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Homographic approximation applied to nonlinear elliptic unilateral problems

M.C. PALMERI

RIASSUNTO: *In questo articolo dimostriamo la disuguaglianza di Lewi-Stampacchia per alcuni problemi unilaterali non lineari, in cui l'operatore è un operatore ellittico di Leray-Lions. La dimostrazione si basa su un metodo di penalizzazione limitata per disequazioni variazionali: l'approssimazione omografica.*

ABSTRACT: *In this paper we prove the Lewi-Stampacchia's inequality for some nonlinear unilateral problems involving elliptic operators of Leray-Lions type. The proof is based on a bounded penalization for elliptic variational inequalities: the homographic approximation.*

– Introduction

Penalization is an interesting method used to prove the existence and some properties of solutions of variational inequalities. To give a general definition of penalization, it consists in replacing the elliptic variational inequality with a class of nonlinear Dirichlet problems which depend upon a small positive parameter.

Several methods of penalization exist. A classical penality method was introduced by J.L. LIONS [11] and it is applied to nonlinear elliptic

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unilateral problems. Let us consider, for example, the following unilateral problem with zero obstacle

$$u \in K_0 : \langle A(u), v-u\rangle \geq \langle f, v-u\rangle \text{ for all } v \in K_0 ,$$
$$K_0 = \{v \in W_0^{1,p}(\Omega) : v \geq 0 \text{ almost everywhere in } \Omega\} ,$$

where A is a coercive, continous, pseudomonotone operator of Leray-Lions type, acting from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$, and $f \in W^{-1,p'}(\Omega)$.

One defines the penalized problem

$$u_\varepsilon \in W_0^{1,p}(\Omega) : A(u_\varepsilon) - \frac{1}{\varepsilon}(u_\varepsilon^-)^{p-1} = f ,$$

that is, a family of equations perturbed by the addition of a penalization term, and then one proves that the approximate sequence u_ε converges, up to a subsequence, weakly in $W_0^{1,p}(\Omega)$ to a solution u of the unilateral problem.

This kind of penalization, involving unbounded mappings in the penalization term, is called *unbounded penalization.*

In this paper we consider a kind of *bounded penalization*, introduced by C. M. Brauner and B. Nicolaenko in the linear case [9]: the homographic approximation. In order to explain the salient features of this technique we describe a simple example.

Let us consider the family of nonlinear problems

$$u_\lambda \in H_0^1(\Omega) : -\Delta u_\lambda + g(x)\frac{u_\lambda}{\lambda + |u_\lambda|} = f(x) + g(x)$$

where $f \in L^q(\Omega)$, $q \geq 2$, is given; g is an arbitrary function in $L^q(\Omega)$, $g \geq 0$.

It is proved that the sequence u_λ converges strongly in $H_0^1(\Omega)$ to a function u. Then, if g is large enough ($g \geq \frac{1}{2}f^-$), u is the solution of the unilateral problem

$$u \in K_0 : \int_\Omega DuD(v-u)\,dx \geq \int_\Omega f(v-u)\,dx \text{ for all } v \in K_0 ,$$
$$K_0 = \{v \in H_0^1(\Omega) : v \geq 0 \text{ almost everywhere in } \Omega\} .$$

Moreover, if $g \geq f^-$, each function u_λ satisfies the constraint $u_\lambda \geq 0$, the sequence u_λ is monotone decreasing and, since $0 \leq \frac{u_\lambda}{\lambda+|u_\lambda|} \leq 1$ for every $\lambda > 0$, we have the well known LEWY-STAMPACCHIA's inequality [10]

$$f \leq -\Delta u \leq f^+ ,$$

that gives a result of regularity for u.

It is natural to extend the above results to nonlinear elliptic problems. We give a contribution in this direction, developing the homographic approximation if $-\Delta$ is replaced by a nonlinear elliptic operator of Leray-Lions type. In short, our main goal is to prove the Lewy-Stampacchia's inequality in the nonlinear case, by using the homographic approximation. In particolar we will consider two situations. First we will study the strongly monotone case and, as in the linear case, will give several results about the approximate sequence; then we will treat the quasilinear case and we will prove that the approximate sequence is monotone decreasing with respect to λ; moreover, we will deduce from the Lewy-Stampacchia's inequality dependence of the regularity of u on the regularity of the obstacle.

We recall that the Lewy-Stampacchia's inequality has been extended by various authors to the case of nonlinear elliptic operators, and used to prove existence and regularity results (see among others A. BENSOUSSAN, J. L. LIONS [4], A. BENSOUSSAN, L. BOCCARDO [2], L. BOCCARDO, G.R. CIRMI [6], L. BOCCARDO, T. GALLOUËT [7], A. MOKRANE, F. MURAT [12], U. MOSCO [13], U. MOSCO, G.M. TROIANELLO [14]).

1 – A nonlinear elliptic unilateral problem

Let A be a nonlinear elliptic differential operator of second order in divergence form

$$A(v) = -\operatorname{div} a(x, v, Dv) + a_0(x, v, Dv) ,$$

where a, a_0 are Carathéodory functions (with values in $\mathbb{R}^N$ and $\mathbb{R}$ respectively) such that for every $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, $\eta \in \mathbb{R}^N$ ($\xi \neq \eta$), and for almost every $x \in \Omega$,

$$|a(x,s,\xi)| \leq \beta\,(k(x) + |s|^{p-1} + |\xi|^{p-1})\,, \tag{1.1}$$

$$|a_0(x,s,\xi)| \leq \beta\,(k(x) + |s|^{p-1} + |\xi|^{p-1})\,, \tag{1.2}$$

$$(a(x,s,\xi) - a(x,s,\eta)) \cdot (\xi - \eta) > 0\,, \quad \xi \neq \eta \tag{1.3}$$

$$a(x,s,\xi) \cdot \xi \geq \alpha\,|\xi|^p\,, \tag{1.4}$$

$$a_0(x,s,\xi)\,s \geq \alpha_0\,|s|^p\,, \tag{1.5}$$

where α, β are positive real constants, $k \in L^{p'}(\Omega)$, and Ω is a bounded open set of $\mathbb{R}^N$.

The operator A is then a coercive, continuous, pseudomonotone operator of Leray-Lions type, acting from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$.

We assume that the obstacle ψ is a measurable function such that the closed convex

$$K_\psi := \{v \in W_0^{1,p}(\Omega)\,:\, v \geq \psi \;\text{ almost everywhere in } \Omega\}$$

is not empty and we pick the datum f in $L^q(\Omega)$ with $q \geq (p^*)'$.

We consider the problem:

$$\begin{aligned} u \in W_0^{1,p}(\Omega):\ \langle A(u), v-u\rangle + \Phi(v-\psi) - \Phi(u-\psi) \geq \\ \geq \langle f, v-u\rangle \forall\, v \in W_0^{1,p}(\Omega) \end{aligned} \tag{1.6}$$

where

$$\Phi(v) = 2\int_\Omega g v^-\,dx, \tag{1.7}$$

and g is a function satisfying

$$g \geq 0,\ g \in L^q(\Omega),\ q \geq (p^*)'\,. \tag{1.8}$$

Let $\lambda > 0$ be fixed. We introduce the *homographic approximation*

$$A(u_\lambda) + g\frac{u_\lambda - \psi}{\lambda + |u_\lambda - \psi|} = f + g \qquad u_\lambda \in W_0^{1,p}(\Omega)\,. \tag{1.9}$$

The existence of a solution of problem (1.9) is a classical result (see J.L. Lions [11])

Theorem 1.1. *The sequence* u_λ *of solutions of* (1.9)*, up to a subsequence, converges strongly in* $W_0^{1,p}(\Omega)$ *to a solution* u *of* (1.6)*.*

Proof. First we multiply (1.9) by u_λ and deduce that u_λ is bounded in $W_0^{1,p}(\Omega)$; hence we have, up to a subsequence, weak convergence in $W_0^{1,p}(\Omega)$ to an element u.

Let us introduce the functional (defined on $L^{q'}(\Omega)$)

$$\Phi_\lambda(v) := \Phi(v) - \lambda \int_\Omega g \log(\lambda + |v|)\, dx = \int_\Omega g[2v^- - \lambda \log(\lambda + |v|)]\, dx\,,$$

whose differential at the point u is

$$\Phi'_\lambda(u) = g\left(\frac{u}{\lambda + |u|} - 1\right)\,.$$

Multiplying (1.9) by $v - u_\lambda$, $v \in W_0^{1,p}(\Omega)$, we deduce

$$\langle A(u_\lambda), v - u_\lambda\rangle + \langle \Phi'_\lambda(u_\lambda - \psi), v - u_\lambda\rangle = \langle f, v - u_\lambda\rangle\,. \tag{1.10}$$

From the convexity of Φ_λ we obtain

$$\Phi_\lambda(v - \psi) - \Phi_\lambda(u_\lambda - \psi) \geq \langle \Phi'_\lambda(u_\lambda - \psi), v - u_\lambda\rangle\,;$$

hence (1.10) implies

$$\langle A(u_\lambda), v - u_\lambda\rangle + \Phi_\lambda(v - \psi) - \Phi_\lambda(u_\lambda - \psi) \geq \langle f, v - u_\lambda\rangle\,. \tag{1.11}$$

We remark that for every $a \in \mathbb{R}$ $|\log(\lambda + |a|)| \leq |\log\lambda| + |a|$, if λ is small enough. Then from

$$\langle A(u_\lambda), u_\lambda - u\rangle \leq \Phi_\lambda(u - \psi) - \Phi_\lambda(u_\lambda - \psi) + \langle f, u_\lambda - u\rangle\,,$$

and from the former estimates we deduce

$$\overline{\lim_{\lambda\to 0}}\langle A(u_\lambda), u_\lambda - u\rangle \leq \Phi(u - \psi) - \underline{\lim_{\lambda\to 0}}\,\Phi(u_\lambda - \psi)\,,$$

finally by Fatou's lemma

$$\overline{\lim_{\lambda\to 0}}\langle A(u_\lambda), u_\lambda - u\rangle \le 0\,,$$

that is

$$\overline{\lim_{\lambda\to 0}}\left(\int_\Omega (a(x,u_\lambda,Du_\lambda) - a(x,u_\lambda,Du))(Du_\lambda - Du)\,dx\right) \le 0\,.$$

Using a standard lemma in the theory of monotone operators, see e.g. lemma 5 of [8], we obtain that u_λ, up to a subsequence, converges strongly to u in $W_0^{1,p}(\Omega)$.

Passing to the limit in (1.11), we complete the proof of the theorem. □

Let us consider the unilateral problem:

$$u \in K_\psi :\ \langle A(u), v-u\rangle \ge \langle f, v-u\rangle,\ \forall\, v \in K_\psi\,. \tag{1.12}$$

It is trivial that a solution of (1.6) that satisfies the constraint $u \ge \psi$ is a solution of (1.12).

One can see that, if $\psi \equiv 0$, the condition $g \ge \frac{1}{2}f^-$ implies that u_λ converges to $u \ge 0$ and the condition $g \ge f^-$ implies $u_\lambda \ge 0$ for every $\lambda > 0$.

The general case needs more attention. We assume

$$\psi \in W^{1,p}(\Omega),\ \psi|_{\partial\Omega} \le 0\,. \tag{1.13}$$

Then we suppose that there exists $\bar a_0 \in L^q(\Omega)$ such that almost everywhere in $\{x \in \Omega :\ s \le \psi(x)\}$

$$\bar a_0(x) - a_0(x,s,\xi) \ge 0\,, \tag{1.14}$$

and that there exists $\bar a \in (L^q(\Omega))^N$, with div $\bar a \in L^q(\Omega)$, such that almost everywhere in $\{x \in \Omega :\ s \le \psi(x)\}$

$$(\bar a(x) - a(x,s,\xi))(D\psi(x) - \xi) \ge \alpha|D\psi(x) - \xi|^p\,. \tag{1.15}$$

Note that if $\psi \equiv 0$, these assumptions are automatically satisfied with $\bar a_0 \equiv 0$, $\bar a \equiv 0$. In the linear case, $a_0(x,s,\xi) = \alpha_0 s$, $a(x,s,\xi) = a(x)\xi$, so that one can take $\bar{a}_0(x) = \alpha_0\psi(x)$, $\bar{a}(x,s,\xi) = a(x)D\psi(x)$. In the strongly monotone nonlinear case, $a(x,s,\xi) = a(x,\xi)$, we can suppose $a(x,D\psi) \in (L^q(\Omega))^N$, $\operatorname{div} a(x,D\psi) \in L^q(\Omega)$ and take $\bar{a}(x,s,\xi) = a(x,D\psi(x))$.

THEOREM 1.2 (g–maximum principle). *If g is a function of $L^q(\Omega)$ satisfying*

$$g(x) \geq \frac{1}{2}(f(x) + \operatorname{div}\bar{a}(x) - \bar{a}_0(x))^-, \text{ almost everywhere in } \Omega, \tag{1.16}$$

then every solution u of (1.6) *belongs to K_ψ.*

PROOF. Choose $v = \psi + (u-\psi)^+ \in W_0^{1,p}(\Omega)$ in (1.6); then

$$\langle -\operatorname{div} a(x,u,Du) + a_0(x,u,Du), (u-\psi)^-\rangle \geq \int_\Omega (f+2g)(u-\psi)^- \, dx\,,$$

and using (1.14), (1.15),

$$\begin{aligned}&\int_\Omega (-\operatorname{div}\bar{a}(x) + \bar{a}_0(x))(u-\psi)^- \, dx - \alpha\times \\ &\times \int_\Omega |D(u-\psi)^-|^p \, dx \geq \int_\Omega (f+2g)(u-\psi)^- \, dx\,.\end{aligned}$$

Finally, since (1.16) implies

$$2g + f + \operatorname{div}\bar{a} - \bar{a}_0 \geq 0\,,$$

we obtain

$$\alpha\int_\Omega |D(u-\psi)^-|^p \, dx \leq 0$$

and so $u \geq \psi$ almost everywhere in Ω. ∎

The $g-$maximum principle ensures that, if g is large enough (in the sense specified by (1.16)), every solution of (1.6) is a solution of (1.12). In this respect, g may be considered a *parameter* of *transition* between the variational inequalities (1.6) and the unilateral problem (1.12).

THEOREM 1.3. *Under the assumptions* (1.13) *to* (1.16), *the sequence u_λ of solutions of* (1.9), *up to a subsequence, converges strongly in $W_0^{1,p}(\Omega)$ to a solution u of the unilateral problem* (1.12).

Theorem 1.3 is a direct consequence of the above results.

However, u_λ *does not belong to* K_ψ *in general.* For example in the simple case $A = -\Delta$, $f \leq 0$ and $\psi \equiv 0$, if we take $g = \frac{1}{2} f^-$ we obtain $u_\lambda \leq 0$. One of the main advantages of the homographic approximation is the following result.

THEOREM 1.4. *Let us assume*

$$g(x) \geq (f(x) + \operatorname{div} \bar{a}(x) - \bar{a}_0(x))^-, \ \textit{almost everywhere in } \Omega\,. \tag{1.17}$$

Then $u_\lambda \in K_\psi$ *for every* $\lambda > 0$.

PROOF. Multiply (1.9) by $(u_\lambda - \psi)^- \in W_0^{1,p}(\Omega)$ to obtain

$$\langle -\operatorname{div} a(x, u_\lambda, Du_\lambda) + a_0(x, u_\lambda, Du_\lambda), (u_\lambda - \psi)^- \rangle \geq \int_\Omega (f+g)(u_\lambda - \psi)^- \, dx\,,$$

and using (1.14), (1.15)

$$\int_\Omega (-\operatorname{div} \bar{a}(x) + \bar{a}_0(x))(u_\lambda - \psi)^- \, dx - \alpha \times \\ \times \int_\Omega |D(u_\lambda - \psi)^-|^p \, dx \geq \int_\Omega (f+g)(u_\lambda - \psi)^- \, dx\,.$$

We remark that (1.17) implies

$$g + f + \operatorname{div} \bar{a} - \bar{a}_0 \geq 0\,,$$

then

$$\alpha \int_\Omega |D(u_\lambda - \psi)^-|^p \, dx \leq 0$$

and so

$$u_\lambda \in K_\psi\,, \ \text{ for every } \ \lambda > 0. \qquad \square$$

As a consequence we can prove the following generalization to the nonlinear case of the Lewy-Stampacchia's inequality.

THEOREM 1.5 (Lewy-Stampacchia's inequality).

$$f \leq A(u) \leq \max\{f\,, -\operatorname{div} \bar{a} + \bar{a}_0\} \ \textit{in} \ L^q(\Omega) \tag{1.18}$$

Proof. We pick g satisfying (1.17); using (1.9) and theorem 1.4 we obtain

$$f \leq A(u_\lambda) \leq f + g \text{ in } \ L^q(\Omega)\,.$$

Then there exists $\chi \in L^q(\Omega)$ such that $A(u_\lambda)$, up to a subsequence, converges to χ weakly in $L^q(\Omega)$ and, as a consequence, strongly in $W^{-1,p'}(\Omega)$. Nevertheless, since u_λ, up to a subsequence, converges to u strongly in $W_0^{1,p}(\Omega)$ and A is continuous, $A(u_\lambda)$ converges to $A(u)$ strongly in $W^{-1,p'}(\Omega)$. We deduce

$$A(u_\lambda) \to A(u) \text{ (up to a subsequence) weakly in } L^q(\Omega)$$

and so

$$f \leq A(u) \leq f + g \text{ in } L^q(\Omega)\,.$$

Inequality (1.18) is obtained using the optimal choice of $g = (f + \operatorname{div} \bar{a} - \bar{a}_0)^-$. ∎

Let us investigate the strongly monotone case. In this specific situation we can obtain new results about the approximate sequence u_λ: u_λ is monotone decreasing as λ decreases to 0; the speed of convergence of u_λ to u in $W_0^{1,p}(\Omega)$ is of order $\lambda^{1/p}$.

We consider the operator A defined, in the case $p \geq 2$, by

$$A(v) = -\operatorname{div} a(x, Dv)$$

where the Carathéodory function a satisfies, for every $\xi, \eta \in \mathbb{R}^N$ and for almost every $x \in \Omega$,

$$a(x,\xi) \cdot \xi \geq \alpha |\xi|^p\,, \tag{1.19}$$

$$|a(x,\xi)| \leq \beta(|h(x)| + |\xi|)^{p-1}\,, \tag{1.20}$$

$$(a(x,\xi) - a(x,\eta)) \cdot (\xi - \eta) \geq \alpha |\xi - \eta|^p\,, \tag{1.21}$$

where α, β are positive real constants, and $h \in L^p(\Omega)$.

Theorem 1.6. *Let A be defined by $A(v) = -\operatorname{div} a(x, Dv)$ where a satisfies conditions* (1.19), (1.20) *and* (1.21). *We assume that*

$$\psi \in W^{1,p}(\Omega),\ \psi|_{\partial\Omega} \leq 0;\ -\operatorname{div} a(x, D\psi) \in L^q(\Omega)$$

and we take $g \in L^q(\Omega)$ *such that*

$$g(x) \geq (f(x) + \operatorname{div} a(x, D\psi(x))^-, \text{ almost everywhere in } \Omega\,. \tag{1.22}$$

Then each u_λ*, solution of* (1.9)*, satisfies the constraint* $u_\lambda \in K_\psi$*, the sequence* u_λ *converges strongly in* $W_0^{1,p}(\Omega)$ *to the solution* u *of the unilateral problem* (1.12) *and the sequence is monotone decreasing as* λ *decreases to* 0*.*

Moreover we have the error estimate

$$\alpha \|u_\lambda - u\|^p_{W_0^{1,p}} \leq \lambda \|g\|_{L^1}\,. \tag{1.23}$$

Proof. First we remark that we can apply theorems 1.3 and 1.4 so that the first part of the result follows.

Let $\lambda_1 < \lambda_2$. Then

$$A(u_{\lambda_1}) + g\frac{u_{\lambda_1} - \psi}{\lambda_1 + u_{\lambda_1} - \psi} = f + g\,,$$

and

$$A(u_{\lambda_2}) + g\frac{u_{\lambda_2} - \psi}{\lambda_2 + u_{\lambda_2} - \psi} = f + g\,.$$

We have

$$\begin{aligned} A(u_{\lambda_2}) - A(u_{\lambda_1}) &= g\left[\frac{u_{\lambda_1} - \psi}{\lambda_1 + u_{\lambda_1} - \psi} - \frac{u_{\lambda_2} - \psi}{\lambda_2 + u_{\lambda_2} - \psi}\right] = \\ &= g\frac{-\psi(\lambda_2 - \lambda_1) + \lambda_2 u_{\lambda_1} - \lambda_1 u_{\lambda_2}}{(\lambda_1 + u_{\lambda_1} - \psi)(\lambda_2 + u_{\lambda_2} - \psi)}, \end{aligned}$$

so that

$$\begin{aligned} &\langle A(u_{\lambda_2}) - A(u_{\lambda_1}), (u_{\lambda_2} - u_{\lambda_1})^- \rangle = \\ &\quad = \int_\Omega g\frac{-\psi(\lambda_2 - \lambda_1) + \lambda_2 u_{\lambda_1} - \lambda_1 u_{\lambda_2}}{(\lambda_1 + u_{\lambda_1} - \psi)(\lambda_2 + u_{\lambda_2} - \psi)}(u_{\lambda_2} - u_{\lambda_1})^-\, dx \geq \\ &\quad \geq \int_\Omega g\frac{(\lambda_2 - \lambda_1)(u_{\lambda_2} - \psi)}{(\lambda_1 + u_{\lambda_1} - \psi)(\lambda_2 + u_{\lambda_2} - \psi)}(u_{\lambda_2} - u_{\lambda_1})^-\, dx \geq 0. \end{aligned}$$

Using the strong monotonicity of A we deduce

$$\alpha\|(u_{\lambda_2}-u_{\lambda_1})^-\|^p_{W_0^{1,p}}\leq 0\,,$$

and so

$$u_{\lambda_1}\leq u_{\lambda_2}\,.$$

It remains to prove estimate (1.23). Since $u_\lambda\in K_\psi$, then

$$\langle A(u),u_\lambda-u\rangle\geq\langle f,u_\lambda-u\rangle$$

and

$$\langle A(u_\lambda),u-u_\lambda\rangle\geq-\lambda\|g\|_{L^1}+\langle f,u-u_\lambda\rangle\,.$$

Using the strong monotonicity of A we deduce

$$\alpha\|u-u_\lambda\|^p_{W_0^{1,p}}\leq\langle A(u_\lambda)-A(u),u_\lambda-u\rangle\leq\lambda\|g\|_{L^1}\,. \qquad \square$$

2 – A quasilinear elliptic unilateral problem

In this section we consider a quasilinear elliptic operator, acting from $H_0^1(\Omega)$ into its dual $H^{-1}(\Omega)$:

$$Q(v)=-\mathrm{div}\,(A(x,v)Dv)$$

where $A(x,s)=(a_{ij}(x,s))$ is a matrix of Carathéodory such that

$$a_{ij}(x,s)\in L^\infty(\Omega\times\mathbb{R}), \tag{2.1}$$

$$A(x,s)\xi\cdot\xi\geq\alpha|\xi|^2,\qquad\forall\,\xi\in\mathbb{R}^N, \tag{2.2}$$

$$|A(x,s)|\leq\beta, \tag{2.3}$$

where $\alpha,\beta>0$;

$$|A(x,s)-A(x,t)|\leq\omega(|s-t|),\qquad\forall\,s,t\in\mathbb{R}, \tag{2.4}$$

where $\omega(s)$ is a function such that

$$\omega \text{ is non decreasing, } \omega(0)=0, \int_{0^+} \frac{ds}{\omega(s)} = +\infty. \tag{2.5}$$

Hypotheses (2.1), (2.2) and (2.3) imply that the operator Q is a coercive, continuous, pseudomonotone operator of Leray-Lions type. Remark that Q is, in general, not a monotone operator since $a(x,s,\xi)=A(x,s)\xi$ depends on s.

We introduce the penalized problem

$$Q(u_\lambda) + g\frac{u_\lambda - \psi}{\lambda + |u_\lambda - \psi|} = f + g \qquad u_\lambda \in H_0^1(\Omega)\,. \tag{2.6}$$

Using theorem 1.1 we deduce that the sequence u_λ of solution of (2.6), up to a subsequence, converges strongly in $H_0^1(\Omega)$ to a solution u of

$$u \in H_0^1(\Omega) : \langle Q(u), v-u\rangle + \Phi(v-\psi) - \Phi(u-\psi) \geq \langle f, v-u\rangle, \forall\, v \in H_0^1(\Omega). \tag{2.7}$$

We can suppose that $\psi \in H^1(\Omega), \psi|_{\partial\Omega} \leq 0$ and that hypothesis (1.15) is satisfied, so that we are able to use theorems 1.2 and 1.4; this gives

$$f \leq Q(u) \leq \max\{f, -\operatorname{div}\bar{a}\} \text{ in } L^q(\Omega)\,, \tag{2.8}$$

where u is a solution of the unilateral problem

$$u \in K_\psi : \langle Q(u), v-u\rangle \geq \langle f, v-u\rangle, \forall\, v \in K_\psi\,. \tag{2.9}$$

Now we prove, again using the homographic approximation, that the inequality (2.8) holds with $\bar{a} = A(x,\psi)D\psi$.

We will use the following result

PROPOSITION 2.1. *Let Q be defined by $Q(v) = -\operatorname{div}((A(x,v)Dv))$ where A satisfies conditions* (2.1), (2.2), (2.3) *and* (2.4).

Then, if

$$\langle Q(u) - Q(v), \int_0^{w^-} \frac{ds}{[\omega(s+\varepsilon)]^2}\rangle \geq 0 \text{ for all } \varepsilon > 0\,,$$

one has $u \geq v$ almost everywhere in Ω, where $u, v \in H^1(\Omega)$ are such that $w^- = (u-v)^- \in H_0^1(\Omega)$.

This proposition is a variant of the comparision result proved in [1]. We give here the proof for the convenience of the reader.

Proof. First we remark that the function $\int_0^{w^-} \frac{ds}{[\omega(s+\varepsilon)]^2}$ belongs to $H_0^1(\Omega)$, for every, $\varepsilon > 0$, and

$$D\Big(\int_0^{w^-} \frac{ds}{[\omega(s+\varepsilon)]^2}\Big) = \frac{Dw^-}{[\omega(w^- + \varepsilon)]^2}\,.$$

We have

$$\int_\Omega \frac{A(x,u)DuDw^-}{[\omega(w^- + \varepsilon)]^2}\,dx \ge \int_\Omega \frac{A(x,v)DvDw^-}{[\omega(w^- + \varepsilon)]^2}\,dx$$

which implies

$$\int_\Omega \frac{A(x,u)D(u-v)Dw^-}{[\omega(w^- + \varepsilon)]^2}\,dx \ge \int_\Omega \frac{[A(x,v)-A(x,u)]DvDw^-}{[\omega(w^- + \varepsilon)]^2}\,dx\,.$$

Then

$$\begin{aligned}-\alpha\int_\Omega \frac{|Dw^-|^2}{[\omega(w^- + \varepsilon)]^2}\,dx &\ge -\int_\Omega \frac{|Dv||Dw^-|}{\omega(w^- + \varepsilon)}\,dx \ge \\ &\ge -\Big(\int_\Omega |Dv|^2\,dx\Big)^{\frac{1}{2}}\Big(\int_\Omega \frac{|Dw^-|^2}{[\omega(w^- + \varepsilon)]^2}\,dx\Big)^{\frac{1}{2}}.\end{aligned}$$

Then Poincaré's inequality gives

$$\int_\Omega\Big[\int_0^{w^-} \frac{ds}{\omega(s+\varepsilon)}\Big]^2 dx \le c_1(\alpha, \|v\|, \Omega)\,.$$

Now, if we define $E_\varrho = \{x \in \Omega\colon\ w^-(x) \ge \varrho\}$ and we assume that meas $E_\varrho > 0$, for some $\varrho > 0$, we have

$$\int_{E_\varrho}\Big[\int_0^{\varrho} \frac{ds}{\omega(s+\varepsilon)}\Big]^2 dx \le c_1, \text{ for every}\,\varepsilon > 0\,,$$

that is a contradiction with (2.5) as ε tends to 0.

Thus $w^- = 0$: that is $u \ge v$. ∎

Theorem 2.1. *Let Q be defined by $Q(v) = -\mathrm{div}\,((A(x,v)Dv))$ where A satisfies conditions* (2.1), (2.2), (2.3) *and* (2.4).

We suppose that

$$\psi \in H^1(\Omega), \psi|_{\partial\Omega} \le 0;\ \mathrm{div}\,(A(x,\psi(x))D\psi(x)) \in L^q(\Omega) \tag{2.10}$$

and we take $g \in L^q(\Omega)$ such that

$$g(x) \ge (f(x) + \mathrm{div}\,(A(x,\psi(x))D\psi(x)))^-, \ \text{almost everywhere in } \Omega. \tag{2.11}$$

Then each u_λ, solution of (2.6), *satisfies the constraint $u_\lambda \in K_\psi$, the sequence u_λ converges strongly in $H_0^1(\Omega)$ to a solution u of the unilateral problem* (2.9) *and the sequence is monotone decreasing as λ decreases to* 0.

Proof. Multiply (2.6) by the function $\int_0^{w_\lambda^-} \frac{ds}{[\omega(s+\varepsilon)]^2}$, with $w_\lambda^- = (u_\lambda - \psi)^-$, to get

$$\langle Q(u_\lambda), \int_0^{w_\lambda^-} \frac{ds}{[\omega(s+\varepsilon)]^2}\rangle \ge \int_\Omega (f+g)\Big(\int_0^{w_\lambda^-} \frac{ds}{[\omega(s+\varepsilon)]^2}\Big)\,dx,$$

so that, using (2.11),

$$\langle Q(u_\lambda) - Q(\psi), \int_0^{w_\lambda^-} \frac{ds}{[\omega(s+\varepsilon)]^2}\rangle \ge 0\,.$$

Proposition 2.1 implies

$$u_\lambda \ge \psi, \ \text{almost everywhere in } \Omega\,.$$

Then u_λ, up to a subsequence, converges strongly in $H_0^1(\Omega)$ to a solution u of (2.9).

Let $\lambda_1 < \lambda_2$. Repeating the proof of theorem 1.5, we obtain

$$\langle Q(u_{\lambda_2}) - Q(u_{\lambda_1}), \int_0^{w^-} \frac{ds}{[\omega(s+\varepsilon)]^2}\rangle \ge 0\,,$$

where $w^- = (u_{\lambda_2} - u_{\lambda_1})^-$. Proposition 2.1 then implies

$$u_{\lambda_2} \ge u_{\lambda_1}\,.$$

As a consequence, for every $\lambda > 0$ fixed, the solution of problem (2.6) is unique. Moreover, since u_λ converges almost everywhere in Ω, the whole sequence u_λ converges strongly in $H_0^1(\Omega)$ to a solution u of (2.9). $\square$

Finally, proceeding as in the proof of theorem 1.5 and using theorem 2.1, we obtain easily the following result.

COROLLARY 2.1 (Lewy-Stampacchia's inequality).

$$f \leq Q(u) \leq \max\{f\,,\, -\mathrm{div}\,(A(x,\psi)D\psi)\} \;\; in\, L^q(\Omega). \tag{2.12}$$

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Indirizzo dell'Autore:

M.C. Palmeri – Dipartimento di Matematica – Università di Roma I – Piazzale A. Moro 2 – 00185 Roma, Italia