Rendiconti di Matematica, Serie VII
Volume 17, Roma (1997), 329-345

q-Bessel functions: the point of view of the generating function method

G. DATTOLI – A. TORRE

Riassunto: *Si dimostra che il metodo della funzione generatrice permette di derivare le proprietà delle funzioni di Bessel di tipo q in maniera piuttosto naturale. Si analizzano i tre diversi tipi di funzioni q-cilindriche di prima specie finora proposte, si discutono le loro funzioni generatrici, le relazioni di ricorrenza da loro soddisfatte ed i relativi teoremi di addizione e moltiplicazione. Si introducono inoltre le relative forme modificate e si accenna infine alla possibilità di considerare q-funzioni di Bessel a più variabili e q-polinomi di tipo Kampé de Fériét.*

Abstract: *We show that the generating function method allows a fairly straightforward understanding of the properties of q-Bessel functions. We analyze three different forms of cylindrical q-Bessel functions so far proposed, discuss their generating functions, the recurrence relations they satisfy, and the relevant addition and multiplication theorems. We also introduce q-Bessel functions of the I-type and touch on the possibility of considering q-Bessel functions with more than one variable, as well as q-Kampé de Fériét polynomials.*

1 – Introduction

Mathematicians have explored the intriguing aspects of the q-analysis for more than 150 years [1]. Within this framework q- or the basic analog of the operations of the ordinary calculus have been introduced, and

Key Words and Phrases: *Bessel functions – Generating functions – q-polynomials*
A.M.S. Classification: 33C10

q-special functions generalizing the conventional functions have been carefully studied.

Basic analogs of Bessel functions (BF's) have been introduced by JACKSON [2], by means of the series ($n = 0, \pm 1, \pm 2 \dots$)

$$\begin{aligned} J_n^{(1)}(x;q) &= \sum_{r=0}^{\infty} \frac{(-1)^r \left[\dfrac{x}{2(1-q)}\right]^{n+2r}}{[r]_q![n+r]_q!}\,, \\ J_n^{(2)}(x;q) &= \sum_{r=0}^{\infty} \frac{(-1)^r \left[\dfrac{x}{2(1-q)}\right]^{n+2r}}{[r]_q![n+r]_q!} q^{r(n+r)}\,, \end{aligned} \tag{1.1a}$$

a third q-BF has been discussed by SWARTHOUW [3] and reads

$$J_n^{(3)}(x;q) = \sum_{r=0}^{\infty} \frac{(-1)^r \left[\dfrac{x}{2(1-q)}\right]^{n+2r}}{[r]_q![n+r]_q!} q^{\frac{r(r+1)}{2}}\,. \tag{1.1b}$$

In the above equations we have defined

$$\begin{aligned} [n]_q &= \frac{1-q^n}{1-q}\,, \\ [n]_q! &= \prod_{r=1}^{n} [r]_q\,, \qquad [0]_q! = 1\,. \end{aligned} \tag{1.2}$$

In the limit $q \to 1$ the functions $J_n^{(k)}((1-q)\cdot x;q)$ $(k=1,2)$ reduce to the ordinary cylindrical BF.

The properties of q-BF's have been widely discussed in refs. [4], [6]. q-analogs of elementary functions have been introduced as well. The q-exponential, for example, is specified by the series [7]

$$e_q(x) = \sum_{r=0}^{\infty} \frac{x^r}{[r]_q!}\,, \tag{1.3}$$

which is uniformly and absolutely convergent for all finite x, when $|q| > 1$, while convergence takes place for $|x| < \frac{1}{(1-q)}$, when $|q| < 1$. The function (1.3) does not have the semigroup property, since $e_q(x) \cdot e_q(y) \neq e_q(x+y)$ and therefore $e_q(x)e_q(-x) \neq 1$. One can, however, introduce the q-complementary exponential, defined by

$$e_{1/q}(x) = \sum_{r=0}^{\infty} \frac{x^r q^{\frac{r(r-1)}{2}}}{[r]_q!}\,, \tag{1.4}$$

for which [3]

$$e_q(x) \cdot e_{1/q}(-x) = 1\,, \tag{1.5}$$

The existence of q-exponentials and q-BF's suggests that a general approach to the theory of basic analogs of special functions may be accomplished using the concepts and the formalism of the generating function method (see also ref. [6]). Before entering into the specific details of the problem it is worth recalling a few notions regarding the q-derivatives, which will be widely exploited in the following.

The q-differential operator is defined, according to Jackson, as [3]

$$D_{(q,x)} = \frac{1}{x}\left[\frac{d}{dx}\right]_q = \frac{1-\exp\left[q\left(\frac{d}{dx}\right)\right]}{(1-q)x}\,. \tag{1.6}$$

As a consequence of the previous definition, we find

$$D_{(q,x)}x^n = [n]_q x^{n-1} \tag{1.7a}$$

and

$$\begin{aligned} D_{(q,x)}e_q(ax) &= ae_q(ax)\,, \\ D_{(q,x)}e_q(ax) &= ae_{1/q}(qax)\,. \end{aligned} \tag{1.7b}$$

Furthermore, the formula of differentiation by parts can be written as

$$D_{(q,x)}[f_1(x)f_2(x)] = f_2(q^{-1}x)D_{(q,x)}f_1(x) + f_1(qx)D_{(q,x)}f_2(x)\,. \tag{1.8}$$

The previous relations will be exploited in the remaining part of the paper, which is organized as follows. The theory of cylindrical q-BF's and associated modified forms is addressed in section 2, using the generating function method as starting point. Within such a context in section 3 we will derive their recurrence relations, and discuss properties such as multiplication theorems, integral representation, etc.

The generating functions are exploited in section 4 to introduce alternative forms of q-Hermite polynomials and are further generalized to present a class of q-polynomials, which is understood as the basic analog of the Kampé de Fériét polynomials [8]. The link of these last polynomials with q-BF's having more than one variable [9] is also pointed out. Finally, a brief discussion of the modified q-BF functions is presented in section 5, where furthermore the q-Bessel differential equation is derived.

2 – Cylindrical q-Bessel functions of first kind

In analogy to the ordinary case, we introduce the generating function

$$\mathcal{G}_1(x;t|q) = e_q\left(\frac{xt}{2}\right) e_q\left(-\frac{xt^{-1}}{2}\right), \tag{2.1}$$

which converges for any finite x when $|q| > 1$ and for $|x| < \left|\frac{2}{1-q}\right|$ when $|q| < 1$.

Expanding the q-exponentials according to (1.3) and (1.4) and noting that for $q = 1$, eq. (2.1) reduces to the ordinary BF generating function, we set

$$\begin{aligned} \mathcal{G}_1(x;t|q) &= \sum_{n=-\infty}^{+\infty} t^n J_n^{(1)}(x|q)\,, \\ J_n^{(1)}(x|q) &= \sum_{s=0}^{\infty} \frac{(-1)^s \left(\dfrac{x}{2}\right)^{n+2s}}{[s]_q![n+s]_q!}\,. \end{aligned} \tag{2.2}$$

The q-BF defined by (2.2) is essentially the first of eqs. (1.1) apart from the unessential factor $(1-q)^{-1}$ in the argument.

In a similar way we can prove that the generating function

$$\mathcal{G}_2(x;t|q) = e_{1/q}\left(\frac{xt}{2}\right) e_{1/q}\left(-\frac{xq}{2t}\right) \tag{2.3}$$

leads to the second of (1.1); in fact, (see also ref. [6])

$$
\begin{aligned}
&\mathcal{G}_2(x;t|q) = \sum_{n=-\infty}^{+\infty} q^{\frac{n(n-1)}{2}} t^n J_n^{(2)}(x|q)\,, \\
&J_n^{(2)}(x|q) = \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{n+2s}}{[s]_q![n+s]_q!} q^{s(n+s)}\,.
\end{aligned} \tag{2.4}
$$

Finally, it is also easily checked that

$$
\mathcal{G}_3(x;t|q) = e_q\left(\frac{xt}{2}\right) e_{1/q}\left(-\frac{xq}{2t}\right) \tag{2.5}
$$

is the generating function of the Swarthouw q-BF, namely,

$$
\begin{aligned}
&\mathcal{G}_3(x;t|q) = \sum_{n=-\infty}^{+\infty} t^n J_n^{(3)}(x|q)\,, \\
&J_n^{(s)}(x|q) = \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{n+2s}}{[s]_q![n+s]_q!} q^{\frac{s(s+1)}{2}}\,,
\end{aligned} \tag{2.6}
$$

and it is worth noting that

$$
J_{-n}^{(3)}(x|q) = J_n^{(3)}\Big(-xq\Big|\frac{1}{q}\Big)\,, \tag{2.7}
$$

which makes the introduction of a fourth generating function unessential, i.e., being

$$
\mathcal{G}_4(x;t|q) = e_{1/q}\left(\frac{qxt}{2}\right) e_q\Big(-\frac{x}{2t}\Big) = \mathcal{G}_3\Big(xq;t\Big|\frac{1}{q}\Big)\,. \tag{2.8}
$$

The recurrences of the q-BF so far introduced can be established in different ways. Keeping the $D_{(q,x)}$ derivative of both sides of the first of eqs. (2.2), we find

$$
\begin{aligned}
&\frac{1}{2}\left[te_q\left(\frac{xt}{2}\right) e_q\left(-\frac{xt^{-1}}{2}\right) - \frac{1}{t}e_q\left(\frac{qxt}{2}\right) e_q\left(-\frac{xt^{-1}}{2}\right)\right] = \\
&\qquad = \sum_{n=-\infty}^{+\infty} t^n D_{(q,x)} J_n^{(1)}(x|q)\,.
\end{aligned} \tag{2.9}
$$

Using the generating function (2.1) and equating the t-like powers, we end up with

$$D_{(q,x)} J_n^{(1)}(x|q) = \frac{1}{2}\left[J_{n-1}^{(1)}(x|q) - q^{\frac{n+1}{2}} J_{n+1}^{(1)}(\sqrt{q}x|q)\right]. \tag{2.10}$$

On the other hand, since

$$\begin{aligned} D_{(q;x)}\mathcal{G}_1(x;t|q) = -\frac{1}{2}\left[\frac{1}{t} e_q\left(-\frac{xt^{-1}}{2}\right) e_q\left(\frac{xt}{2}\right) + \right.\\ \left. - t\, e_q\left(-\frac{qxt^{-1}}{2}\right) e_q\left(\frac{xt}{2}\right)\right], \end{aligned} \tag{2.11}$$

we can establish the further recurrence

$$D(q,x) J_n^{(1)}(x|q) = \frac{1}{2}\left[q^{\frac{n-1}{2}} J_{n-1}^{(1)}(\sqrt{q}x|q) - J_{n+1}^{(1)}(x|q)\right], \tag{2.12}$$

which along with (2.10) leads to the difference equation

$$q^{-\frac{n-1}{2}} J_{n-1}^{(1)}(\sqrt{q}x|q) + q^{\frac{n+1}{2}} J_{n+1}^{(1)}(\sqrt{q}x|q) = J_{n-1}^{(1)}(x|q) + J_{n+1}^{(1)}(x|q). \tag{2.13}$$

The second recurrence of $J_n^{(1)}(x|q)$ can be obtained either deriving the first of (2.2) with respect to $D_{(q,t)}$ or manipulating the series definition.

Multiplying both sides of the second of (2.2) by $[n]_q$ and noting that

$$[n]_q = [n+s]_q - q^n [s]_q, \tag{2.14}$$

we obtain

$$2\frac{[n]_q}{x} J_n^{(1)}(x|q) = J_{n-1}^{(1)}(x|q) + q^n J_{n+1}^{(1)}(x|q), \tag{2.15}$$

which is reminiscent of the recurrence of the ordinary case. The recurrences (2.10) and (2.12), although connecting the nearest-neighbor indices, link q-BF's with different arguments. It is, however, worth noting that (see Appendix)

$$q^{-\frac{n-1}{2}} J_{n-1}^{(1)}(\sqrt{q}x|q) = J_{n-1}^{(1)}(x|q) + (1-q)\frac{x}{2} J_n^{(1)}(x|q), \tag{2.16}$$

so that, combining eqs. (2.16) and (2.13), we find

$$\text{(2.17a)}\qquad \left[D_{(q,x)} - (1-q)\frac{x}{4}\right] J_n^{(1)}(x|q) = -\frac{1}{2}\left[J_{n-1}^{(1)}(x|q) - J_{n+1}^{(1)}(x|q)\right]$$

and

$$\text{(2.17b)}\qquad q^{\frac{n+1}{2}} J_{n+1}^{(1)}(\sqrt{q}x|q) = -(1-q)\frac{x}{2} J_n^{(1)}(x|q) + J_{n+1}^{(1)}(x|q)\,.$$

Further comments will be presented in the concluding remarks.

To establish the recurrences of $J_n^{(2)}(x|q)$ we note that setting

$$\text{(2.18)}\qquad y = x\sqrt{q}\,,\quad \upsilon = \frac{t}{\sqrt{q}}\,,$$

we obtain from eqs. (2.3) and (2.4)

$$\text{(2.19a)}\qquad e_{1/q}\left(y\frac{\upsilon}{2}\right) e_{1/q}\left(-\frac{y}{2\upsilon}\right) = \sum_{n=-\infty}^{+\infty} J_n^{(1)}\left(y\middle|\frac{1}{q}\right)\,,$$

or, what is the same,

$$\text{(2.19b)}\qquad J_n^{(1)}\left(\sqrt{q}x\middle|\frac{1}{q}\right) = q^{\frac{n^2}{2}} J_n^{(2)}(x|q)\,.$$

It is, therefore, evident that

$$\text{(2.20a)}\qquad \begin{aligned} &\left[D_{(q^{-1},y)} - (1-q^{-1})\frac{y}{4}\right] J_n^{(1)}\left(y\middle|\frac{1}{q}\right) = \\ &= \frac{1}{2}\left[J_{n-1}^{(1)}\left(y\middle|\frac{1}{q}\right) - J_{n+1}^{(1)}\left(y\middle|\frac{1}{q}\right)\right]\,. \end{aligned}$$

and that

$$\text{(2.20b)}\qquad 2\frac{[n]_{q^{-1}}}{y} J_n^{(1)}\left(y\middle|\frac{1}{q}\right) = J_{n-1}^{(1)}\left(y\middle|\frac{1}{q}\right) + q^{-n} J_{n+1}^{(1)}\left(y\middle|\frac{1}{q}\right)\,.$$

From this last identity we also infer that $J_n^{(3)}(x|q)$ satisfies the same recurrence (2.15). The possibility of combining (2.20a) and (2.20b) to get a second-order difference equation for $J_n^{(2)}(x|q)$ will be discussed in the concluding section.

Regarding the recurrences of the function $J_n^{(2)}(x|q)$, we can proceed as before. It is, however, worth noting that we can extend the definition (2.6) as follows:

$$J_{n,\beta}^{(3)}(x|q) = \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{n+2s}}{[s]_q![n+s]_q!} q^{\frac{s(s+\beta)}{2}}, \tag{2.21}$$

where β is not necessarily an integer. According to eq. (2.21), we find

$$J_{n,1}^{(3)}(q^\alpha x|q) = q^{\alpha n} J_{n,4\alpha+1}(x|q). \tag{2.22}$$

Keeping the $D_{(q,x)}$ derivative of the second of eq. (2.6), we find

$$D_{(q,x)} J_{n,1}^{(3)}(x|q) = \frac{1}{2}\left[J_{n-1,1}^{(3)}(x|q) - q^{n+2} J_{n+1,5}^{(3)}(x|q)\right]. \tag{2.23a}$$

The second recurrence relation is given by

$$2\frac{[n]_q}{x} J_{n,1}^{(3)}(x|q) = \frac{1}{2}\left[J_{n-1,1}^{(3)}(x|q) - q^n J_{n+1,3}^{(3)}(x|q)\right]. \tag{2.23b}$$

The examples we have provided give an idea of the wealth of properties of the q-BF's. An idea of their behavior is also offered by figs. (1-3).

3 – Multiplication and addition theorems

We have already noted that the functions $J_n^{(1)}(x|q)$ and $J_n^{(2)}(x|q)$ are linked by relations of the type (2.19). The link between, e.g., $J_n^{(2)}(x|q)$ and $J_n^{(3)}(x|q)$ can be obtained noting that (see eq. (1.5))

$$\mathcal{G}_2(x;t;q) = \mathcal{G}_3(x;t;q) \cdot e_{1/q}\left(\frac{xq}{2t}\right) \cdot e_{1/q}\left(-\frac{xq}{2t}\right). \tag{3.1}$$

Accordingly, we can write

$$\begin{aligned} J_n^{(2)}(x|q) &= q^{-\frac{n(n-1)}{2}} \cdot \sum_{m=0}^{\infty} \left(\frac{x}{2}\right)^m q^{\frac{m}{2}(m-1)} J_{n-m}^{(3)}(x|q) \cdot A_m(q), \\ A_m(q) &= \sum_{s=0}^{m} \frac{(-1)^s q^{s(s-m)}}{[m-s]_q![s]_q!}. \end{aligned} \tag{3.2}$$

An analogous procedure can be exploited to state the multiplication theorem. In fact, using eqs. (2.1) and (2.5), we can write

$$\mathcal{G}_1(\lambda x; t|q) = \mathcal{G}_3(x; \lambda t|q) \cdot e_q\left(\frac{xq}{2\lambda t}\right) e_q\left(-\frac{\lambda x}{2t}\right), \tag{3.3}$$

which can be exploited to derive the following theorem:

$$\begin{aligned} J_n^{(1)}(\lambda x|q) &= \lambda^n \sum_{m=0}^{\infty} q^m \cdot A_m(\lambda|q) \cdot \left(\frac{x}{2}\right)^m J_{n+m}^{(3)}(x|q), \\ A_m(\lambda|q) &= \sum_{s=0}^{m} \frac{(-1)^s \lambda^{2s}}{q^s [m-s]_q! [s]_q!}. \end{aligned} \tag{3.4}$$

Further examples can be discussed, but are omitted for the sake of conciseness.

Before discussing the addition theorems, we state a straightforward but important identity

$$e_q\left(\frac{x}{2}t\right) e_q\left(\frac{yt}{2}\right) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} O_n(x, y|q), \tag{3.5a}$$

where

$$\begin{aligned} O_n(x, y|q) &= \sum_{s=0}^{n} \begin{bmatrix} n \\ s \end{bmatrix}_q \left(\frac{x}{2}\right)^{n-s} \left(\frac{y}{2}\right)^s \\ \begin{bmatrix} n \\ s \end{bmatrix}_q &= \frac{[n]_q!}{[n-s]_q! [s]_q!} \end{aligned} \tag{3.5b}$$

It is clear that $O_n(x, y|q)$ are polynomials generalizing the ordinary binomial form; furthermore, they satisfy the relations

$$D_{(q,x)} O_n(x, y|q) = D_{(q,y)} O_n(x, y|q) = [n]_q O_{n-1}(x, y|q). \tag{3.6}$$

According to (3.5a), we can also introduce q-BF's with four variables according to the generating function

$$e_q\left(\frac{x}{2}t\right) e_q\left(\frac{yt}{2}\right) e_q\left(-\frac{x'}{2t}\right) e_q\left(-\frac{y'}{2t}\right) = \sum_{n=-\infty}^{+\infty} t^{n(1)} J_n^{(1)}(x, y, x', y'|q), \tag{3.7a}$$

where

$$ {}^{(1)}J_n^{(1)}(x,y,x',y'|q) = \sum_{s=0}^{\infty} \frac{(-1)^s O_{n-s}(x,y|q) O_s(x',y'|q)}{[n-s]_q![s]_q!} \,. \tag{3.7b} $$

According to the above results, we can prove the theorem

$$ \begin{gathered} \sum_{\ell=-\infty}^{+\infty} J_{n-\ell}^{(1)}(x|q) J_\ell^{(1)}(y|q) = {}^{(1)}J_n^{(1)}(x,y|q)\,, \\ {}^{(1)}J_n^{(1)}(x,y|q) = {}^{(1)}J_n^{(1)}(x,y,x,y|q)\,. \end{gathered} \tag{3.8} $$

Multiplying the l.h.s. of (3.8) by t^n and then summing over n, we find

$$ \begin{aligned} &\sum_{n=-\infty}^{+\infty} t^n \sum_{\ell=-\infty}^{+\infty} J_{n-\ell}^{(1)}(x|q) J_\ell^{(1)}(y|q) = \\ &= \sum_{\ell=-\infty}^{+\infty} t^\ell \left(\sum_{n=-\infty}^{+\infty} t^{n-\ell} J_{n-\ell}^{(1)}(x|q) \right) J_\ell^{(1)}(y|q) = \\ &= e_q\left(\frac{xt}{2}\right) e_q\left(-\frac{x}{2t}\right) e_q\left(\frac{yt}{2}\right) e_q\left(-\frac{y}{2t}\right)\,. \end{aligned} \tag{3.9} $$

The identity (3.8) follows, therefore, from eqs. (3.5) and (3.7).

The same technique can be exploited to prove an extension of the Graf addition formula, namely,

$$ \sum_{\ell=-\infty}^{\infty} \xi^\ell J_{n-\ell}^{(1)}(x|q) J_\ell^{(1)}(y|q) = {}^{(1)}J_n^1 \left(x, y\xi, x, \frac{y}{\xi} \middle| q \right). \tag{3.10} $$

Addition formulae for q-BF's have been discussed by FLOREANINI and VINET [6] within the context of a different formalism employing an algebraic point of view and the use of ${}_r\phi_s\ (\dots)$ hypergeometric functions.

4 – The generating function and q-Kampé de Fériét polynomials

Let us consider the generating function

$$\mathcal{F}_1(x,y;t|q) = e_q(xt)e_q(yt^2) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} H_n^{(1)}(x,y|q)\,, \tag{4.1a}$$

where

$$H_n^{(1)}(x,y|q) = [n]_q! \sum_{s=0}^{[n/2]} \frac{x^{n-2s}y^s}{[n-2s]_q![s]_q!}\,, \tag{4.1b}$$

and the symbol $[n/2]$ denotes the truncated part of $n/2$. The polynomials (4.1b) are the q-analog of the Kampé de Fériét polynomials [8] and their recurrences are reported below:

$$\begin{aligned} D_{(q,x)} H_n^{(1)}(x,y|q) &= [n]_q H_{n-1}^{(1)}(x,y|q)\,, \\ D_{(q,y)} H_n^{(1)}(x,y|q) &= [n]_q [n-1]_q H_{n-2}^{(1)}(x,y|q) \\ [n]_q H_n^{(1)}(x,y|q) &= x D_{(q,x)} H_n^{(1)}(x,y|q) + \\ &\quad + q^n \cdot y D_{(q,y)} \left[H_n^{(1)}\left(x, \frac{y}{q^2}\right) + H_n^{(1)}\left(x, \frac{y}{q}\middle| q\right)\right]. \end{aligned} \tag{4.2}$$

The last expression can also be recast as

$$\begin{aligned} H_n^{(1)}(x,y|q) = x H_{n-1}^{(1)}(x,y|q) + [n-1]_q q^{n-2} y H_{n-2}^{(1)}\left(x, \frac{y}{q^2}\middle| q\right) + \\ + [n-1]_q q^{n-1} y H_{n-2}^{(1)}\left(x, \frac{y}{q}\middle| q\right)\,, \end{aligned}$$

or equivalently,

$$\begin{aligned} H_n^{(1)}(x,y|q) = x H_{n-1}^{(1)}(x,y|q) + [n-1]_q y H_{n-2}^{(1)}(qx,y|q) + \\ + [n-1]_q q y H_{n-2}^{(1)}(qx,qy|q)\,. \end{aligned}$$

Without entering into the specific importance of q-type Hermite polynomials, we note that, in analogy to ref. [9], they can be exploited to introduce q-GBF's. It is, indeed, easily proved that

$$ {}^{(2)}J_n^{(1)}(x,y|q) = \sum_{s=0}^{\infty} \frac{H_{n+s}^{(1)}\left(\frac{x}{2},\frac{y}{2}\Big|q\right) H_s^{(1)}\left(-\frac{x}{2},-\frac{y}{2}\Big|q\right)}{[n+s]_q![s]_q!} \qquad (n \geq 0) \tag{4.3} $$

Or, what is the same,

$$ {}^{(2)}J_n^{(1)}(x,y|q) = \sum_{\ell=-\infty}^{+\infty} J_{n+2\ell}^{(1)}(x|q) J_\ell^{(1)}(y|q) \,, \tag{4.4} $$

whose properties will be discussed elsewhere. The theory of q-BF's is, indeed, rich enough to require a separate treatment.

5 – Concluding remarks

In section 2 we mentioned cylindrical q-BF's of the first kind only. However, there is no prescription against the introduction of modified forms, defined through the generating function

$$ \begin{aligned} \mathcal{G}_1(x;t|q) &= e_q\left(\frac{xt}{2}\right) e_q\left(\frac{x}{2t}\right) = \sum_{n=-\infty}^{+\infty} t^n I_n^{(1)}(x|q) \,, \\ I_n^{(1)}(x|q) &= \sum_{s=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2s}}{[n+s]_q![s]_q!} \,. \end{aligned} \tag{5.1} $$

The properties of $I_n^{(1)}(x|q)$ are listed below

$$ \begin{aligned} I_{-n}^{(1)}(x|q) &= I_n^{(1)}(x|q) \,, \\ I_n^{(1)}(-x|q) &= (-1)^n I_n^{(1)}(x|q) \,, \\ I_n^{(1)}(ix|q) &= i^n J_n^{(1)}(x|q) \,, \end{aligned} \tag{5.2a} $$

and

$$ \begin{aligned} \frac{2[n]_q}{x} I_n^{(1)}(x|q) &= I_{n-1}^{(1)}(x|q) - q^n I_n^{(1)}(x|q) \,, \\ \left[D_{(q,x)} + (1-q)\frac{x}{4}\right] I_n^{(1)}(x|q) &= \frac{1}{2}\left[I_{n-1}^{(1)}(x|q) + I_{n+1}^{(1)}(x|q)\right] \end{aligned} \tag{5.2b} $$

Furthermore, fig. 4 yields an idea of their behavior as functions of x and q. We omit the discussion of the functions $I_n^{(2,3)}(x|q)$ for the sake of conciseness.

The recurrence relations can now be exploited to derive the q-analog of the Bessel equation. Combining, indeed, eqs. (2.15) and (2.17a), we find [6]

$$\begin{aligned}&\left\{\frac{[n]_q}{x} - D_{(q,x)} + (1-q)\frac{x}{4}\right\} J_n^{(1)}(x|q) = \frac{(1+q^n)}{2} J_{n+1}^{(1)}(x|q)\,,\\ &\left\{\frac{[n]_q}{x} + q^n\left[D_{(q,x)} + (1-q)\frac{x}{4}\right]\right\} J_n^{(1)}(x|q) = \frac{(1+q^n)}{2} J_{n-1}^{(1)}(x|q)\,.\end{aligned} \tag{5.3a}$$

Defining the shifting operators

$$\begin{aligned}\hat{E}_{+,n} &= \frac{2}{(1-q^n)}\left\{\frac{[n]_q}{x} - D_{(q,x)} + (1-q)\frac{x}{4}\right\}\,,\\ \hat{E}_{-,n} &= \frac{2}{(1+q^n)}\left\{\frac{[n]_q}{x} + q^n\left[D_{(q,x)} - (1-q)\frac{x}{4}\right]\right\}\,,\end{aligned} \tag{5.3b}$$

which turn $J_n^{(1)}(x|q)$ into $J_{n+1}^{(1)}(x|q)$ and $J_{n-1}^{(1)}(x|q)$ respectively, and noting that

$$\hat{E}_{-,n+1}\hat{E}_{+,n} J_n^{(1)}(x|q) = J_n^{(1)}(x|q)\,, \tag{5.3c}$$

we end up with

$$\begin{aligned}&\Big\{q^{n+1}D^2_{(q,x)} + \Big[\frac{[n]_q^2}{x}(1-q) + \frac{q^n}{x} - q^{n+1}(1-q^2)\frac{x}{4}\Big]D_{(q,x)} +\\ &- \frac{[n]_q^2}{x^2} - \frac{1-q^2}{4}\Big([n]_q[n+1]_q + q^n + q^{n+1}(1-q)^2\frac{x^2}{16}\Big) +\\ &+ \frac{(1+q^n)(1+q^{n+1})}{4}\Big\} J_n^{(1)}(x|q) = 0\,,\end{aligned} \tag{5.4}$$

which reduces to the ordinary Bessel equation in the limit $q \to 1$.

It is interesting to note that, unlike the ordinary case, in the large x limit, eq. (5.4) does not indicate that $J_n^{(1)}(x|q)$ is an oscillating function with decreasing amplitudes. The presence of the term $\frac{(1-q)^2}{16}x^2$ raises the doubt, supported by fig. 1, that the amplitudes increase. However, this problem will be dealt with in a forthcoming paper.

The q-Bessel equations for $I_n^{(1)}(x|q)$ and $J_n^{(2)}(x|q)$ can be obtained using similar methods.

In this paper we have presented the theory of q-BF's, using the generating function method as starting point. We have seen that, within such a context, one can verify the treatment of the various forms so far introduced and that one can go a step further in proposing modified forms and generalized q-BF's with more than one variable. An alternative treatment, based on an algebraic point of view, is that of Floreanini and Vinet [6], [10]. Their approach, certainly more general than the present analysis, should be considered as an effort to frame the q-special functions within the framework of the theory of quantum groups. Albeit less ambitious, the present analysis offers a possibility to understand the details of q-BF theory in a more classical sense, without any recourse to group theoretical concepts and to the generalized hypergeometric functions.

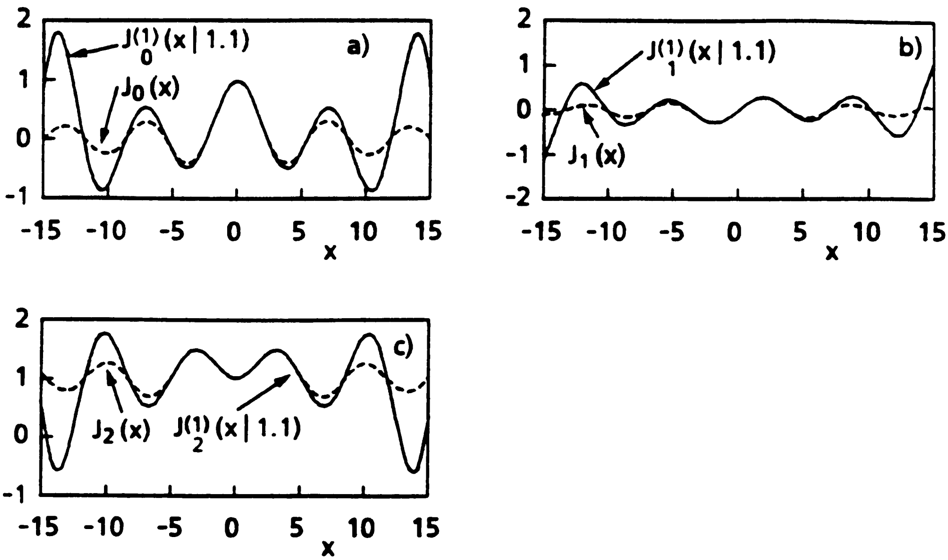

Fig. 1 Comparison between the first three q-BF's $J_n^{(1)}(x|q)$ ($n = 0, 1, 2,\ q = 1.1$) and the ordinary BF.

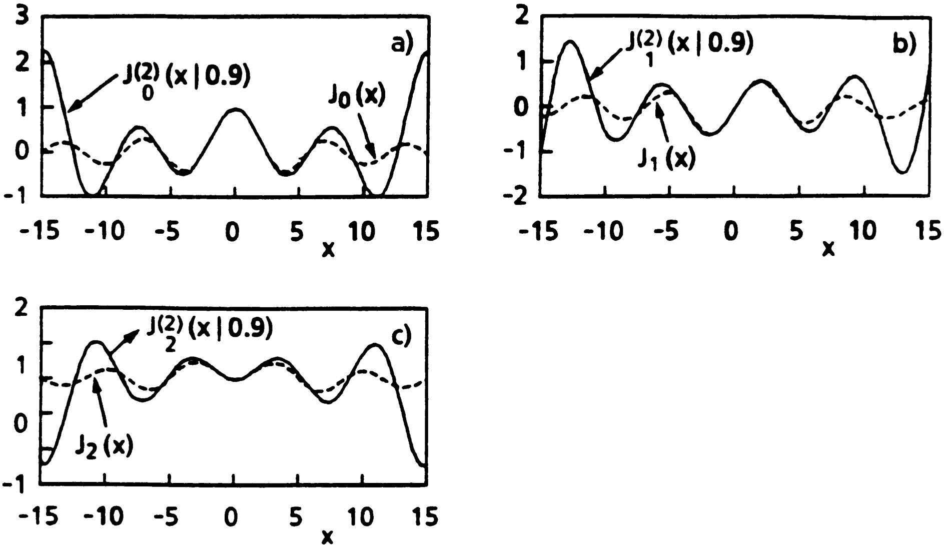

Fig. 2 Same as fig. 1 for $J_n^{(2)}(x|q)$ $(n = 0, 1, 2,\ q = 0.9)$.

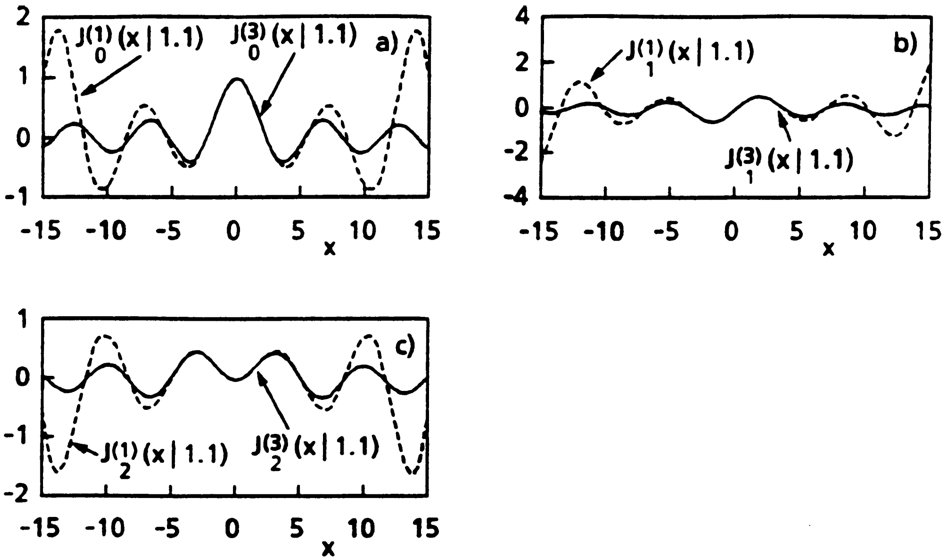

Fig. 3 Comparison between $J_n^{(3)}(x|q)$ and $J_n^{(1)}(x|q)$ $(n = 0, 1, 2,\ q = 1.1)$.

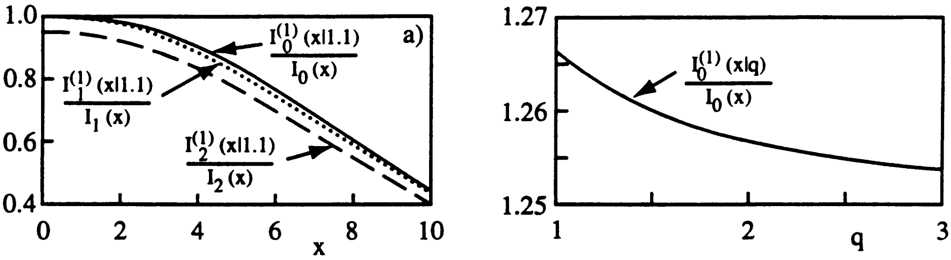

Fig. 4 a) $\frac{I_n^{(1)}(x|q)}{I_n(x)}$ vs. x ($n = 0, 1, 2$; $q = 1.1$). b) $I_0^{(1)}(1|q)$ vs. q.

Appendix

The identity (2.16) can be proved as follows.

Since

$$q^{-\frac{n-1}{2}} \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x\sqrt{q}}{2}\right)^{n-1+2s}}{[s]_q![n-1+s]_q!} = \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{n-1+2s} q^s}{[s]_q![n-1+s]_q!}\,, \tag{A.1}$$

using the identity

$$q^s = 1 - (1-q)[s]_q\,, \tag{A.2}$$

we find,

$$q^{-\frac{n-1}{2}} \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x\sqrt{q}}{2}\right)^{n-1+2s}}{[s]_q![n-1+s]_q!} = J_n^{(1)}(x|q) + (1-q)\frac{x}{2} j_n^{(1)}(x|q)\,. \tag{A.3}$$

Acknowledgements

The authors recognize that this research has been stimulated by Dr. S. Lorenzutta, whose kind interest and help is gratefully acknowledged.

REFERENCES

[1] G.E. Andrews: *q-series: their development and application in analysis, number theory combinations, physics and computer algebra*, Conference Board of Mathematical Science N. 66 (Am. Math. Soc., Providence (R.I.) USA, (1986).

[2] F.H. Jackson: *The application of basic numbers to bessel's and Legendre functions*, Proc. London Math. Soc., **2** (1903-1904), 192-220.

[3] R.F. Swarthouw: *An addition theorem and some product formulas for the Hahn-Exton q-Bessel functions*, Can. J. Math., **44** (1992), 867-879.

[4] M.E.H. Ismail: *The zeros of basic Bessel functions, the function* $J_{v+\alpha x}(x)$ *and associated orthogonal polynomials*, J. Math. Anal. Appl., **86** (1982), 1-19.

[5] M.E.H. Ismail: *The basic Bessel functions and polynomials*, SIAM J. Math. Anal., **12** (1981), 454.

[6] M. Rahman: *An integral representation and some transformation properties of q-Bessel functions*, Can. J. Math., **40** (1988), 1203.

[7] R. Floreanini – L. Vinet: *Addition formulas for q-Bessel functions*, J. Math. Phys., **33** (1992), 2984.

[8] N. Ya. Vilenkin – A.U. Klimyk: *Representation of Lie groups and special functions*, Vol. 3, Kluwer Academic Publisher, Boston (1992), 3.

[9] P. Appell – J. Kampé de Fériét: *Fonctions Hypérgeométriques et Hypersphériques, Polynomes d'Hermite*, Gauthiers-Villars, Paris (1926).

[10] G. Dattoli – C. Chiccoli – S. Lorenzutta – G. Maino – A. Torre: *Computers*, Math. Appl., **28** (1994), 71-83.

[11] G. Dattoli – C. Chiccoli – S. Lorenzutta – G. Maino – A. Torre: *Generalized Bessel functions and generalized Hermite polynomials*, J. Math. Anal. Appl., **178** (1993), 509-516.

[12] G. Dattoli – A. Torre: *Theory and application of generalized Bessel functions*, Aracne, Roma (1996)

[13] R. Floreanini – L. Vinet: *Quantum algebras and q-special functions*, Annals of Physics, **221** (1993), 53-70.

Lavoro pervenuto alla redazione il 17 luglio 1996
ed accettato per la pubblicazione il 23 aprile 1997.
Bozze licenziate il 30 maggio 1997

Indirizzo degli Autori:

G. Dattoli – A. Torre – ENEA, Dipartimento Innovazione, Settore Fisica Applicata, Centro Ricerche Energia Frascati, P.O. Box 65 - 00044 Frascati, Roma (Italia)