 \in X$, we have

$$\hat{A}(x_1) = t_1 \quad \text{and} \quad \varsigma_\alpha = t_1\,.$$

Therefore, the following inequality

$$\xi_\alpha \leq \alpha < \varsigma_\alpha \leq \alpha_i$$

is obvious, since $\alpha \neq \varsigma_\alpha$. As $\hat{A}(x_1) = \varsigma_\alpha > \alpha$, by the definition of operation uppag(α) , we have $\hat{A}^\alpha(x_1) = 1$. Also, $\hat{A}(x_1) = \varsigma_\alpha \leq \alpha_i$ imply that $\hat{A}^{\alpha_i}(x_1) = \varsigma_\alpha$.

Hence, we have

$$\hat{A}^{\alpha_i}(x_1) < \hat{A}^\alpha(x_1) .$$

Therefore $\hat{A}^{\alpha_i} \neq \hat{A}^\alpha$. On the other hand, if $\alpha_i < \xi_\alpha$, then there exists $t_0 \in \mathrm{Im}_{\alpha^-}\tau$ such that $t_0 > \alpha_i$. Then we get the following inequality

$$\alpha_i < t_0 \leq \xi_\alpha \leq \alpha .$$

Now, since $t_0 \in \mathrm{Im}_{\alpha^-}\tau$, for some $\check{A} \in \tau$ and $x_0 \in X$, $\check{A}(x) = t_0$. So, we have $\check{A}^{\alpha_i}(x_0) = 1$ and $\check{A}^\alpha(x_0) = t_0$. So that we have

$$\check{A}^\alpha(x_0) < \check{A}^{\alpha_i}(x_0) .$$

Therefore $\check{A}^{\alpha_i} \neq \check{A}^\alpha$. Consequently in both cases we contradict $(*)$. Hence $\alpha_i \in [\xi_\alpha, \varsigma_\alpha[$. Again, since the pair $(\tau^{\alpha_i}, \tau_{\beta_i})$ is identical decomposition with $(\tau^\alpha, \tau_\beta)$ we have

$$(**) \qquad A_{\beta_i} = A_\beta \quad \forall\, A \in \tau .$$

Moreover, since $\xi_\beta < \beta \leq \varsigma_\beta$, we can show that

$$A_\beta = A_{\varsigma_\beta} \quad \forall\, A \in \tau .$$

We have to prove $\beta_i \in [\xi_\beta, \varsigma_\beta]$. For, if not, then

$$\beta_i < \xi_\beta \quad \text{or} \quad \varsigma_\beta < \beta_i .$$

Now, if $\beta_i < \xi_\beta$ then, there exists $t_0 \in \mathrm{Im}_{\beta^-}\tau$, such that

$$\beta > t_0 > \beta_i .$$

Since $t_0 \in \mathrm{Im}_{\beta^-}\tau$, there exists $\hat{A} \in \tau$ and $x_0 \in X$, such that $\hat{A}(x_0) = t_0$. This in view of the definition of the operation lowag(α) , imply that $\hat{A}_{t_0}(x_0) = t_0$, $\hat{A}_\beta(x_0) = 0$ and $\hat{A}_{\beta_i}(x_0) = t_0$, and thus, we get

$$\hat{A}_\beta(x_0) < \hat{A}_{\beta_i}(x_0) .$$

This contradicts $(**)$. On the other hand, if $\varsigma_\beta < \beta_i$ then, since $\mathrm{Im}\tau$ is lower well ordered, the subset $\mathrm{Im}_{\beta+}\tau$ has a least element, say t_1. That is $t_1 \in \mathrm{Im}_{\beta+}\tau$ and $\varsigma_\beta = t_1$. So there exists $\check{A} \in \tau$ and $x_1 \in X$, such that $\check{A}(x_1) = t_1$. Now $\check{A}_{t_1}(x_1) = t_1$ and $\check{A}_{\beta_i}(x_1) = 0$. Hence we have

$$\check{A}_{\beta_i}(x_1) < \check{A}_{\varsigma_\beta}(x_1) = \check{A}_{t_1}(x_1)\,.$$

That is

$$\check{A}_{\beta_i}(x_1) < \check{A}_{\beta}(x_1) \text{ since } \beta \le \varsigma_\beta = t_1\,.$$

This again contradicts $(**)$. Therefore in both cases, we arrive at a contradiction. Consequently $\beta_i \in [\xi_\beta, \varsigma_\beta]$. This completes the proof of the theorem.

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