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A characterization of hypergraphs which are products of a finite number of edges

In memory of Giuseppe Tallini

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Riassunto: *In questo articolo si dimostra che un ipergrafo è il prodotto di un numero finito di spigoli se e soltanto se è intervallo-regolare, ha ogni spigolo "gated" ed ha almeno un vertice di grado finito. In particolare, ciò dà luogo ad una caratterizzazione dei grafi di Hamming e degli ipercubi.*

Abstract: *We prove that a hypergraph is a product of a finite number of edges if and only if it is interval-regular, satisfies the gated-edge property and has a vertex of finite degree. As a consequence, we get a characterization of Hamming graphs.*

1 – Introduction

The concept of a *hypergraph* is a generalization of that of a graph. In much the same way as the theory of functions of several complex variables is something completely new with respect to that of one complex variable, hypergraph theory is something different and much more rich than graph theory. Given a set V of *vertices*, an *edge* of a (simple) graph on V is

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a set of *two* vertices, while an edge of a hypergraph on V is *any* subset of V. In other words, hypergraphs are precisely the *geometric spaces* (cf. [26]), the main object of *combinatorial geometry.* This is the general theory of sets equipped with some distinguished subsets, which are called in different ways — *subspaces, flats, blocks, edges, hyperedges, code words etc.*— according to the different theories and contexts.

One aspect of the theory of hypergraphs —popularized and enriched by many contributions of BERGE [3], [4]— is the extension of theorems about graphs to hypergraphs. The problem is to find a suitable formulation of the theorems for hypergraphs in such a way that they contain the graph case as a special case.

It may happen that these generalizations clarify the deep meanings of the correspondent statements of graph theory and lead to new developments of the theory itself. This is the case of the theorem considered in this paper.

Both (connected) graphs and hypergraphs can be considered as metric spaces. This point of view has been very successeful for graph theory (cf. [5], [28]), and could be useful also for hypergraphs.

In this paper, using the concept of *gated set* in a metric space (cf. [17]), we consider, from a metric point of view, hypergraphs which are products of a finite number of edges. This class of hypergraphs is a "good" extension of a very famous class of graphs, the celebrated *hypercubes.* The hypercube Q_n is the graph whose set of vertices is the set of ordered n-tuples of 0's and 1's, and two n-tuples are adjacent if they differ in one component. A vertex of Q_n can be identified with the characteristic function of a subset of a fixed set X with n elements, so that Q_n appears to be the undirected graph associated with the Hasse diagram of the lattice $P(X)$ of subsets of X. The hypercube Q_n is from a theoretical and practical point of view one of the most fascinating graphs, see [21]. This is the reason why many characterizations of Q_n have been given; the first one was due to FOLDES [18]; see [7], [9], [15], [25], [27] for overviews. Another crucial problem is to characterize those graphs which are isometrically embeddable in a hypercube; this problem was solved in two different ways by DJOKOVIC [16] and by GRAHAM and WINKLER [19].

The importance of hypercubes suggests to study some "good" generalizations of them. One is the so called "q-analogue of Q_n", introduced in [12], [13] as the undirected graph associated with the Hasse diagram of the lattice of subspaces of the n-dimensional vector space over the finite field with q elements; a further generalization in this direction is obtained by considering, instead of the vector space, any finite (and possibly reducible) projective space, cf. [14].

Another generalization of Q_n, going in the same direction developed in this paper, is the so called cube-hypergraph $Q(n,t)$, introduced in [10], [11]. This is the hypergraph having $\{0,1,\dots,t-1\}^n$ as set of vertices and having as edges any set $\{(a_1,a_2,\dots,x_k,\dots,a_n)\}$, where the a_j's are fixed and x_k runs over $\{0,1,\dots,t-1\}$. When t is a prime number, then the vertex set of $Q(n,t)$ can be identified with the set of points of the n-dimensional affine space over the field with t elements and the edge set of $Q(n,t)$ can be identified with the set of lines parallel to the coordinate axes. Note that $Q(n,2)=Q_n$. The cube-hypergraph $Q(n,t)$ has been characterized in [11], and the hypergraphs which are isometrically embeddable in $Q(n,t)$ are characterized in [10]; when $t=2$, these results give well known theoremes for Q_n. The hypergraph $Q(n,t)$ can be also introduced as a suitable *Cayley hypergraph*, via a group theoretical approach considered in [6].

As the cube-hypergraph $Q(n,t)$ turns out to be the product of the n edges $X_1=\ldots=X_n=\{0,1,\dots,t-1\}$, it is quite natural to consider all those hypergraphs which are product of n arbitrary edges $X_1,\dots,X_n$, where the X_k's are not necessarily finite and not necessarily of the same cardinality. This level of generality is optimal, in the sense that it clarifies completely the structure of the $Q(n,t)$'s and in particular the structure of the hypercubes Q_n's. In this paper a characterization of such general hypergraphs is given.

Another link between graphs and hypergraphs is presented in Section 4. With any hypergraph one can associate its adjacency graph and with any graph one can associate its clique-hypergraph. This leads to a method for carrying certain theorems from graphs to hypergraphs and conversely. For instance our mentioned characterization gives a new characterization of the (general) n-dimensional Hamming graphs, i.e. of graphs which are product of n possibly infinite complete graphs. Finite Hamming graphs of dimension n are another important generalization of the hypercubes Q_n's; characterizations and properties of them have been intensively investigated in the recent literature, cf. [1], [2], [8], [23], [24], [27], [30].

2 – Basic concepts

A *hypergraph* on a set V is a pair $H = (V, E)$ where E is a set of subsets of V. The elements of V and of E are called *vertices* and *edges* respectively. H is *simple* if any two distinct edges of H are not in the inclusion relation. H is called *t-uniform* if every edge of H has t vertices. A 2-uniform hypergraph is a simple graph, and conversely. The *degree*, $d(x)$, of a vertex x is the number of edges containing x. H is called *locally finite* if any vertex has a finite degree. H is called *connected* if for any two distinct vertices x and y there exists an *xy-path*, that is a sequence $p(x,y) = x_1Y_1x_2Y_2 \dots x_dY_dx_{d+1}$, where the x_i's are vertices and the Y_i's are edges such that $x_1 = x$, $x_{d+1} = y$ and $\{x_i, x_{i+1}\} \in Y_i$, for any $i = 1, \dots, d$. The integer d is the *length* of $p(x,y)$. An *xy-geodesic* is an xy-path of minimum length. The *distance* $d(x,y)$ between x and y is the length of any xy-geodesic. The *diameter* of H, denoted by $\mathrm{diam}(H)$, is the maximum distance between any pair of vertices of H. In the following any hypergraph under consideration will be simple and connected.

Denote by $\Gamma(x,y)$ the set of xy-geodesics and by $\gamma(x,y)$ its cardinality. Then denote by $N_1(x,y)$ the set of neighbours of x which are on at least one xy-geodesic. In other words:

$$N_1(x,y) := \{z \in V : d(x,z) = 1, d(y,z) = d(x,y) - 1\}.$$

Following a terminology introduced for graphs by Mulder [25], H is said to be *interval-regular* when one of the following equivalent conditions is fulfilled (cf. [11]):

(a) $\gamma(x,y) = d(x,y)!, \forall\, x,y \in V$.

(b) $|N_1(x,y)| = d(x,y), \forall\, x,y \in V$.

Following a terminology used in [17] for subsets of a metric space, we shall say that an edge X of H is *gated* if for any vertex y of H there exists exactly one vertex x of X such that $d(y,x) = d(y,X)$. This vertex x (the "gate" of X for y, i.e. the vertex of X nearest to y) will be denoted by $\nu(X,y)$; if $d(x,y) = j$, then $d(z,y) = j+1$ for any $z \in X - \{x\}$. We shall say that H is a *gated-edge hypergraph* (or that H has the *gated-edge property*) if any edge X of H is gated. When H reduces to a graph, the gated-edge property reduces to bipartiteness.

Note that two distinct edges of a gated-edge hypergraph H have at most one vertex in common, i.e. H is *semilinear*. Note also that any gated-edge hypergraph is *triangle free*.

An *isomorphism* of hypergraphs, from $H = (V, E)$ to $H' = (V', E')$, is a bijective map $f : V \to V'$ such that

$$(k \subset V) \qquad k \in E \iff k^f \in E'.$$

We shall say that a hypergraph $H = (V, E)$ is a *product of* n *edges* (where n is an integer ≥ 1) if V is the cartesian product of n sets $X_1, X_2, \ldots, X_n$ of cardinalities all greater than 1, i.e. $V = \prod_{i=1}^{n} X_i$, and E is the set of those subsets of V each of which is of type $\{(a_1, \ldots, x_j, \ldots, a_n) : x_j \in X_j\}$, where $j \in \{1, \ldots, n\}$, and $a_i \in X_i$.

This hypergraph will be denoted by $\mathop{\square}_{i=1}^{n} X_i$, according to the fact that it is the *cartesian product* of the hypergraphs $(X_i, \{X_i\})$, cf. [10]. When all the X_i's have the same cardinality $t \geq 2$, then the t-uniform hypergraph $\mathop{\square}_{i=1}^{n} X_i$ reduces to the cube-hypergraph $Q(n, t)$ introduced in [10], [11]. Also, if $t = 2$, then $Q(n, 2) = Q_n$ is the celebrated hypercube of dimension n, which is the graph with vertex set consisting of all the ordered n-tuples of 0's and 1's and in which two vertices are adjacent if and only if they differ in exactly one component.

In this paper, we give a characterization of hypergraphs which are products of a finite number of edges. This characterization immediately implies the characterization of Q_n given by Foldes [18] and the characterization of $Q(n, t)$ given in [11]. Note that here the edges may be infinite sets (possibly of distinct cardinalities) and that also $E(H)$ may be an infinite set; as an example from geometry, take the hypergraph whose vertices are the points of the n-dimensional euclidean space and whose edges are the lines parallel to the coordinate axes.

3 – The characterization

Theorem A. *A connected hypergraph H is, up to isomorphism, the product of n edges if and only if the following conditions hold:*

(i) *H has the gated-edge property;*

(ii) *H is interval regular;*

(iii) *H has a vertex of degree n.*

PROOF. ($\Longrightarrow$). Suppose $H = \mathop{\square}\limits_{i=1}^{n} X_i$. Each vertex of H has obviously degree n. Let $a = (a_i)$ and $b = (b_i)$ be two distinct vertices of H. W.l.o.g. we can assume that a and b differ precisely in each of their first d components, $1 \le d \le n$. We have the following ab-path:

$$\begin{aligned}(a_1, \ldots, a_n)&\{(x_1, a_2, \ldots, a_n) : x_1 \in X_1\}\\ &(b_1, a_2, \ldots, a_n)\{(b_1, x_2, \ldots, a_n) : x_2 \in X_2\}\\ &(b_1, b_2, a_3, \ldots, a_n) \ldots \{(b_1, b_2, \ldots, b_{d-1}, x_d, \ldots, a_n) : x_d \in X_d\}\\ &(b_1, b_2, \ldots, b_d, a_{d+1}, \ldots, a_n)\,.\end{aligned}$$

This is obviously an ab-geodesic (of length d), uniquely determined by giving a, b and the permutation $(1, 2, \ldots, d)$: this permutation shows in which order the components of a must be changed to those of b. Therefore the number of ab-geodesics equals the number $d!$ of all permutations of the set $\{1, 2, \ldots, d\}$. Then H is connected and interval regular. Moreover H has the gated-edge property, because for any vertex $y = (y_1, \ldots, y_n)$ and any edge $X = \{a_1\} \times \ldots \times X_i \times \ldots \times \{a_n\}$, the only vertex $x \in X$ such that $d(x, y) = d(x, X)$ is the vertex $x = (a_1, \ldots, y_i, \ldots, a_n)$.

($\Longleftarrow$). Suppose now that H is a connected hypergraph satisfying conditions (i)-(iii). Fix a vertex $O \in V(H)$ such that $d(O) = n$, and let $X_1, X_2, \ldots, X_n \in E(H)$ be the edges of H through O; by semilinearity, any two distinct of them intersect only in O.

We shall prove that H is isomorphic to $\mathop{\square}\limits_{i=1}^{n} X_i$, showing that the following map:

$$f : V(H) \to V(\mathop{\square}\limits_{i=1}^{n} X_i) : x \mapsto x^f = (x_1, x_2, \ldots, x_n), \text{ where } x_i = \nu(X_i, x)\,,$$

is a hypergraph isomorphism.

Given a vertex $v = (v_1, v_2, \ldots, v_n) \in V(\mathop{\square}\limits_{i=1}^{n} X_i)$, define *weight* of v the integer $w(v) := |\{i : v_i \neq O\}|$. It is clear that for any $x \in V(H)$ we have that $\nu(X_i, x) \neq O$ iff $\nu(X_i, x) \in N_1(O, x)$, so that:

$$w(x^f) = |N_1(O, x)| = \text{ (by interval-regularity) } d(x, O)\,.$$

Now we prove that f is bijective, i.e. that: "for every $v \in V(\mathop{\square}\limits_{i=1}^{n} X_i)$:

$(1)_v$ there exists exactly one $z \in V(H)$ such that $z^f = v$".

We use induction on $w(v)$. If $w(v) = 0$, i.e. if $v = (O, O, \dots, O)$, then $(1)_v$ is true with $z = O$. If $w(v) = 1$, we can assume w.l.o.g. that $v = (v_1, O, \dots, O)$ with $v_1 \neq O$; then $(1)_v$ is true with $z = v_1$. If $w(v) = 2$, we can assume w.l.o.g. that $v = (v_1, v_2, O, \dots, O)$ with $v_1 \neq O \neq v_2$. As H is triangle free, we get $d(v_1, v_2) = 2$. Then $\gamma(v_1, v_2) = 2! = 2$, and the set of geodesics between v_1 and v_2 must be of type $\Gamma(v_1, v_2) = \{v_1X_1OX_2v_2, v_1Y_1xY_2v_2\}$ for uniquely determined $Y_1, Y_2 \in E(H)$ and $x \in V(H)$ with $d(x, O) = 2$. Then $(1)_v$ is true with $z = x$.

Let $w(v) = d \geq 3$. W.l.o.g. we can assume $v = (v_1, \dots v_d, O, \dots, O)$ with $v_1 \neq O, \dots, v_d \neq O$. Set $v' = (v_1, \dots, v_{d-1}, O, \dots, O)$. Then $w(v') = d - 1$ and there exists by induction exactly one $y \in V(H)$ such that $y^f = v'$. We have $\nu(X_d, y) = O$ and $d(O, y) = w(y^f) = w(v') = d-1$, so that $d(y, v_d) = d$ by the gated-edge property. Therefore $N_1(y, O)$ is a $(d-1)$-set contained in the d-set $N_1(y, v_d)$.

Let z be the unique element of $N_1(y, v_d) \setminus N_1(y, O)$. It follows that $d(z, v_d) = d - 1$ and $d(z, O) \neq d - 2$. It is not $d(z, O) = d - 1$, otherwise by $d(z, v_d) = d - 1$ and by the gated edge property it would follow the existence of $w_d \in X_d \setminus \{O\}$ such that $d(z, w_d) = d-2$, and hence $d(y, w_d) \leq d - 1$, which contradicts $\nu(X_d, y) = O$. Then $d(z, O) = d$.

It follows that $\nu(X_d, z) = v_d$. For $1 \leq i \leq d - 1$ we have that $\nu(X_i, y) = v_i$, so that $d(y, v_i) = d(y, O) - 1 = d - 2$; hence $d(z, v_i) < d = d(z, O)$ and then $\nu(X_i, z) = v_i$. Finally $\nu(X_i, z) = O$ for every $i > d$, otherwise $w(z^f) > d$. In conclusion $z^f = v$, and this proves that f is surjective.

Now, to complete the proof of $(1)_v$, we show that if $z^f = z'^f = v = (v_1, \dots, v_d, O, \dots, O)$, then $z = z'$. Set $N_1(z, O) = \{a_1, a_2, ..., a_d\}$ and $N_1(z', O) = \{b_1, b_2, \dots, b_d\}$. Each a_i^f and each b_j^f is a "vector" of weight $d - 1$ obtained from v changing to O one of its first d components; there are precisely d such vectors, and by induction hypothesis each of them is the image under f of exactly one vertex of H. It follows $N_1(z, O) = N_1(z', O)$. Let $y \in N_1(z, O) = N_1(z', O)$. W.l.o.g. we can assume that $y^f = (v_1, \dots, v_{d-1}, O, \dots, O)$. We shall prove that $z = z'$ showing that

$$z, z' \in N_1(y, v_d) \setminus N_1(y, O) \text{ and } |N_1(y, v_d) \setminus N_1(y, O)| = 1.$$

In order to prove that $z, z' \in N_1(y, v_d)$, we have to prove that

$$d(z, v_d) = d(z', v_d) = d - 1 \text{ and that } d(y, v_d) = d;$$

these follow from

$$d(z,O)=d(z',O)=w(z^f)=d, \qquad \nu(X_d,z)=\nu(X_d,z')=v_d$$

and from

$$d(y,O)=w(y^f)=d-1, \qquad \nu(X_d,y)=O$$

respectively.

We have $z,z'\notin N_1(y,O)$, otherwise we would get $d(z,O)=d(z',O)=d(y,O)-1=d-2$. Finally $|N_1(y,v_d)\setminus N_1(y,O)|=d(y,v_d)-d(y,O)=1$.

In conclusion, f is a bijection.

Now we prove that if $X\in E(H)$ then $X^f\in E(\mathop{\square}\limits_{i=1}^{n}X_i)$. We can assume w.l.o.g. that

$$X=\{x,y,\dots\},\quad \nu(X,O)=x,\quad d(x,O)=d-1,\quad d(z,O)=d \quad \text{for all } z\in X\setminus\{x\},$$
$$x^f=(v_1,\dots,v_{d-1},O,\dots,O),\quad y^f=(v_1,\dots,v_{d-1},y_d,O,\dots,O),$$
$$v_1\neq O,\dots,v_{d-1}\neq O,\quad y_d\neq O.$$

Let us prove that for all $z\in X\setminus\{x\}$, we have:

$$\nu(X_i,z)=v_i, \quad \text{for } 1\le i\le d-1, \tag{2}$$

$$\nu(X_d,z)\neq O, \tag{3}$$

$$\nu(X_j,z)=O, \quad \text{for } d<j\le n. \tag{4}$$

In fact, (2) immediately follows from the equalities $d(z,O)=d$, $d(z,v_i)=d-1 (1\le i\le d-1)$. For proving (3) and (4), suppose on the contrary that there exists $s>d$ such that $\nu(X_s,z)=z_s\neq O$. We have $d(z,z_s)=d-1$ since $d(z,O)=d$. Therefore $d(y,z_s)\leqslant d$. But we have that $\nu(X_s,y)=O$ and $d(y,O)=d$, so that $d(y,z_s)=d+1$, a contradiction.

From (2), (3) and (4) we have that $z\in X$ implies that $z^f\in\{(v_1,\dots,v_{d-1},z_d,O,\dots,O):z_d\in X_d\}$, i.e. X^f is included in the edge $X':=\{(v_1,\dots,v_{d-1},z_d,O,\dots,O):z_d\in X_d\}$ of $\mathop{\square}\limits_{i=1}^{n}X_i$. Let us prove that conversely $X'\subset X^f$. As f is bijective any $v=(v_1,\dots,v_{d-1},x_d,O,\dots,O)$ with $x_d\in X_d$ is of type $v=z^f$ for exactly one $z\in V(H)$. We have to prove that $z \in X$. This is obvious if $x_d = y_d$, because in this case $z = y \in X$. Then let $x_d \neq y_d$. From $d(x, O) = d - 1$ and $\nu(X_d, x) = O \neq x_d$ it follows that

$$d(x, x_d) = d, \qquad x \in X;$$

from $d(y, O) = d$ and $\nu(X_d, y) = y_d \neq x_d$, it follows that

$$d(y, x_d) = d, \qquad y \in X.$$

Then, by the gated-edge property, there exists $z \in X$ such that

$$d(z, x_d) = d - 1, \qquad z \in X.$$

Hence $\nu(X_d, z) = x_d$. Then, by (2) and (4), we get $z^f = v$ with $z \in X$, as required.

We prove now that, conversely, any edge $X' = \{(v_1, v_2, \dots, x_j, \dots, v_n) : x_j \in X_j\}$ of $\mathop{\square}\limits_{i=1}^{n} X_i$ is the image $X' = X^f$ of an edge $X \in E(H)$. Let us consider the following two vertices of X':

$$v' = (v_1, \dots, v_{j-1}, O, v_{j+1}, \dots, v_n) \text{ and}$$
$$v = (v_1, \dots, v_{j-1}, y_j, v_{j+1}, \dots, v_n) \text{ with } y_j \neq O.$$

There are uniquely determined vertices $x, y \in V(H)$ such that $x^f = v'$ and $y^f = v$. We shall prove that x and y belong to the same edge X of H; it will follow that X^f is the unique edge of $\mathop{\square}\limits_{i=1}^{n} X_i$ containing x^f and y^f, i.e. that $X^f = X'$.

Note that $\nu(X_j, x) = O$, so that $d(x, y_j) = d + 1$, where $d := d(x, O)$. Now, using an argument similar to that used for proving the surjectivity of f, we have that the pre-image y of v is determined as the unique vertex of $N_1(x, y_j) \setminus N_1(x, O)$. As $y \in N_1(x, y_j)$, we get $d(x, y) = 1$, i.e. x and y belong to the same edge of H. This completes the proof of the theorem. □

Remark. In Theorem A, condition (iii) can be replaced by the following condition:
(iii′) $\operatorname{diam}(H) = n$.

Actually, (iii′) immediately follows from (i)-(iii), by Theorem A. We prove now that conversely (i), (ii), (iii′) implies (iii). Let x and y be vertices of H at the maximum distance: $d(x,y) = \mathrm{diam}(H) = n$. Then $|N_1(x,y)| = n$ by (ii) and (b); each edge X of H through x contains exactly one vertex of $N_1(x,y)$ because X is gated by (i); each vertex of $N_1(x,y)$ belongs to exactly one edge of H through x, because H is semilinear by (i). Therefore x has degree n.

4 – A link between hypergraphs and graphs

Now, we want to show that the above Theorem A allows to get a characterization of Hamming graphs. To do this, we need some considerations.

Given a graph $G = (V, E)$ and denoted by $\mathbb{E}$ the set of cliques (maximal complete subgraphs) of G, define the *clique hypergraph of* G to be the hypergraph $C(G) := (V, \mathbb{E})$.

It is clear that $C(G)$ is connected iff G is connected, and that $d_{C(G)}(x,y) = d_G(x,y)$ for any two vertices $x, y \in V$; it follows that G is interval-regular iff $C(G)$ is interval-regular. Moreover, $C(G)$ has the gated-edge property if and only if G has the *gated-clique* property, i.e. for any $y \in V$ and for any clique X of G there exists exactly one $x \in X$ such that $d(y, X) = d(y, x)$.

Given any hypergraph $H = (V, \mathbb{E})$, define the *adjacency graph of* H to be the graph $A(H) := (V, E)$, where

$$(x, y \in V) \qquad xy \in E :\Longleftrightarrow\ d_H(x,y) = 1.$$

Note that for any graph G, we have $A(C(G)) = G$. If $X \in \mathbb{E}$, then X is a complete subgraph (not necessarily maximal) of $A(H)$.

It is clear that $A(H)$ is connected iff H is connected, and that $d_{A(H)}(x,y) = d_H(x,y)$ for any two vertices $x, y \in V$; it follows that H is interval-regular iff $A(H)$ is interval-regular.

An edge $X \in \mathbb{E}$ will be called 1-*saturated* if it contains any vertex $y \in V$ having distance 1 from all vertices of $X - \{y\}$:

$$(y \in V) \qquad [x \in X - \{y\} \implies d_H(x,y) = 1] \implies y \in X.$$

An edge $X \in \mathbb{E}$ is 1-saturated if and only if X is a clique of $A(H)$. We shall say that a hypergraph H is 1-saturated iff any edge of H is 1-saturated. With this terminology, a graph turns out to be 1-saturated iff it is triangle free; for such a graph G, $A(G) = G$ and $C(A(G)) = G$. For any hypergraph H, we have that $C(A(H)) = H$ iff H is 1-saturated. If an edge $X \in \mathbb{E}$ is gated, then X is 1-saturated (but the converse is false); so any hypergraph having the gated-edge property is 1-saturated.

Let H be any 1-saturated hypergraph. It is clear that the hypergraph H has the gated-edge property iff the graph $A(H)$ has the gated-clique property.

A turns the class of hypergraphs which are product of a finite number of edges into the class of Hamming graphs, and C turns this class into the first.

Applying the above links between hypergraphs and graphs, one obtains that the following characterization of Hamming graphs is equivalent to Theorem A.

Theorem B. *A connected graph G is, up to isomorphism, a Hamming graph if and only if the following conditions hold:*
(i) *H has the gated-clique property;*
(ii) *H is interval regular;*
(iii) *H has a vertex of degree n.*

Remark. By the Remark after Theorem A, it follows that in Theorem B, condition (iii) can be replaced by the following condition:
(iii′) $\mathrm{diam}(G) = n$.

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