Rendiconti di Matematica, Serie VII
Volume 17, Roma (1997), 403-421

Orthogonal polynomials related to the unit circle and differential-difference equations

A. CACHAFEIRO – C. SUÁREZ

Riassunto: *Si determinano le successioni di polinomi ortogonali sul cerchio unità che verificano la seguente equazione differenziale alle differenze:*

$$(z-\alpha)(z-\beta)\frac{\phi_n'(z)}{n} = (z+\alpha_n)\phi_n(z) + \beta_n\phi_{n-1}(z)\,.$$

Le soluzioni sono i polinomi di Szegö ed i polinomi i cui nuclei monici normalizzati sono ortogonali. Se ne deduce che sul cerchio unità l'equazione considerata e la condizione di Hahn non sono tra loro equivalenti.

Abstract: *In this paper we obtain the orthogonal polynomial sequences, related to the unit circle, that verify the following differential-difference equation:*

$$(z-\alpha)(z-\beta)\frac{\phi_n'(z)}{n} = (z+\alpha_n)\phi_n(z) + \beta_n\phi_{n-1}(z)\,.$$

Since these solutions are the Szegö polynomials and those whose normalized monic kernels are orthogonal, we conclude that on the unit circle the above equation and Hahn's condition are not equivalent.

1 – Introduction

Let $\mathbb{P}$ be the space of polynomials with complex coefficients and let u be a linear continuous functional on $\mathbb{P}$. We say that u is regular (positive

---

Key Words and Phrases: *Orthogonal polynomials - unit circle - classical polynomials - semiclassical polynomials*

A.M.S. Classification: 33C45 - 42C05

Research supported by the DGICYT under grant number PB93-1169 and Consellería de Educación e Ordenación Universitaria de la Xunta de Galicia.

definite) if the principal minors of the Hankel matrix associated to the moment sequence $u_n = u(x^n)$ for $n \in \mathbb{N}$ are nonsingular (positive). In the positive definite case there exists a unique positive measure μ such that $u_n = \int_{\mathbb{R}} x^n d\mu$ for $n \in \mathbb{N}$.

A regular or positive definite functional u defines a symmetric bilinear form ϕ_u on $\mathbb{P} \times \mathbb{P}$, with respect to the shift operator is symmetric. Therefore, there exists a unique sequence of monic orthogonal polynomials (*M.O.P.S.*) respect to ϕ_u, that is, a family of polynomials $\{P_n(x)\}_{n\in\mathbb{N}}$ that satisfies:

$$\deg P_n(x) = n \tag{1}$$

$$\phi_u(P_n(x), P_m(x)) = u(P_n(x)P_m(x)) = k_n\delta_{n,m} \tag{2}$$

with $k_n \neq 0 \; \forall n \in \mathbb{N}$.

The most extensively studied and widely applied orthogonal systems are called classical polynomials. This class includes the well-known polynomial systems of Jacobi, Laguerre, Hermite and Bessel. From the differential point of view these four classical families of orthogonal polynomials can be characterized by the following equivalent properties:

C-1) They are the only sequences related to a Hankel infinite regular matrix whose derivatives are again orthogonal polynomials or are reducible to them by a linear change of independent variable (see [7]).

C-2) The classical families are the only *M.O.P.S.* related to a Hankel, infinite and regular matrix that satisfy an ordinary differential equation:

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x,n)y(x) = 0$$

with $a_k(x) \in \mathbb{P}_k \;\; k = 0, 1, 2$ (see [2]).

C-3) They are also the only *M.O.P.S.* related to a Hankel, infinite and regular matrix whose moment functional satisfies the distributional equation $D(\phi u) + \psi u = 0$ where $\phi(x), \psi(x) \in \mathbb{P}_2$ with $\deg \psi(x) = 1$ (see [4]).

C-4) Finally, in [1], Al-Salam and Chihara have shown that these classical polynomials are the only orthogonal solutions of the differential-difference equation:

$$\phi(x)y_n'(x) = (a_nx + b_n)y_n(x) + c_ny_{n-1}(x)$$

with $\phi(x) \in \mathbb{P}_2$.

On the unit circle $\mathbb{T}$ the classical families in the Hahn sense (condition C-1 before) were characterized in [9]. They are the orthogonal polynomials related to the Lebesgue measure on $\mathbb{T}$ and it is easy to prove that C-2 is only satisfied by the same sequence. Therefore on the unit circle the Hahn and Bochner characterizations are equivalent.

In this paper we study the corresponding condition C-4; that is we solve the following problem: Find the orthogonal polynomial sequences $\{\phi_n(z)\}_{n\in\mathbb{N}}$ on the unit circle such that there exists a polynomial $A(z)$ with degree less or equal than two satisfying

$$A(z)\phi_n'(z) = (a_n z + b_n)\phi_n(z) + c_n\phi_{n-1}(z)\,.$$

Since the problem for polynomials $A(z)$ of degree one was solved in [11], in the present paper we restrict ourselves to the case $\deg A(z) = 2$.

2 – Preliminary results

Let Λ be the space of Laurent polynomials, that is, $\Lambda = \operatorname{span}\{z^k\}_{k\in\mathbb{Z}}$ and let $L : \Lambda \to \mathbb{C}$ be a linear hermitian and regular functional. If we denote the moments by $c_n = L(z^n)$ for $n \in \mathbb{Z}$ we say that:

L is hermitian if $c_{-n} = \overline{c_n}$ for all $n \in \mathbb{N}$.

L is regular (positive definite) if the principal submatrices of the moment matrix are nonsingular (positive), that is,

$$\forall n \geq 0 \;\; \Delta_n = \det\left(L(z^{i-j})\right)_{i=0,\ldots,n;j=0,\ldots,n} \neq 0 (> 0)\,.$$

In the positive definite case there exists a Borel finite and positive measure with infinite support on $[0, 2\pi]$ such that $\forall n \; c_n = \int_0^{2\pi} e^{in\theta} d\mu(\theta)$.

If we denote by $\{\phi_n(z)\}_{n\in\mathbb{N}}$ the monic orthogonal polynomial sequence related to L $(M.O.P.S.(L))$, then it is known that $\{\phi_n(z)\}_{n\in\mathbb{N}}$ satisfy the following recurrence relations which are equivalent:

(R1) $$\phi_n(z) = z\phi_{n-1}(z) + \phi_n(0)\phi^*_{n-1}(z)$$

(R2) $$\phi_n(z) = (1 - |\phi_n(0)|^2) z\phi_{n-1}(z) + \phi_n(0)\phi^*_n(z)$$

(R3) $$\phi^*_n(z) = \phi^*_{n-1}(z) + \overline{\phi_n(0)} z\phi_{n-1}(z)$$

(R4) $$\phi^*_n(z) = (1 - |\phi_n(0)|^2)\phi^*_{n-1}(z) + \overline{\phi_n(0)}\phi_n(z)$$

where the $*$-operator is defined for a polynomial $P(z)$ of degree n by $P^*(z) = z^n \overline{P}(\frac{1}{z})$. Also it is known that in the positive definite case, the zeros of the corresponding orthogonal polynomials are inside the unit disk D and therefore $\forall n \geq 1$ $|\phi_n(0)| < 1$.

Conversely, Favard's theorem, in [5], says that given a sequence of complex numbers $\{a_n\}_{n\geq 1}$ with $|a_n| < 1$ $\forall n \geq 1$ there exists a Borel finite and positive measure with infinite support on $[0, 2\pi]$ such that the sequence defined by (R1) with $\phi_n(0) = a_n$ is orthogonal with respect to μ.

In the regular case the zeros of the corresponding orthogonal polynomials are not on $\mathbb{T}$ and therefore $\forall n \geq 1$ $|\phi_n(0)| \neq 1$ (see [11]).

The corresponding theorem of Favard for the regular case can be formulated as follows:
Given a sequence of complex numbers $\{a_n\}_{n\geq 1}$ with $|a_n| \neq 1$ $\forall n \geq 1$ there exists a regular hermitian functional L on Λ such that the sequence defined by (R1) with $\phi_n(0) = a_n$ is the *M.O.P.S.*(L).

3 – The difference-differential equation

The following lemma is crucial in our proof of the main result in theorem 3:

LEMMA 1. *If* $\{\phi_n(z)\}_{n\in\mathbb{N}}$ *is a M.O.P.S. related to* $\mathbb{T}$ *that verifies the following difference-differential equation:*

$$(z-\alpha)(z-\beta)\frac{\phi_n'(z)}{n} = (z+\alpha_n)\phi_n(z) + \beta_n\phi_{n-1}(z) \tag{3.1}$$

for $n \geq 1$, *then it holds:*

$$\begin{aligned} &\big((z-\alpha)(z-\beta)(1+\overline{\alpha_n}z)|\phi_n(0)|^2 - (1-\overline{\alpha}z)(1-\overline{\beta}z)A_n(z)\big)\phi_n^*(z) = \\ &= \big((1-\overline{\alpha}z)(1-\overline{\beta}z)B_n(z) - (z-\alpha)(z-\beta)\overline{\beta_n}z^2|\phi_n(0)|^2\big)\phi_{n-1}^*(z) \end{aligned} \tag{3.2}$$

for $n \geq 2$ *with* $A_n(z), B_n(z) \in \mathbb{P}_1$.

Proof. By taking derivatives in relation (R2) and by applying that (see [11])

$$z(\phi_n^*(z))' = n\phi_n^*(z) - (\phi_n'(z))^*$$

we get:

$$\phi_n'(z) = (1 - |\phi_n(0)|^2)\left(\phi_{n-1}(z) + z\phi_{n-1}'(z)\right) + \\ + \frac{\phi_n(0)}{z}\left(n\phi_n^*(z) - (\phi_n'(z))^*\right) .$$

If we substitute this last expression in (3.1) we obtain:

$$\begin{aligned} &\frac{(1 - |\phi_n(0)|^2)}{n}(z-\alpha)(z-\beta)z\phi_{n-1}(z) + \\ &+ \frac{(1 - |\phi_n(0)|^2)}{n}(z-\alpha)(z-\beta)z^2\phi_{n-1}'(z) + \\ &+ \phi_n(0)(z-\alpha)(z-\beta)\phi_n^*(z) + \\ &- \frac{\phi_n(0)}{n}(z-\alpha)(z-\beta)(\phi_n'(z))^* = \\ &= z(z+\alpha_n)\phi_n(z) + \beta_n z\phi_{n-1}(z) . \end{aligned} \tag{3.3}$$

By substituting again (3.1) for $n-1$ in (3.3) we get

$$\begin{aligned} \frac{\phi_n(0)}{n}(z-\alpha)(z-\beta)(\phi_n'(z))^* = &\frac{(1-|\phi_n(0)|^2)}{n}(z-\alpha)(z-\beta)z\phi_{n-1}(z) + \\ &+ (1 - |\phi_n(0)|^2)\frac{n-1}{n}z^2(z+\alpha_{n-1})\phi_{n-1}(z) + \\ &+ (1 - |\phi_n(0)|^2)\frac{n-1}{n}\beta_{n-1}z^2\phi_{n-2}(z) + \\ &+ \phi_n(0)(z-\alpha)(z-\beta)\phi_n^*(z) + \\ &- z(z+\alpha_n)\phi_n(z) - \beta_n z\phi_{n-1}(z) \qquad n \geq 2 . \end{aligned} \tag{3.4}$$

If we multiply by $\overline{\phi_n(0)}$ and use relations (R3) and (R4)

$$\begin{aligned}
&\frac{|\phi_n(0)|^2}{n}(z-\alpha)(z-\beta)(\phi_n'(z))^* = \\
&= \frac{(1-|\phi_n(0)|^2)}{n}(z-\alpha)(z-\beta)\left(\phi_n^*(z)-\phi_{n-1}^*(z)\right)+ \\
&\quad + (1-|\phi_n(0)|^2)\frac{n-1}{n}z(z+\alpha_{n-1})\left(\phi_n^*(z)-\phi_{n-1}^*(z)\right)+ \\
&\quad + |\phi_n(0)|^2(z-\alpha)(z-\beta)\phi_n^*(z)+ \\
&\quad + (1-|\phi_n(0)|^2)\frac{n-1}{n}\beta_{n-1}\overline{\phi_n(0)}z^2\phi_{n-2}(z)+ \\
&\quad - z(z+\alpha_n)\Big(\phi_n^*(z)-(1-|\phi_n(0)|^2)\phi_{n-1}^*(z)\Big)+ \\
&\quad - \beta_n\left(\phi_n^*(z)-\phi_{n-1}^*(z)\right)\,.
\end{aligned} \tag{3.5}$$

From (R2), by using (R3) we get that

$$\overline{\phi_n(0)}z^2\phi_{n-2}(z) = \frac{\phi_n^*(z)-(1+\overline{\phi_n(0)}\phi_{n-1}(0)z)\phi_{n-1}^*(z)}{(1-|\phi_{n-1}(0)|^2)}$$

and by substituting in (3.5) we finally obtain:

$$\frac{|\phi_n(0)|^2}{n}(z-\alpha)(z-\beta)(\phi_n'(z))^* = A_n(z)\phi_n^*(z)+B_n(z)\phi_{n-1}^*(z) \tag{3.6}$$

with

$$\begin{aligned}
A_n(z) = &\frac{(1-|\phi_n(0)|^2)}{n}(z-\alpha)(z-\beta)+(1-|\phi_n(0)|^2)\frac{n-1}{n}z(z+\alpha_{n-1})+ \\
&+\frac{(n-1)\Big(1-|\phi_n(0)|^2\Big)}{n\Big(1-|\phi_{n-1}(0)|^2\Big)}\beta_{n-1}+ \\
&+|\phi_n(0)|^2(z-\alpha)(z-\beta)-\beta_n-z(z+\alpha_n)
\end{aligned}$$

and

$$\begin{aligned}
B_n(z) = &\beta_n-\frac{(1-|\phi_n(0)|^2)}{n}(z-\alpha)(z-\beta)-(1-|\phi_n(0)|^2)\frac{n-1}{n}z(z+\alpha_{n-1})+ \\
&-\frac{(n-1)\Big(1-|\phi_n(0)|^2\Big)}{n\Big(1-|\phi_{n-1}(0)|^2\Big)}\beta_{n-1}\Big(1+\overline{\phi_n(0)}\phi_{n-1}(0)z\Big)+ \\
&+(1-|\phi_n(0)|^2)z(z+\alpha_n)\,.
\end{aligned}$$

On the other hand, from (3.1), by applying the $*$-operator we have

$$(1-\overline{\alpha}z)(1-\overline{\beta}z)\frac{(\phi_n'(z))^*}{n} = (1+\overline{\alpha_n}z)\phi_n^*(z) + \overline{\beta_n}z^2\phi_{n-1}^*(z)\,. \tag{3.7}$$

Finally from (3.6) and (3.7) by eliminating $(\phi_n'(z))^*$ we conclude the result. ∎

Theorem 1. *If* $\{\phi_n(z)\}_{n\in\mathbb{N}}$ *is a M.O.P.S. related to* $\mathbb{T}$ *that verifies* (3.1) *then*

$$\phi_n(0)=0 \;\; \forall n\geq 5.$$

Proof. From lemma 1 we have that for $n\geq 2$

$$P_{3,n}(z)\phi_n^*(z) = Q_{4,n}(z)\phi_{n-1}^*(z)$$

with $\deg P_{3,n}(z)\leq 3$ and $\deg Q_{4,n}(z)\leq 4$.

Assume that there exists $n\geq 5$ such that $\phi_n(0)\neq 0$. Then let

$$n_0=\min\{n\in\mathbb{N} : n\geq 5, \phi_n(0)\neq 0\}$$

As $\phi_{n_0}(0)\neq 0$ then $\phi_{n_0}^*(z)$ and $\phi_{n_0-1}^*(z)$ have no common roots and therefore $P_{3,n_0}(z)=0$ which implies $Q_{4,n_0}(z)=0$, that is:

$$(z-\alpha)(z-\beta)(1+\overline{\alpha_{n_0}}z)|\phi_{n_0}(0)|^2 = (1-\overline{\alpha}z)(1-\overline{\beta}z)A_{n_0}(z) \tag{3.8}$$

and

$$(z-\alpha)(z-\beta)\overline{\beta_{n_0}}|\phi_{n_0}(0)|^2 = (1-\overline{\alpha}z)(1-\overline{\beta}z)B_{n_0}(z)\,. \tag{3.9}$$

By comparing degrees in (3.9) we get $\overline{\beta_{n_0}}=0$ which implies $B_{n_0}(z)=0$ and therefore

$$(n_0-1)\beta_{n_0-1} = -\alpha\beta(1-|\phi_{n_0-1}(0)|^2)$$

and

$$\alpha+\beta-(n_0-1)\alpha_{n_0-1}+\alpha\beta\overline{\phi_{n_0}(0)}\phi_{n_0-1}(0)+n_0\alpha_{n_0}=0\,. \tag{3.10}$$

Then going back to (3.1) it follows $(z-\alpha)(z-\beta)\frac{\phi'_{n_0}(z)}{n_0} = (z+\alpha_{n_0})\phi_{n_0}(z)$ and so α or β must be a root of $\phi_{n_0}(z)$. Assume $\phi_{n_0}(\alpha) = 0$ which implies $\alpha \neq 0$ and $|\alpha| \neq 1$.

From (3.6), by applying the $*$-operator

$$(1-\overline{\alpha}z)(1-\overline{\beta}z)\frac{|\phi_{n_0}(0)|^2}{n_0}\phi'_{n_0}(z) = A^*_{n_0}(z)\phi_{n_0}(z)\,.$$

If we eliminate $\phi'_{n_0}(z)$ between these two last relations we deduce that

$$(1-\overline{\alpha}z)(1-\overline{\beta}z)|\phi_{n_0}(0)|^2(z+\alpha_{n_0}) = (z-\alpha)(z-\beta)A^*_{n_0}(z)\,. \tag{3.11}$$

Taking $z = \alpha$ in (3.11) it follows

$$(1-\alpha\overline{\beta})(\alpha+\alpha_{n_0}) = 0\,. \tag{3.12}$$

– If $\alpha\overline{\beta} = 1$ then $\beta = \frac{1}{\overline{\alpha}}$ with $|\beta| \neq 1$. By substituting in (3.1) it holds that

$$(z-\alpha)\Big(z-\frac{1}{\overline{\alpha}}\Big)\frac{\phi'_{n_0}(z)}{n_0} = (z+\alpha_{n_0})\phi_{n_0}(z).$$

Since $\phi_n(z)$ cannot have symmetric roots with respect to $\mathbb{T}$, then $\alpha_{n_0} = -\frac{1}{\overline{\alpha}}$ and $\phi_{n_0}(z) = (z-\alpha)^{n_0}$.

– If $\alpha_{n_0} = -\alpha$, then $(z-\beta)\frac{\phi'_{n_0}(z)}{n_0} = \phi_{n_0}(z)$. This implies $\phi_{n_0}(z) = (z-\beta)^{n_0}$ with $|\beta| \neq 1$ and since $\phi_{n_0}(\alpha) = 0$ then $\alpha = \beta$.

Next we try to determine n_0, taking into account that $\phi_{n_0}(z) = (z-\alpha)^{n_0}$ with $\alpha \neq 0$ and $|\alpha| \neq 1$. We distinguish between two cases:

i) If $n_0 > 5$, then $\phi_5(0) = ... = \phi_{n_0-1}(0) = 0$ and $\phi_{n_0}(0) \neq 0$.

In the positive definite case, that is, if $|\alpha| < 1$ then $\phi_{n_0-1}(z)$ has simple roots and $\phi_{n_0-1}(0) \neq 0$, (see [10]), which is a contradiction.

In the regular case, $\phi_{n_0-1}(z)$ must have simple roots in D (see [10]) and therefore this implies that $n_0 = 6$ and $\phi_6(z) = (z-\alpha)^6$.

– In case $\alpha\overline{\beta} = 1$ then

$$z(z-\alpha)\Big(z-\frac{1}{\overline{\alpha}}\Big)\frac{\phi'_5(z)}{5} = (z^2+\alpha_5 z+\beta_5)\phi_5(z)$$

As $\phi_5(\frac{1}{\overline{\alpha}}) \neq 0$, (see [10]), and $\phi_5(\alpha) \neq 0$ then $z\frac{\phi_5'(z)}{5} = \phi_5(z)$, which implies $\phi_5(z) = z^5$ and so $\phi_5(0) = 0$ which is a contradiction.

– In case $\alpha_6 = -\alpha$ which implies $\beta = \alpha$ we have $\phi_5(\frac{1}{\overline{\alpha}}) \neq 0$. Also since $\phi_6(0) \neq 0$ then $\phi_5(\alpha) \neq 0$. Therefore $\phi_5(z) = z^5$ and so $\phi_5(0) = 0$ which is a contradiction.

ii) If $n_0 = 5$ we have $\phi_5(z) = (z-\alpha)^5$ with $\alpha \neq 0$, $B_5(z) = 0$, $\beta_5 = 0$ and $\alpha_5 = -\beta$. If we rewrite (3.8) we have $(z-\alpha)(z-\beta)|\phi_5(0)|^2 = (1-\overline{\alpha}z)A_5(z)$ from which we deduce $\beta = \frac{1}{\overline{\alpha}}$. Indeed if $\beta = \alpha$ then $\alpha = \frac{1}{\overline{\alpha}}$ which implies $|\alpha| = 1$ (contradiction).

Next we prove that relation (3.1) for $n = 5$ is not true and thus $\phi_n(0) = 0 \;\; \forall n \geq 5$ as we wanted to prove:

In this situation, from (R2) we obtain that:

$$\phi_4(z) = z^4 + \frac{5\alpha(1-|\alpha|^8)}{|\alpha|^{10}-1}z^3 - \frac{10\alpha^2(1-|\alpha|^6)}{|\alpha|^{10}-1}z^2 + \\ + \frac{10\alpha^3(1-|\alpha|^4)}{|\alpha|^{10}-1}z + \frac{5\alpha^4(1-|\alpha|^2)}{|\alpha|^{10}-1}$$

and

$$\phi_3(z) = z^3 + \frac{50\alpha|\alpha|^6(|\alpha|^2-1)(1-|\alpha|^4) - 5\alpha(1-|\alpha|^8)(|\alpha|^{10}-1)}{(|\alpha|^{10}-1)^2(|\phi_4(0)|^2-1)}z^2 + \\ + \frac{-50\alpha^2|\alpha|^4(|\alpha|^2-1)(1-|\alpha|^6) + 10\alpha^2(1-|\alpha|^6)(|\alpha|^{10}-1)}{(|\alpha|^{10}-1)^2(|\phi_4(0)|^2-1)}z + \\ + \frac{25\alpha^3|\alpha|^2(|\alpha|^2-1)(1-|\alpha|^8) - 10\alpha^3(1-|\alpha|^4)(|\alpha|^{10}-1)}{(|\alpha|^{10}-1)^2(|\phi_4(0)|^2-1)}$$

Since $\beta_4 = \frac{\alpha}{4\overline{\alpha}}(|\phi_4(0)|^2-1)$ and $4\alpha_4 = \alpha - \frac{4}{\overline{\alpha}} - \frac{5\alpha|\alpha|^8(|\alpha|^2-1)}{|\alpha|^{10}-1}$, if we identify the coefficients of z in

$$(z-\alpha)\Big(z-\frac{1}{\overline{\alpha}}\Big)\frac{\phi_4'(z)}{4} = (z+\alpha_4)\phi_4(z) + \beta_4\phi_3(z)$$

we deduce that $|\alpha| = 1$ which is impossible. □

Theorem 2. *If* $\{\phi_n(z)\}_{n\in\mathbb{N}}$ *is a M.O.P.S. related to* $\mathbb{T}$ *that verifies* (3.1) *then* $\phi_4(0) = 0$ *and* $\phi_3(0) = 0$.

Proof. At first, we prove that $\phi_4(0) = 0$.
Since $\phi_5(0) = 0$ then $\phi_5(z) = z\phi_4(z)$.
We write (3.1) for $n = 5$ obtaining:

$$(z-\alpha)(z-\beta)\frac{\phi_4(z) + z\phi_4'(z)}{5} = (z+\alpha_5)z\phi_4(z) + \beta_5\phi_4(z),$$

which implies

$$z(z-\alpha)(z-\beta)\frac{\phi_4'(z)}{5} = \left(\frac{4}{5}z^2 + \Big[\alpha_5 + \frac{1}{5}(\alpha+\beta)\Big]z + \beta_5 - \frac{\alpha\beta}{5}\right)\phi_4(z).$$

If $\phi_4(0) \neq 0$ then $\beta_5 = \frac{\alpha\beta}{5}$ and

$$(z-\alpha)(z-\beta)\frac{\phi_4'(z)}{5} = \left(\frac{4}{5}z + \alpha_5 + \frac{1}{5}(\alpha+\beta)\right)\phi_4(z)\,.$$

Therefore $\phi_4(z)$ must have α or β as a root. If we assume that $\phi_4(\alpha) = 0$ then from (3.1) for $n = 4$ we deduce that $\beta_4 = 0$.

Let us consider (3.2) for $n = 4$.
If $(z-\alpha)(z-\beta)(1+\overline{\alpha_4}z)|\phi_4(0)|^2 - (1-\overline{\alpha}z)(1-\overline{\beta}z)A_4(z) \neq 0$ we conclude that $\phi_4^*(z)$ and $\phi_3^*(z)$ must have a common root which is impossible if $\phi_4(0) \neq 0$.

Then assume

$$(z-\alpha)(z-\beta)(1+\overline{\alpha_4}z)|\phi_4(0)|^2 = (1-\overline{\alpha}z)(1-\overline{\beta}z)A_4(z)\,, \tag{3.13}$$

which implies $B_4(z) = 0$.

From $B_4(z) = 0$ we obtain that $\beta_3 = -\frac{\alpha\beta}{3}(1-|\phi_3(0)|^2)$ and $4\alpha_4 = 3\alpha_3 - (\alpha+\beta) - \alpha\beta\overline{\phi_4(0)}\phi_3(0)$.

From (3.13), following the same argument as in theorem 1, we deduce that

- If $1 = \overline{\alpha}\beta$ then $\alpha_4 = -\frac{1}{\overline{\alpha}}$, $\phi_4(z) = (z-\alpha)^4$, $A_4(z) = -\frac{|\phi_4(0)|^2}{\overline{\alpha}}(z-\alpha)$ and thus $3\alpha_3 = -\frac{3}{\overline{\alpha}} + \alpha + \alpha\overline{\alpha}^3\phi_3(0)$.

On the other hand by identifying coefficients in $A_4(z)$ we get

$$3\alpha_3 = \frac{\alpha + 3\alpha|\alpha|^8 - \frac{3}{\overline{\alpha}} - \frac{|\alpha|^8}{\overline{\alpha}}}{1-|\alpha|^8}\,.$$

From both expressions of α_3, by taking into account that $\phi_3(0) = \frac{4\alpha^3(|\alpha|^2-1)}{1-|\alpha|^8}$ we obtain $\alpha = 0$ (contradiction).

– If $\alpha_4 = -\alpha$ then $\beta = \alpha$ and $\phi_4(z) = (z-\alpha)^4$.

By substituting the last in (3.13) it holds that $\alpha = \frac{1}{\overline{\alpha}}$ which is impossible.

Therefore $\phi_4(0) = 0$.

Next, we prove that $\phi_3(0) = 0$.

Following the same argument as in the previous case it holds that

$$z(z-\alpha)(z-\beta)\frac{\phi_3'(z)}{4} = \left(\frac{3}{4}z^2 + \Big[\alpha_4 + \frac{1}{4}(\alpha+\beta)\Big]z + \beta_4 - \frac{\alpha\beta}{4}\right)\phi_3(z).$$

If $\phi_3(0) \neq 0$ then $\beta_4 = \frac{\alpha\beta}{4}$ and $(z-\alpha)(z-\beta)\frac{\phi_3'(z)}{4} = \left(\frac{3}{4}z + \alpha_4 + \frac{1}{4}(\alpha+\beta)\right)$ $\phi_3(z)$.

Therefore $\phi_3(z)$ must have α or β as a root. If we assume that $\phi_3(\alpha) = 0$ then from (3.1) for $n = 3$ we deduce that $\beta_3 = 0$ and $\phi_3(z)$ and $\phi_3'(z)$ must have, at least, a common root.

From (3.2) for $n = 3$ we obtain

$$\Big((z-\alpha)(z-\beta)(1+\overline{\alpha_3}z)|\phi_3(0)|^2 - (1-\overline{\alpha}z)(1-\overline{\beta}z)A_3(z)\Big)\phi_3^*(z) =$$
$$= \Big((1-\overline{\alpha}z)(1-\overline{\beta}z)B_3(z)\Big)\phi_2^*(z)\,.$$

By identifying the degrees of both members we have that

$$\overline{\alpha_3}|\phi_3(0)|^2 - \overline{\alpha}\overline{\beta}a_3 = 0,$$

where a_3 is the principal coefficient of $A_3(z)$.

Then if we suppose the following polynomial of degree less or equal than 2 is different from zero $(z-\alpha)(z-\beta)(1+\overline{\alpha_3}z)|\phi_3(0)|^2 - (1-\overline{\alpha}z)(1-\overline{\beta}z)A_3(z) \neq 0$, it follows that

$$\phi_3^*(z) \quad \text{divides} \quad (1-\overline{\alpha}z)(1-\overline{\beta}z)B_3(z),$$

then $\phi_3^*(\frac{1}{\overline{\beta}}) = 0$ and

$$\text{(3.14)}\quad \phi_2^*(z) \text{ divides } (z-\alpha)(z-\beta)(1+\overline{\alpha_3}z)|\phi_3(0)|^2-(1-\overline{\alpha}z)(1-\overline{\beta}z)A_3(z)\,.$$

On the other hand from (3.1) for $n = 3$, taking into account that $\beta_3 = 0$ and $\phi_3(\beta) = 0$, we deduce that

$$\phi_3(z) = (z-\alpha)^2(z-\beta) \quad \text{and} \quad \alpha_3 = -\alpha, \qquad \text{or}$$
$$\phi_3(z) = (z-\alpha)(z-\beta)^2 \quad \text{and} \quad \alpha_3 = -\beta$$

Suppose the first situation (the second is similar), from (3.14) it holds that $\phi_2^*(\frac{1}{\overline{\alpha}}) = 0$. Therefore $\phi_2(\alpha) = 0$ which is impossible.

If

$$\text{(3.15)}\qquad (z-\alpha)(z-\beta)(1+\overline{\alpha_3}z)|\phi_3(0)|^2-(1-\overline{\alpha}z)(1-\overline{\beta}z)A_3(z) = 0$$

then $B_3(z) = 0$ and from expression of $B_n(z)$ in (3.6) it holds that $\beta_2 = -\frac{\alpha\beta(1-|\phi_2(0)|^2)}{2}$ and $3\alpha_3 = 2\alpha_2 - (\alpha+\beta) - \alpha\beta\overline{\phi_3(0)}\phi_2(0)$.

Let $z = \frac{1}{\overline{\alpha}}$ then $\beta\overline{\alpha} = 1$ or $\alpha_3 = -\alpha$.

- If $\alpha_3 = -\alpha$, from (3.1) for n=3 it holds that $\phi_3(z) = (z-\beta)^3$ then $\alpha = \beta$ and from (3.15) we conclude that $\alpha = \frac{1}{\overline{\alpha}}$.
- If $\overline{\alpha}\beta = 1$, from (3.1), for $n = 3$ it holds that $\phi_3(z) = (z-\alpha)^3$. From (3.15) we get $A_3(z) = -\frac{\alpha}{\overline{\alpha}^2}|\phi_3(0)|^2(z-\overline{\alpha})$. In this case $2\alpha_2 = -\frac{2}{\overline{\alpha}} + \alpha - \alpha\overline{\alpha}^2\phi_2(0)$.

  On the other hand, by identifying coefficients in $A_3(z)$ we get

$$2\alpha_2 = \frac{3}{1-|\alpha|^6}\left(-\frac{\alpha}{\overline{\alpha}^2}|\alpha|^6 + \alpha|\alpha|^6 + \frac{1}{\overline{\alpha}}|\alpha|^6 - \frac{1}{\overline{\alpha}} + \frac{(1-|\alpha|^6)(1-|\alpha|^2)}{3\overline{\alpha}}\right)$$

  From both expressions of α_2, we deduce $\phi_2(0) = \frac{3\alpha^2(|\alpha|^2-\alpha)}{|\alpha|^6-1}$ and from the recurrence relations $\phi_2(0) = \frac{3\alpha^2(|\alpha|^2-1)}{|\alpha|^6-1}$ which implies $\alpha = 1$. Then necessarily $\phi_3(0) = 0$. $\square$

Theorem 3. *The M.O.P.S.* $\{\phi_n(z)\}$ *such that verifies* (3.1) *are the following*

$$\phi_n(z) = z^n \tag{3.16}$$

$$\phi_n(z) = z^{n-1}(z-\alpha)\,\forall\, n \geq 1 \quad \textit{with} \quad |\alpha| \neq 1 \quad \textit{and} \quad \alpha \neq 0. \tag{3.17}$$

$$\phi_n(z) = z^{n-2}(z-\alpha)^2 \quad \forall\, n \geq 2,\ \phi_1(z) = z - \frac{2\alpha}{1+|\alpha|^2} \tag{3.18}$$

$$\textit{with } |\alpha| \neq 1 \textit{ and } \alpha \neq 0\,.$$

$$\phi_n(z) = z^{n-2}(z-\alpha)(z-\beta) \quad n \geq 2,\ \textit{and} \tag{3.19}$$

$$\phi_1(z) = z + \frac{\alpha(1-|\beta|^2) + \beta(1-|\alpha|^2)}{|\alpha\beta|^2 - 1},$$

$$\textit{with } |\alpha| \neq 1, |\beta| \neq 1, \alpha \neq \beta \textit{ and } \alpha\beta \neq 0\,.$$

Proof. Since $\phi_3(z) = z\phi_2(z)$ then from (3.1) for $n = 3$

$$(z-\alpha)(z-\beta)\frac{\phi_2(z) + z\phi_2'(z)}{3} = (z^2 + \alpha_3 z + \beta_3)\phi_2(z)\,. \tag{3.20}$$

Next, we distinguish the following cases

- If $\phi_2(\alpha) \neq 0$ and $\phi_2(\beta) \neq 0$ then $(z-\alpha)(z-\beta) = (z^2 + \alpha_3 z + \beta_3)$ and from (3.20) it holds that $\phi_2(z) = z^2$.
- If $\phi_2(\alpha) = \phi_2(\beta) = 0$ then from (3.1) for n=2 taking $z = \alpha$ and $z = \beta$ it holds that $\phi_2(0) = 0$ or $\phi_2(z) = (z-\alpha)(z-\beta)$ with $\alpha \neq \beta$.
- If $\phi_2(\alpha) = 0$ and $\phi_2(\beta) \neq 0$ then from (3.1) for n=2 taking $z = \alpha$ it holds that $\phi_2(0) = 0$ or $\phi_2(z) = (z-\alpha)^2$.

To sum up, we have the following possibilities for $\phi_2(z)$

(i) $\phi_2(0) = 0$.
(ii) $\phi_2(z) = (z-\alpha)(z-\beta)$ with $|\alpha| \neq 1$ $|\beta| \neq 1$, $\alpha\beta \neq 0$ and $\alpha \neq \beta$.
(iii) $\phi_2(z) = (z-\alpha)^2$, $|\alpha| \neq 1$ and $\alpha \neq 0$.

In the case (i) $\phi_2(z) = z\phi_1(z)$. If we distinguish the following situations:

- If $\phi_1(\alpha) \neq 0$ and $\phi_1(\beta) \neq 0$ then $(z-\alpha)(z-\beta) = (z^2 + \alpha_2 z + \beta_2)$ and $\phi_1(z) = z$.

– If $\phi_1(\alpha) = \phi_1(\beta) = 0$ then $\phi_1(z) = (z - \alpha)$.
– If $\phi_1(\alpha) = 0$ and $\phi_1(\beta) \neq 0$ then $\phi_1(z) = z - \alpha$. From (3.1) for $n = 1$ it holds that $\beta_1 = 0$ and $\alpha_1 = -\beta$.

In the case (ii), from (3.1) for $n = 2$

$$(z-\alpha)(z-\beta)\Big[z - \frac{\alpha+\beta}{2}\Big] = (z+\alpha_2)(z-\alpha)(z-\beta) + \beta_2\phi_1(z)\,.$$

Taking $z = \alpha$ and $z = \beta$ in the last expression we obtain $\beta_2 = 0$ and $\alpha_2 = -\frac{\alpha+\beta}{2}$.

Well now, by applying the descending recurrence relation it holds that

$$\phi_1^*(z) = \frac{|\alpha\beta|^2 - 1 + (\overline{\alpha} + \overline{\beta} - |\alpha|^2\overline{\beta} - |\beta|^2\overline{\alpha})z}{|\alpha\beta|^2 - 1}\,,$$

then

$$\phi_1(z) = z + \frac{\alpha(1-|\beta|^2) + \beta(1-|\alpha|^2)}{|\alpha\beta|^2 - 1}\,.$$

In the case (iii) from (3.1) for $n = 2$

$$(z-\alpha)(z-\beta)(z-\alpha) = (z+\alpha_2)(z-\alpha)^2 + \beta_2\phi_1(z)\,,$$

putting $z = \alpha$ it holds $\beta_2\phi_1(\alpha) = 0$.

Taking derivatives in the previous relation and putting $z = \alpha$ in the resulting expression we get $\alpha_2 = -\beta$.

By applying the descending recurrence relation we obtain as in (ii)

$$\phi_1(z) = z - \frac{2\alpha}{1+|\alpha|^2}\,. \qquad \square$$

Remark. If we take into account that the only sequences $\{\phi_n(z)\}$ such that the sequences of monic kernel polynomials $\{\tilde{K}_n(z,c)\}$ are orthogonal are the following

$$\phi_n(z) = z^{n-2}(z - \frac{1}{\bar{c}})(z-b) \quad \text{for each} \quad n \geq 2$$

with $0 \neq |c| \neq 1$ and $|c| \neq |b| \neq 1$, (see [11] and [3]), we conclude:

Corollary 1. *If $\{\phi_n(z)\}$ is a M.O.P.S. related to $\mathbb{T}$ such that verifies* (3.1) *then $\{\phi_n(z)\}$ is the sequence related to the Lebesgue measure or $\{\phi_n(z)\}$ is such that the sequence of normalized kernels $\{\tilde{K}_n(z, \frac{1}{\bar{\alpha}})\}$ are orthogonal with $0 \neq |\alpha| \neq 1$.*

Remark. The sequences of polynomials $\{\phi_n(z)\}$ given in theorem 3 are solutions of the difference-differential equation (3.1). Next, we obtain the difference-differential equation for each case.

By substituting in (3.1) after some easy calculations we have:

– If $\{\phi_n(z)\}$ is given by (3.16) then

$$(z-\alpha)(z-\beta)\frac{\phi_n'(z)}{n} = (z-(\alpha+\beta))\,\phi_n(z) + \alpha\beta\phi_{n-1}(z).$$

In this case the difference-differential equation can be reduced to the following

$$\frac{\phi_n'(z)}{n} = \phi_{n-1}(z)$$

– If $\{\phi_n(z)\}$ is given by (3.17) then

$$(z-\alpha)(z-\beta)\frac{\phi_n'(z)}{n} = \left(z-(\beta+\frac{n-1}{n}\alpha)\right)\phi_n(z) + \frac{n-1}{n}\alpha\beta\phi_{n-1}(z).$$

In this case the difference-differential equation can be reduced to the following

$$(z-\alpha)\frac{\phi_n'(z)}{n} = \phi_n(z) - \frac{n-1}{n}\alpha\phi_{n-1}(z).$$

– If $\{\phi_n(z)\}$ is given by (3.18) then

$$(z-\alpha)(z-\beta)\frac{\phi_n'(z)}{n} = \left(z-(\beta+\frac{n-2}{n}\alpha)\right)\phi_n(z) + \\ + \frac{n-2}{n}\alpha\beta\phi_{n-1}(z), \forall\, n \geq 2$$

and $\alpha_1 = \frac{\alpha-\beta-|\alpha|^2(\alpha+\beta)}{1+|\alpha|^2}$, $\beta_1 = \frac{\alpha(|\alpha|^2-1)\left(\beta(|\alpha|^2+1)-2\alpha\right)}{(1+|\alpha|^2)^2}$

– At last, if $\{\phi_n(z)\}$ is given by (3.19) then

$$(z-\alpha)(z-\beta)\frac{\phi_n'(z)}{n} = \left(z - \frac{n-1}{n}(\alpha+\beta)\right)\phi_n(z) + \\ + \frac{n-2}{n}\alpha\beta\phi_{n-1}(z), \forall\, n \geq 2\,.$$

and

$$(z-\alpha)(z-\beta)\phi_1'(z) = (z+\alpha_1)\phi_1(z) + \beta_1$$

with

$$\alpha_1 = \frac{\beta|\alpha|^2(1-|\beta|^2) + \alpha|\beta|^2(1-|\alpha|^2)}{|\alpha\beta|^2 - 1}$$

and

$$\beta_1 = \frac{\alpha\beta(|\alpha\beta|^2-1)^2 - \alpha\beta|\alpha|^2(1-|\beta|^2)^2}{(|\alpha\beta|^2-1)^2} + \\ + \frac{-(\alpha^2|\beta|^2 + \beta^2|\alpha|^2)(1-|\alpha|^2)(1-|\beta|^2) - \alpha\beta|\beta|^2(1-|\alpha|^2)^2}{(|\alpha\beta|^2-1)^2}$$

4 – The smallest class

Definition 1. Given a linear, regular and hermitian functional L, we say that L is semiclassical if there exist two polynomials $A(z) \neq 0$ and $B(z)$ such that

$$D\,(A(z)L) = B(z)L$$

The derivative of a linear hermitian functional means

$$< DL, P(z) >= -i < L, zP'(z) > \qquad \forall P \in \Lambda.$$

If $\deg A(z) = p'$ and $\max\{p'-1, \deg\left((p'-1)A(z) + iB(z)\right)\} = q$, we say that L belongs to the class (p', q).

Remark. It is obvious that if L is a functional that belongs to the class (p', q) then L is a functional that belongs to the class $(p'+1, q+1)$.

It is known that the sequences $\{\phi_n(z)\}$ given in theorem 3 are semiclassical families.

– If $\{\phi_n(z)\}$ is given by (3.16) then $A(z) = 1$ and $B(z) = 0$ then $(p', q) = (0, 0)$.
– If $\{\phi_n(z)\}$ is given by (3.17) then $A(z) = (z-\alpha)(1-\overline{\alpha}z)$ and $B(z) = iA(z)$ then $(p', q) = (2, 1)$.
– If $\{\phi_n(z)\}$ is given by (3.18) then $A(z) = (z-\alpha)^2(1-\overline{\alpha}z)^2$ and $B(z) = 2iA(z)$ then $(p', q) = (4, 4)$.
– If $\{\phi_n(z)\}$ is given by (3.19) then $A(z) = (z-\alpha)(z-\beta)(1-\overline{\alpha}z)(1-\overline{\beta}z)$ and $B(z) = 2iA(z)$ then $(p', q) = (4, 4)$.

Next, we study if the class can be reduced. At first, we recall some results about semiclassical functionals (see [11]).

Theorem 4. *Let $S(z)$ be the series given as follows:*

$$S(z) = \sum_{k=0}^{+\infty} \overline{c_k} z^k.$$

If L is a regular functional such that $D(A(z)L) = B(z)L$, then:

$$zA(z)S'(z) + i\left(B(z) - izA'(z)\right)S(z) = C(z) \tag{4.1}$$

where $C(z)$ is a polynomial such that $\deg C(z) \leq \max(p', q)$.

Theorem 5. *Let L be a regular functional such that $D(A(z)L) = B(z)L$ and assume that*

$$z - z_0 \quad \textit{divides} \quad \gcd(A(z), B(z) - izA'(z)).$$

Then L verifies

$$D\left(\tilde{A}(z)L\right) = \tilde{B}(z)L \tag{4.2}$$

with $A(z) = (z - z_0)\tilde{A}(z)$ and $\tilde{B}(z) = iz\tilde{A}'(z) + \frac{B(z)-izA'(z)}{z-z_0}$ if and only if $C(z_0) = 0$.

DEFINITION 2. Given a semiclassical functional L. We say that $D\,(A(z)L) = B(z)L$ is a smallest class for L if

$$\gcd\left(A(z), B(z) - izA'(z), C(z)\right) = 1,$$

where $C(z)$ is defined in theorem 4.

COROLLARY 2. *If* $\{\phi_n(z)\}$ *is given by* (3.16) *the smallest class is* $(0,0)$.
If $\{\phi_n(z)\}$ *is given by* (3.17) *the smallest class is* $(2,1)$.
If $\{\phi_n(z)\}$ *is given by* (3.18) *the smallest class is* $(2,2)$ *with* $\tilde{A}(z) = (z-\alpha)(1-\overline{\alpha}z)$ *and* $\tilde{B}(z) = i\left[(|\alpha|^2+1)z - 2\alpha\right]$.
If $\{\phi_n(z)\}$ *is given by* (3.19) *the smallest class is* $(4,4)$.

PROOF. The first statement is obvious and the second statement it follows taking into account that the class $(1,0)$ is the empty set (see [11]).

Next, we prove the third statement.

Let $\{\phi_n(z)\}$ the *M.O.P.S.* given by (3.18). In this situation $B(z) - izA'(z) = 2i(1-\overline{\alpha}z)(z-\alpha)(\overline{\alpha}z^2-\alpha)$.

On the other hand, taking into account that $L\left(\phi_n(z)\right) = 0\ \forall n \geq 1$ it holds

$$c_1 = \frac{2\alpha}{1+|\alpha|^2}$$

and

$$c_n - 2\alpha c_{n-1} + \alpha^2 c_{n-2} = 0 \quad \forall n \geq 2.$$

On the assumption $c_0 = 1$ we solve the previous difference equation and we obtain

$$c_n = \frac{\alpha^n}{1+|\alpha|^2}\left((n+1) + (1-n)|\alpha|^2\right) \quad \forall n \geq 0,$$

hence

$$S(z) = \frac{1}{1+|\alpha|^2}\left(\frac{1+|\alpha|^2 - 2\overline{\alpha}|\alpha|^2 z}{(1-\overline{\alpha}z)^2}\right).$$

From theorem 4, $C(z) = \frac{2}{1+|\alpha|^2}(z-\alpha)(1-\overline{\alpha}z)\left(|\alpha|^2 z + \alpha(|\alpha|^2+1)\right)$ and from theorem 5 it follows the result.

In the last case, we obtain $B(z) - izA'(z) = i(\overline{\alpha}z^2 - \alpha)$ whose roots belong to $\mathbb{T}$. Then $\gcd(A(z), B(z) - izA'(z)) = 1$ and the class cannot be reduced. □

REFERENCES

[1] W. A. Al-Salam – T. S. Chihara: *Another characterization of the classical orthogonal polynomials*, SIAM J. Math. Anal. **3** (1972), 65-70.

[2] S. Bochner: *Uber Sturm-Liouvillsche Polynomsysteme*, Math. Zeit. **29** (1929), 730-736.

[3] A. Cachafeiro – C. Suárez: *Kernels on the unit circle. Orthogonality*, J. Comput. Appl. Math. **72** (1996), 407-420.

[4] T. S. Chihara: *An Introduction to Orthogonal Polynomials*, Gordon & Breach, New York (1978).

[5] T. Erdelyi – J. S. Geronimo – P.Nevai – J. Zhang: *A simple proof of "Favard's theorem" on the unit circle*, Atti. Sem. Mat. Fis. Univ. Modena **29** (1991), 41-46.

[6] U. Grenander – G. Szegö: *Toeplitz forms and their applications*, University of California Press, Berkeley (1958).

[7] W. Hahn: *Uber die jacobischen polynome und zwei verwandte Polynomklassen*, Math. Zeit. **39** (1935), 634-638.

[8] F. Marcellán: *Orthogonal Polynomials and Toeplitz matrices: Some applications*, Seminario Matemático García de Galdeano, Universidad de Zaragoza (1989), 31-57.

[9] F. Marcellán – P. Maroni: *Orthogonal Polynomials on the unit circle and their derivatives*, Constr. Approx. **7** (1991), 341-348.

[10] J. M. Montaner: *Matrices de Toeplitz banda y polinomios ortogonales sobre la circunferencia unidad*, Doctoral Dissertation, Universidad de Zaragoza (1992).

[11] C. Tasis: *Propiedades diferenciales de los polinomios ortogonales relativos a la circunferencia unidad*, Doctoral Dissertation, Universidad de Cantabria (1989).

Lavoro pervenuto alla redazione il 12 marzo 1996
ed accettato per la pubblicazione il 4 dicembre 1996.
Bozze licenziate il 10 luglio 1997

Indirizzo degli Autori:

A. Cachafeiro – C. Suárez – Departamento de Matemática Aplicada – Universidad de Vigo – 36.280 Vigo – Spain