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Sobolev-type orthogonal polynomials and their zeros

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Riassunto: *Nello spazio $\mathcal{P}$ dei polinomi in una variabile, essendo σ un funzionale dei momenti quasi-definito su $\mathcal{P}$, si considera la forma bilineare simmetrica $\phi(\cdot,\cdot)$ definita in $\mathcal{P}\times\mathcal{P}$ da $\phi(p,q) := \langle\sigma, pq\rangle + \lambda p^{(r)}(a)q^{(r)}(a) + \mu p^{(s)}(b)q^{(s)}(b)$, dove λ, μ, a, b sono numeri complessi e r, s sono interi non negativi. Si stabilisce una condizione necessaria e sufficiente affinché esista un sistema $\{R_n(x)\}_{n=0}^{\infty}$ di polinomi ortogonali relativi a ϕ. Si discutono le proprietà algebriche di $\{R_n(x)\}_{n=0}^{\infty}$ e si riconosce che, quando σ è semiclassico, $\{R_n(x)\}_{n=0}^{\infty}$ deve soddisfare un'equazione differenziale del secondo ordine con coefficienti polinomiali. Nel caso in cui σ sia definito positivo e λ, μ, a, b siano reali, si analizzano le relazioni tra gli zeri di $\{R_n(x)\}_{n=0}^{\infty}$ e gli zeri del sistema $\{P_n(x)\}_{n=0}^{\infty}$ dei polinomi ortogonali rispetto a σ.*

Abstract: *When σ is a quasi-definite moment functional on $\mathcal{P}$, the space of polynomials in one variable, we consider a symmetric bilinear form $\phi(\cdot,\cdot)$ on $\mathcal{P}\times\mathcal{P}$ defined by $\phi(p,q) := \langle\sigma, pq\rangle + \lambda p^{(r)}(a)q^{(r)}(a) + \mu p^{(s)}(b)q^{(s)}(b)$, where λ, μ, a, b are complex numbers and r, s are non-negative integers. We find a necessary and sufficient condition under which there is an orthogonal polynomial system $\{R_n(x)\}_{n=0}^{\infty}$ relative to ϕ and discuss their algebraic properties. When σ is semi-classical, we show that $\{R_n(x)\}_{n=0}^{\infty}$ must satisfy a second order differential equation with polynomial coefficients. When σ is positive-definite and λ, μ, a, b are real, we investigate the relations between zeros of $\{R_n(x)\}_{n=0}^{\infty}$ and of the system of the orthogonal polinomiels relative to σ.*

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1 – Introduction

Let $\mathcal{P}$ be the vector space of polynomials with complex coefficients in one variable and σ a quasi-definite moment functional on $\mathcal{P}$ (Chihara [5, page 16, Def. 3.2]). We consider a symmetric bilinear form on $\mathcal{P} \times \mathcal{P}$:

$$\phi(p,q) := \langle \sigma, pq \rangle + \lambda p^{(r)}(a) q^{(r)}(a) + \mu p^{(s)}(b) q^{(s)}(b) \tag{1.1}$$

where $\lambda, \mu, a, b \in \mathcal{C}$ with $\lambda \neq 0$ and r, s are nonnegative integers with $0 \leq r \leq s$ ($r < s$ if $a = b$). As in the case of moment functionals, we call $\phi(\cdot,\cdot)$ to be quasi-definite if there is a sequence of polynomials $\{R_n(x)\}_{n=0}^{\infty}$, which are orthogonal relative to $\phi(\cdot,\cdot)$. When $r = s = 0$, $\{R_n(x)\}_{n=0}^{\infty}$ are ordinary orthogonal polynomials relative to the moment functional $\sigma + \lambda\delta(x-a) + \mu\delta(x-b)$ and when $r+s \geq 1$, $\{R_n(x)\}_{n=0}^{\infty}$ are Sobolev-type orthogonal polynomials.

In this work, we view $\phi(\cdot,\cdot)$ as mass point perturbation of σ. Ever since H. L. Krall [10] found three new orthogonal polynomials (satisfying fourth order differential equations), which are orthogonal relative to classical weights plus one or two masses at the end points of the interval of orthogonality, many authors handled the problem of adding one or more mass points to a quasi-definite moment functional. When σ is positive-definite and $\lambda, \mu \geq 0$ and $r = s = 0$, see [6], [9]. When σ is semi-classical and $\mu = r = 0$, see [8,14]. When $r = s = 0$, see [7], [11]. When σ and λ, a are real and $r \geq 1$, $\mu = 0$, see [8], [16], [17]. There are also many results on various aspects of Sobolev-type orthogonal polynomials when σ is positive-definite and λ and $\mu \geq 0$; see [12], [13] and references therein.

We first find a necessary and sufficient condition for $\phi(\cdot,\cdot)$ to be quasi-definite (see theorem 2.1), which generalizes results in [7], [11], [14], [16] and then investigate algebraic properties of $\{R_n(x)\}_{n=0}^{\infty}$ in connection with $\{P_n(x)\}_{n=0}^{\infty}$, orthogonal polynomials relative to σ. When σ is semi-classical, we show that $\{R_n(x)\}_{n=0}^{\infty}$ must be solutions of a second order differential equation with polynomial coefficients, whose degrees are independent of n. Finally, when σ is positive-definite and λ, μ, a, b are real

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so that $R_n(x), n \geq 0$, are real polynomials, we discuss the behavior of zeros of $R_n(x)$ in connection with zeros of $P_n(x)$.

2 – Orthogonal polynomials $\{R_n(x)\}_{n=0}^{\infty}$

For any $p(x)$ in $\mathcal{P}$, we let $\deg(p)$ be the degree of $p(x)$ with the convention that $\deg(0) = -1$.

By a polynomial system (PS), we mean a sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ with $\deg(P_n) = n$, $n \geq 0$.

For any symmetric bilinear form $\phi(\cdot,\cdot)$ on $\mathcal{P} \times \mathcal{P}$, we let

$$\phi_{i,j} := \phi(x^i, x^j), \qquad i \text{ and } j \geq 0\,,$$

the moments of $\phi(\cdot,\cdot)$ and call $\phi(\cdot,\cdot)$ to be quasi-definite if

$$\Delta_n(\phi) := det[\phi_{i,j}]_{i,j=0}^{n} \neq 0, \qquad n \geq 0\,.$$

It's easy to see that $\phi(\cdot,\cdot)$ is quasi-definite if and only if there is a unique monic PS $\{R_n(x)\}_{n=0}^{\infty}$ such that

$$\phi(R_m, R_n) = k_n \delta_{mn}, \qquad m \text{ and } n \geq 0\,,$$

where $k_n, n \geq 0$, are non-zero constants. In this case, we call $\{R_n(x)\}_{n=0}^{\infty}$ the monic orthogonal polynomial system (MOPS) relative to $\phi(\cdot,\cdot)$.

In the following, we always assume that $\phi(\cdot,\cdot)$ is given by (1.1), where σ is quasi-definite. Let $\{P_n(x)\}_{n=0}^{\infty}$ be the MOPS relative to σ and

$$K_n(x,y) := \sum_{j=0}^{n} \frac{P_j(x)P_j(y)}{\langle \sigma, P_j^2 \rangle}, \qquad n \geq 0\,,$$

the n-th kernel polynomial for $\{P_n(x)\}_{n=0}^{\infty}$. We denote $\partial_x^i \partial_y^j K_n(x,y)$ by $K_n^{(i,j)}(x,y)$.

THEOREM 2.1. *The symmetric bilinear form $\phi(\cdot,\cdot)$ is quasi-definite if and only if*

$$(2.1) \qquad d_n := \begin{vmatrix} 1 + \lambda K_n^{(r,r)}(a,a) & \mu K_n^{(r,s)}(a,b) \\ \lambda K_n^{(r.s)}(a,b) & 1 + \mu K_n^{(s,s)}(b,b) \end{vmatrix} \neq 0, \qquad n \geq 0\,.$$

When $\phi(\cdot,\cdot)$ is quasi-definite, we have

$$
\begin{aligned}
R_n(x) = P_n(x) - \frac{\lambda}{d_{n-1}} & \begin{vmatrix} P_n^{(r)}(a) & \mu K_{n-1}^{(r,s)}(a,b) \\ P_n^{(s)}(b) & 1+\mu K_{n-1}^{(s,s)}(b,b) \end{vmatrix} K_{n-1}^{(0,r)}(x,a) + \\
& - \frac{\mu}{d_{n-1}} \begin{vmatrix} 1+\lambda K_{n-1}^{(r,r)}(a,a) & P_n^{(r)}(a) \\ \lambda K_{n-1}^{(r,s)}(a,b) & P_n^{(s)}(b) \end{vmatrix} K_{n-1}^{(0,s)}(x,b)
\end{aligned} \tag{2.2}
$$

and

$$
\phi(R_n,R_n) = \frac{d_n}{d_{n-1}}\langle \sigma, P_n^2\rangle, \qquad n \ge 0\,, \tag{2.3}
$$

where $K_{-1}(x,y)\equiv 0$ and $d_{-1}=1$.

Proof. Assume that $\phi(\cdot,\cdot)$ is quasi-definite and expand $R_n(x)$ as

$$
R_n(x) = P_n(x) + \sum_{j=0}^{n-1} C_j^n P_j(x), \qquad n \ge 1,
$$

where

$$
C_j^n = \frac{\langle \sigma, P_j R_n\rangle}{\langle \sigma, P_j^2\rangle} = \frac{-\lambda P_j^{(r)}(a)R_n^{(r)}(a) - \mu P_j^{(s)}(b) R_n^{(s)}(b)}{\langle \sigma, P_j^2\rangle}, \quad 0 \le j \le n-1\,.
$$

So, we obtain

$$
R_n(x) = P_n(x) - \lambda R_n^{(r)}(a) K_{n-1}^{(0,r)}(x,a) - \mu R_n^{(s)}(b) K_{n-1}^{(0,s)}(x,b)\,, \quad n \ge 0\,, \tag{2.4}
$$

and

$$
\begin{pmatrix} 1+\lambda K_{n-1}^{(r,r)}(a,a) & \mu K_{n-1}^{(r,s)}(a,b) \\ \lambda K_{n-1}^{(r,s)}(a,b) & 1+\mu K_{n-1}^{(s,s)}(b,b) \end{pmatrix} \begin{pmatrix} R_n^{(r)}(a) \\ R_n^{(s)}(b) \end{pmatrix} = \begin{pmatrix} P_n^{(r)}(a) \\ P_n^{(s)}(b) \end{pmatrix}, n \ge 0. \tag{2.5}
$$

We also have from (2.4)

$$
\begin{aligned}
\phi(R_n,P_j) &= \langle \sigma, R_n P_j\rangle + \lambda R_n^{(r)}(a) P_j^{(r)}(a) + \mu R_n^{(s)}(b) P_j^{(s)}(b) = \\
&= \langle \sigma, P_n P_j\rangle - \lambda R_n^{(r)}(a)\langle \sigma, K_{n-1}^{(0,r)}(x,a) P_j(x)\rangle + \\
&\quad - \mu R_n^{(s)}(b) \langle \sigma, K_{n-1}^{(0,s)}(x,b) P_j(x)\rangle + \\
&\quad + \lambda R_n^{(r)}(a) P_j^{(r)}(a) + \mu R_n^{(s)}(b) P_j^{(s)}(b) = \\
&= \{\langle \sigma, P_n^2\rangle + \lambda R_n^{(r)}(a) P_j^{(r)}(a) + \\
&\quad + \mu R_n^{(s)}(b) P_j^{(s)}(b)\}\delta_{jn}, \qquad 0 \le j \le n\,,
\end{aligned} \tag{2.6}
$$

since $\langle \sigma, K_{n-1}^{(0,r)}(x,a) P_j(x)\rangle = (1-\delta_{jn}) P_j^{(r)}(a)$, $0 \le j \le n$.

We now show that $d_n \neq 0$, $n \geq 0$, by induction on n. For $n = 0, d_0 = \frac{\phi(1,1)}{\langle\sigma,1\rangle} \neq 0$. Assume $d_n \neq 0, 0 \leq n \leq m$ for some integer $m \geq 0$. Then the system of equations (2.5) is uniquely solvable for $0 \leq n \leq m+1$ so that we have from (2.6)

$$
\begin{aligned}
\phi(R_n,R_n) &= \phi(R_n,P_n) = \langle\sigma,P_n^2\rangle + \lambda R_n^{(r)}(a)P_n^{(r)}(a)+ \\
&\quad + \mu R_n^{(s)}(b)P_n^{(s)}(b) = \\
&= \langle\sigma,P_n^2\rangle + \frac{\lambda P_n^{(r)}(a)}{d_{n-1}}\begin{vmatrix} P_n^{(r)}(a) & \mu K_{n-1}^{(r,s)}(a,b) \\ P_n^{(s)}(b) & 1+\mu K_{n-1}^{(s,s)}(b,b)\end{vmatrix} + \\
&\quad + \frac{\mu P_n^{(s)}(b)}{d_{n-1}}\begin{vmatrix} 1+\lambda K_{n-1}^{(r,r)}(a,a) & P_n^{(r)}(a) \\ \lambda K_{n-1}^{(r,s)}(a,b) & P_n^{(s)}(b)\end{vmatrix}, \\
&\quad 0 \leq n \leq m+1\,.
\end{aligned} \tag{2.7}
$$

On the other hand, we have

$$
\begin{aligned}
d_n &= d_{n-1} + \frac{\lambda P_n^{(r)}(a)}{\langle\sigma,P_n^2\rangle}\begin{vmatrix} P_n^{(r)}(a) & \mu K_{n-1}^{(r,s)}(a,b) \\ P_n^{(s)}(b) & 1+\mu K_{n-1}^{(s,s)}(b,b)\end{vmatrix} + \\
&\quad + \frac{\mu P_n^{(s)}(b)}{\langle\sigma,P_n^2\rangle}\begin{vmatrix} 1+\lambda K_{n-1}^{(r,r)}(a,a) & P_n^{(r)}(a) \\ \lambda K_{n-1}^{(r,s)}(a,b) & P_n^{(s)}(b)\end{vmatrix}, \qquad n \geq 0.
\end{aligned} \tag{2.8}
$$

Hence, (2.3) holds for $0 \leq n \leq m+1$ and $d_{m+1} \neq 0$. Therefore, by induction, $d_n \neq 0$,$n \geq 0$, and (2.3) holds for all $n \geq 0$. We also have (2.2) from (2.4) and (2.5).

Conversely, we assume that $d_n \neq 0, n \geq 0$, and define $R_n(x)$ by (2.2). Then $\{R_n(x)\}_{n=0}^{\infty}$ is a monic PS and (2.5) holds. We also have (2.4) so that (2.6) and (2.7) hold. Hence, $\phi(\cdot,\cdot)$ is quasi-definite since $\{R_n(x)\}_{n=0}^{\infty}$ is the MOPS relative to $\phi(\cdot,\cdot)$. □

Remark 2.1. Several special cases of theorem 2.1 are proved in [7], [12], [14], [16] under various stronger restrictions on $\phi(\cdot,\cdot)$. See also [6], [8], [9], [17] for other special examples.

We now set

$$\Psi(x)=\begin{cases} x & \text{if} \quad \mu=0, r=0 \\ (x-a)^{r+1} & \text{if} \quad \mu=0, r\geq 1 \\ x & \text{if} \quad \mu\neq 0, r=s=0 \\ (x-b)^{s+1} & \text{if} \quad \mu\neq 0, 0=r<s, a\neq b \\ & \text{or} \quad \mu\neq 0, 0\leq r<s, a=b \\ (x-a)^{r+1}(x-b)^{s+1} & \text{if} \quad \mu\neq 0,\ r,s\geq 1. \end{cases}$$

and $\deg(\Psi)=u$ $(1\leq u\leq r+s+2)$. Then

$$\phi(\Psi p,q)=\langle\sigma,\Psi pq\rangle=\phi(p,\Psi q),\quad p(x) \text{ and } q(x)\in\mathcal{P}.$$

Proposition 2.2 (Recurrence Relations). *The* MOPS $\{R_n(x)\}_{n=0}^{\infty}$ *satisfies* $2u+1$*-term recurrence relations:*

$$\Psi(x)R_n(x)=\sum_{j=n-u}^{n+u}C_j^n R_j(x),\qquad (n\geq u) \tag{2.9}$$

where

$$C_j^n=\frac{\phi(\Psi R_n,R_j)}{\phi(R_j,R_j)}=\frac{\langle\sigma,\Psi R_jR_n\rangle}{\phi(R_j,R_j)}, \quad n-u\leq j\leq n+u\ (C_{n+u}^n=1,\ C_{n-u}^n\neq 0) \tag{2.10}$$

and

$$\frac{C_{n-j}^{n-i}}{\phi(R_{n-i},R_{n-i})}=\frac{C_{n-i}^{n-j}}{\phi(R_{n-j},R_{n-j})}, \quad 0\leq i\leq n-u \text{ and } i-u\leq j\leq i+u\,. \tag{2.11}$$

Proof. We may expand $\Psi(x)R_n(x)$ as

$$\Psi(x)R_n(x) = \sum_{j=0}^{n+u} C_j^n R_j(x)\,.$$

Then

$$C_i^n \phi(R_i, R_i) = \phi(\Psi R_n, R_i) = \phi(R_n, \Psi R_i) = \langle \sigma, \Psi R_n R_i \rangle, \quad 0 \le i \le n+u,$$

so that

$$C_i^n = \frac{\phi(R_n, \Psi R_i)}{\phi(R_i, R_i)} = \begin{cases} 0 & \text{if } i < n-u \\ \frac{\langle \sigma, \Psi R_n R_i \rangle}{\phi(R_i, R_i)} & \text{if } i \ge n-u\,, \end{cases}$$

and

$$C_{n-u}^n = \frac{\phi(R_n, \Psi R_{n-u})}{\phi(R_{n-u}, R_{n-u})} \ne 0, \quad C_{n+u}^n = 1.$$

Now, it's easy to deduce (2.11) from (2.10). $\square$

Explicit expressions for C_j^n in terms of coefficients of the three term recurrence relations for $\{P_n(x)\}_{n=0}^\infty$ are given in [7], [11], [14] in case $r = s = 0$ and in [15] in case $\mu = 0$, $r = 1$.

We now let

$$L_n(x, y) = \sum_{j=0}^{n} \frac{R_j(x) R_j(y)}{\phi(R_j, R_j)}, \quad n \ge 0\,,$$

be the n-th kernel polynomial for $\{R_n(x)\}_{n=0}^\infty$.

Proposition 2.3 (cf. propositions 5.1 and 5.2 in [15]). *For any integers $n, k \ge 0$ and any polynomial $p(x)$ of degree at most n, we have* (reproducing property):

$$\phi(L_n^{(0,k)}(x, y), p(x)) = p^{(k)}(y) \tag{2.12}$$

and

$$\begin{aligned} L_n(x,y) &= \\ &= K_n(x,y) - \frac{\lambda}{d_n} \begin{vmatrix} K_n^{(r,0)}(a,y) & \mu K_n^{(r,s)}(a,b) \\ K_n^{(s,0)}(b,y) & 1 + \mu K_n^{(s,s)}(b,b) \end{vmatrix} K_n^{(0,r)}(x,a) + \\ &\quad - \frac{\mu}{d_n} \begin{vmatrix} 1 + \lambda K_n^{(r,r)}(a,a) & K_n^{(r,0)}(a,y) \\ \lambda K_n^{(r,s)}(a,b) & K_n^{(s,0)}(b,y) \end{vmatrix} K_n^{(0,s)}(x,b)\,. \end{aligned} \tag{2.13}$$

PROOF. Let $p(x) \in \mathcal{P}$ be such that $\deg(p) \leq n$. Then

$$p(x) = \sum_{i=0}^{n} \frac{\phi(p, R_i)}{\phi(R_i, R_i)} R_i(x)$$

so that

$$\begin{aligned}
\phi(L_n^{(0,k)}(x,y), p(x)) &= \sum_{i=0}^{n} \frac{\phi(p, R_i)}{\phi(R_i, R_i)} \phi(L_n^{(0,k)}(x,y), R_i(x)) = \\
&= \sum_{i=0}^{n} \frac{\phi(p, R_i)}{\phi(R_i, R_i)} \sum_{j=0}^{n} \frac{R_j^{(k)}(y)}{\phi(R_j, R_j)} \phi(R_j(x), R_i(x)) = \\
&= \sum_{i=0}^{n} \frac{\phi(p, R_i)}{\phi(R_i, R_i)} R_i^{(k)}(y) = p^{(k)}(y).
\end{aligned}$$

Expand $L_n(x,y)$ as $L_n(x,y) = \sum_{j=0}^{n} A_j^{(n)}(y) P_j(x)$, where

$$\begin{aligned}
A_j^{(n)}(y) &= \frac{\langle \sigma, L_n(x,y) P_j(x) \rangle}{\langle \sigma, P_j^2 \rangle} \\
&= \frac{P_j(y)}{\langle \sigma, P_j^2 \rangle} - \frac{\lambda}{\langle \sigma, P_j^2 \rangle} L_n^{(r,0)}(a,y) P_j^{(r)}(a) \\
&\qquad - \frac{\mu}{\langle \sigma, P_j^2 \rangle} L_n^{(s,0)}(b,y) P_j^{(s)}(b) \quad \text{by (2.12).}
\end{aligned}$$

Hence, we have

$$\begin{aligned}
L_n(x,y) = K_n(x,y) - \lambda L_n^{(r,0)}(a,y) K_n^{(0.r)}(x,a) + \\
- \mu L_n^{(s,0)}(b,y) K_n^{(0,s)}(x,b)
\end{aligned} \tag{2.14}$$

so that

$$\begin{pmatrix} 1+\lambda K_n^{(r,r)}(a,a) & \mu K_n^{(r,s)}(a,b) \\ \lambda K_n^{(r,s)}(a,b) & 1+\mu K_n^{(s,s)}(b,b) \end{pmatrix} \begin{pmatrix} L_n^{(r,0)}(a,y) \\ L_n^{(s,0)}(b,y) \end{pmatrix} = \begin{pmatrix} K_n^{(r,0)}(a,y) \\ K_n^{(s,0)}(b,y) \end{pmatrix}. \tag{2.15}$$

Now, (2.13) follows immediately from (2.14) and (2.15). □

From the recurrence relation (2.9) and (2.11), we can easily deduce the following Christoffel-Darboux type formula for $\{R_n(x)\}_{n=0}^{\infty}$.

Proposition 2.4 (cf. proposition 4.3 in [15]). *The* MOPS $\{R_n(x)\}_{n=0}^{\infty}$ *satisfies*

$$(\Psi(x)-\Psi(y))L_n(x,y)= \\ =\sum_{j=1}^{t}\sum_{i=n-j+1}^{n}\frac{C_i^{i+j}}{\phi(R_{i+j},R_{i+j})}(R_{i+j}(x)R_i(y)-R_i(x)R_{i+j}(y)),\ n\geq 0.$$

We now set

$$\pi(x)=\begin{cases}(x-a)^{s+1} & \text{if } a=b \text{ and } \lambda\neq 0,\mu\neq 0\\ (x-a)^{(r+1)\operatorname{sgn}|\lambda|}(x-b)^{(s+1)\operatorname{sgn}|\mu|} & \text{if } a\neq b\end{cases} \tag{2.16}$$

and $\deg(\pi)=t$ $(0\leq t\leq r+s+2)$. Then

$$\phi(\pi p,q)=\langle\sigma,\pi pq\rangle=\phi(p,\pi q),\quad p(x) \text{ and } q(x)\in\mathcal{P}.$$

Proposition 2.5 (quasi-orthogonality). *The* MOPS $\{R_n(x)\}_{n=0}^{\infty}$ *is quasi-orthogonal of order* $2t$ *relative to* σ*:*

$$\pi(x)R_n(x)=\sum_{j=n-t}^{n+t}D_j^nP_j(x),\quad (n\geq t)\,, \tag{2.17}$$

where

$$D_j^n=\frac{\langle\sigma,\pi P_jR_n\rangle}{\langle\sigma,P_j^2\rangle}=\frac{\phi(R_n,\pi P_j)}{\langle\sigma,P_j^2\rangle},\ \ n-t\leq j\leq n+t,\ (D_{n+t}^n=1,D_{n-t}^n\neq 0)\,.$$

Proof. We may expand $\pi(x)R_n(x)$ as

$$\pi(x)R_n(x)=\sum_{j=0}^{n+t}D_j^nP_j(x)\,.$$

Then

$$D_j^n\langle\sigma,P_j^2\rangle=\langle\sigma,\pi R_nP_j\rangle=\phi(R_n,\pi P_j),\quad 0\leq j\leq n+t\,,$$

so that the conclusion follows. ∎

3 – Second order differential equations

The MOPS $\{P_n(x)\}_{n=0}^{\infty}$ satisfies three term recurrence relations:

$$P_{n+1}(x) = (x-b_n)P_n(x) - c_n P_{n-1}(x)\,, \qquad n \ge 0 \quad (P_{-1}(x)=0 \text{ and } \; c_n \ne 0, n \ge 1)\,. \tag{3.1}$$

In the following, we will denote $\pi_k(x,n)$ a polynomial such that its coefficients may depend on n but the degree is at most k, independent of n. The polynomial $\pi_k(x,n)$ may not be the same in different contexts.

We have by (3.1) and induction on $k = 1, 2, \cdots$ that

$$P_{n+k}(x) = \pi_k(x,n)P_n(x) + \pi_{k-1}(x,n)P_{n-1}(x), \quad n \ge 0\,, \tag{3.2}$$

and

$$P_{n-k-1}(x) = \pi_{k-1}(x,n)P_n(x) + \pi_k(x,n)P_{n-1}(x), \quad n \ge k+1. \tag{3.3}$$

We now assume that σ is a semi-classical moment functional satisfying

$$(A(x)\sigma)' - B(x)\sigma = 0, \tag{3.4}$$

where $\deg(A) \ge 0$, $\deg(B) \ge 1$. We set $\deg(A) = \alpha$ and $\beta = \max(\deg(A)-2, \deg(B)-1)$. Then $\{P_n(x)\}_{n=0}^{\infty}$ satisfies the so-called structure relations [18]

$$A(x)P_n'(x) = \sum_{n-\beta-1}^{n+\alpha-1} Q_j^n P_j(x), \quad n \ge \beta+1\,. \tag{3.5}$$

By (3.2) and (3.3), we may review (3.5) and (2.17) as

$$A(x)P_n'(x) = \pi_{\beta+1}(x,n)P_n(x) + \pi_{\beta+2}(x,n)P_{n-1}(x) \tag{3.6}$$

and

$$\pi(x)R_n(x) = \pi_t(x,n)P_n(x) + \pi_{t-1}(x,n)P_{n-1}(x). \tag{3.7}$$

Differentiating (3.7) and then multiplying by $A(x)$ yields via (3.1) and (3.6)

$$\begin{aligned}A(x)\pi(x)R_n'(x)+A(x)\pi'(x)R_n(x)=\\ =\pi_{t+\beta+1}(x,n)P_n(x)+\pi_{t+\beta+2}(x,n)P_{n-1}(x).\end{aligned} \tag{3.8}$$

We then obtain from (3.7) and (3.8)

$$\pi_{2t+\beta+2}(x,n)P_n(x)=\pi_{2t+\alpha-1}(x,n)R_n'(x)+\pi_{2t+\beta+2}(x,n)R_n(x) \tag{3.9}$$

and

$$\pi_{2t+\beta+2}(x,n)P_{n-1}(x)=\pi_{2t+\alpha}(x,n)R_n'(x)+\pi_{2t+\beta+1}(x,n)R_n(x)\,. \tag{3.10}$$

Differentiating (3.8) and then multiplying by $A(x)$ yields via (3.1) and (3.6)

$$\begin{aligned}A^2\pi R_n''(x)+A(A'\pi+2A\pi')R_n'(x)+A(A'\pi'+A\pi'')R_n(x)\\ =\pi_{t+2\beta+4}(x,n)P_n(x)+\pi_{t+2\beta+5}(x,n)P_{n-1}(x).\end{aligned} \tag{3.11}$$

From (3.9), (3.10), and (3.11), we finally obtain:

THEOREM 3.1. *When σ is a semi-classical moment functional satisfying* (3.4)*, the MOPS $\{R_n(x)\}_{n=0}^{\infty}$ satisfies a second order differential equation with polynomial coefficients of the form*

$$S(x,n)R_n''(x)+T(x,n)R_n'(x)+U(x,n)R_n(x)=0\,, \tag{3.11}$$

where $\deg(S)\le 3t+2\alpha+\beta+2, \deg(T)\le 3t+\alpha+2\beta+5$, *and* $\deg(U)\le 3t+3\beta+6$.

Theorem 3.1 was proved in [16] for $\mu=0$ and $r=1$ and in [17] for positive-definite σ, $\mu=0,\lambda>0$. Explicit construction of the differential equation (3.11) can be found in [7], [16] for σ to be the Bessel moment functional and in [17] for σ to be the Hermite moment functional.

4 – Zeros of $R_n(x)$

We now consider the quasi-definite bilinear form $\phi(\cdot,\cdot)$ as in (1.1) for which we assume further that σ is positive-definite and λ, μ, a, b are real so that $\{R_n(x)\}_{n=0}^{\infty}$ is the real MOPS relative to $\phi(\cdot,\cdot)$. We let

$$x_{n,1} < x_{n,2} < \cdots < x_{n,n}$$

be the zeros of $P_n(x), n \geq 1$, and

$$\xi = \lim_{n\to\infty} x_{n,1} \ (\geq -\infty), \qquad \eta = \lim_{n\to\infty} x_{n,n} \ (\leq \infty)$$

so that $I = [\xi, \eta]$ is the true interval of orthogonality for σ (see [5, page 29, Definition 5.2]).

Then the quasi-orthogonality (2.17) of $\{R_n(x)\}_{n=0}^{\infty}$ relative to σ implies that $\pi(x)R_n(x)$ has at least $n-t$ nodal zeros (i.e., zeros of odd multiplicity) in $\overset{\circ}{I} = (\xi, \eta)$. Hence, $R_n(x)$ has at least $n-t-2$ nodal zeros in $\overset{\circ}{I}$. Zeros of $R_n(x)$ are handled by many authors [1]-[4], [11], [15], [19], [20], [21], in all of which, except [1], $\phi(\cdot,\cdot)$ is assumed to be positive-definite.

Let $N = N(n)(0 \leq N \leq n)$ be the number of nodal zeros of $R_n(x)$ in $\overset{\circ}{I}$ and $\Phi_N(x)$ the monic polynomial of degree N having simple zeros at nodal zeros of $R_n(x)$ in $\overset{\circ}{I}$. Then $\Phi_N(x)R_n(x)$ has the constant sign on $\overset{\circ}{I}$, which we may assume to be positive.

In order to find a lower bound for $N(n)$, we consider

Case I: $\mu = 0$, Case II: $a = b, \lambda \neq 0, \mu \neq 0$, and Case III: $a \neq b, \lambda \neq 0, \mu \neq 0$ separately.

In the following, we always assume $n \geq r+1$ unless stated otherwise since if $0 \leq n \leq r$ then $R_n(x) = P_n(x)$ and $N(n) = n$.

Case I: $\mu = 0$.

Then $\pi(x) = (x-a)^{r+1}$ so that $N(n) \geq n-r-2$.

Proposition 4.1. *Assume $\mu = 0$ and $a \in \overset{\circ}{I}$. If a is a nodal zero of $R_n(x)$ or r is odd, then $N(n) \geq n-r-1$.*

Proof. When a is a nodal zero of $R_n(x)$, it is trivial. Hence, we assume r is odd. Then $\pi(x)\Phi_N(x)R_n(x) \geq 0$ on I so that by (2.17)

$$\langle \sigma, \pi\Phi_N R_n\rangle = \sum_{n-r-1}^{n+r+1} D_j^n \langle \sigma, \Phi_N P_j\rangle > 0.$$

Hence, $\deg(\pi\Phi_N) = N(n) + r + 1 \geq n$. ∎

Proposition 4.2. *Assume $\mu = 0$ and $a \notin \mathring{I}$. Then $N(n) \geq n-1$ so that $R_n(x)$ has n simple real zeros. Moreover, zeros of $P_n(x)$ and $R_n(x)$ interlace each other.*

Proof. Assume $a \geq \eta$. (Proof for $a \leq \xi$ runs the same.) Let

$$l_{n,k}(x) = \frac{P_n(x)}{x - x_{n,k}} \quad \text{and} \quad \pi_{n,k}(x) = l_{n,k}(x)R_n(x), \quad 1 \leq k \leq n.$$

Then by Gauss' quadrature formula, there are positive constants $A_{j,k}^n$, $1 \leq j,\ k \leq n$, such that

$$\langle \sigma, \pi_{n,k}\rangle = \sum_{j=1}^{n} A_{j,k}^n \pi_{n,k}(x_{n,j}) = A_{k,k}^n P_n'(x_{n,k})R_n(x_{n,k})\,. \tag{4.1}$$

On the other hand, we also have from (1.1) and (2.5)

$$\langle \sigma, \pi_{n,k}\rangle = -\lambda l_{n,k}^{(r)}(a)R_n^{(r)}(a) = \frac{-\lambda}{d_{n-1}} l_{n,k}^{(r)}(a)P_n^{(r)}(a)\,. \tag{4.2}$$

Since $\operatorname{sgn} P_n'(x_{n,k}) = (-1)^{n-k}, l_{n,k}^{(r)}(a) > 0$, and $P_n^{(r)}(a) > 0$, we have from (4.1) and (4.2)

$$\operatorname{sgn} R_n(x_{n,k}) = (-1)^{n-k+1}\operatorname{sgn}(\lambda d_{n-1}), \quad 1 \leq k \leq n\,. \tag{4.3}$$

Hence, $R_n(x)$ has at least one and, in fact, exactly one nodal zero in each interval $(x_{n,k}, x_{n,k+1}), 1 \leq k \leq n-1$. ∎

Proposition 4.1 and proposition 4.2 are also obtained in [20] in case $\lambda > 0$ and $r = 1$. MEIJER [20] also showed that a may be chosen so that $R_n(x)$ has two complex zeros for large enough λ. In this case, a must be in $\overset{\circ}{I}$ by proposition 4.2.

Proposition 4.2 implies that $R_n(x)$ has n simple zeros

$$z_{n,1} < z_{n,2} < \cdots < z_{n,n}$$

all of which, except $z_{n,1}$ or $z_{n,n}$, lie in $(x_{n,1}, x_{n,n})$. The location of $z_{n,1}$ or $z_{n,n}$ not in $(x_{n,1}, x_{n,n})$ depends on $\mathrm{sgn}(\lambda d_{n-1})$, $\mathrm{sgn} R_n(\xi)$, and $\mathrm{sgn} R_n(\eta)$.

Below, we handle the case $a \geq \eta$. The case $a \leq \xi$ can be handled similarly.

LEMMA 4.3. *Assume $\mu = 0$ and $a \geq \eta$. Then*

$$\lambda d_{n-1} < 0 \quad \text{if and only if} \quad \frac{-1}{K_{n-1}^{(r,r)}(a,a)} < \lambda < 0\,; \tag{4.4}$$

$$\lambda d_{n-1} > 0 \quad \text{if and only if} \quad \lambda < \frac{-1}{K_{n-1}^{(r,r)}(a,a)} \ \textit{or}\ \lambda > 0\,. \tag{4.5}$$

PROOF. Since $a \geq \eta$ and $P_j(x)$ is a monic polynomial with all zeros in $(-\infty, a)$, $P_j^{(r)}(a) = 0$ for $0 \leq j < r$ and $P_j^{(r)}(a) > 0$ for $j \geq r$. Hence $K_{n-1}^{(r,r)}(a,a) > 0$ so that $d_{n-1} = 1 + \lambda K_{n-1}^{(r,r)}(a,a) > 0$ if $\lambda > 0$. Now (4.4) and (4.5) follow immediately. ∎

PROPOSITION 4.4. *Assume $\mu = 0$ and $a \geq \eta$.*

(i) *If $\frac{-1}{K_{n-1}^{(r,r)}(a,a)} < \lambda < 0$ and $(-1)^{n+1} R_n(\xi) \geq 0$, then*

$$z_{n,1} \leq \xi < x_{n,1} < z_{n,2} < \cdots < z_{n,n} < x_{n,n} < \eta.$$

(ii) *If $\frac{-1}{K_{n-1}^{(r,r)}(a,a)} < \lambda < 0$ and $(-1)^{n+1} R_n(\xi) < 0$, then*

$$\xi < z_{n,1} < x_{n,1} < z_{n,2} < \cdots < z_{n,n} < x_{n,n} < \eta.$$

(iii) *If* $\lambda < \frac{-1}{K_{n-1}^{(r,r)}(a,a)}$ *or* $\lambda > 0$ *and* $R_n(\eta) > 0$*, then*

$$\xi < x_{n,1} < z_{n,1} < \cdots < x_{n,n} < z_{n,n} < \eta.$$

(iv) *If* $\lambda < \frac{-1}{K_{n-1}^{(r,r)}(a,a)}$ *or* $\lambda > 0$ *and* $R_n(\eta) \leq 0$*,* $R_n(a) > 0$*, then*

$$\xi < x_{n,1} < z_{n,1} < \cdots < x_{n,n} < \eta \leq z_{n,n} < a.$$

(v) *If* $\lambda < \frac{-1}{K_{n-1}^{(r,r)}(a,a)}$ *or* $\lambda > 0$ *and* $R_n(\eta) \leq 0$*,* $R_n(a) \leq 0$*, then*

$$\xi < x_{n,1} < z_{n,1} < \cdots < x_{n,n} < \eta \leq a \leq z_{n,n}.$$

Hence, $N(n) = n$ *only in cases* (ii) *and* (iii)*.*

Proof. (i) and (ii): Assume $\frac{-1}{K_{n-1}^{(r,r)}(a,a)} < \lambda < 0$. Then $\lambda d_{n-1} < 0$ by lemma 4.3 and $R_n(x_{n,n}) > 0$ by (4.3). Hence, $z_{n,1} < x_{n,1}$ and $\operatorname{sgn} R_n(x_{n,1}) = (-1)^{n+1}$ from which (i) and (ii) follow immediately. In particular, we note that $\xi > -\infty$ in case (i).

(iii), (iv), and (v): Assume $\lambda < \frac{-1}{K_{n-1}^{(r,r)}(a,a)}$ or $\lambda > 0$. Then $\lambda d_{n-1} > 0$ by lemma 4.3 and $R_n(x_{n,n}) < 0$ by (4.3). Hence, $x_{n,n} < z_{n,n}$ and (iii), (iv), and (v) follow immediately. ∎

Special case of proposition 4.2 and proposition 4.4 is also handled by Meijer [19] when $\lambda > 0$, $a = 0$, $r \geq 1$, and $[\xi, \eta] = [0, \infty]$.

Proposition 4.5. *Assume* $\mu = 0$ *and* $a \geq \eta$*. If* $\lambda < \frac{-1}{K_{n-1}^{(r,r)}(a,a)}$ *or* $\lambda > 0$*, then zeros of* $R_n(x)$ *also interlace with zeros of* $P_{n-1}(x)$*. More precisely, we have*

$$x_{n,k} < z_{n,k} < x_{n-1,k} < x_{n,k+1} < z_{n,k+1}, \qquad 1 \leq k \leq n-1. \tag{4.6}$$

PROOF. Assume $\lambda < \frac{-1}{K_{n-1}^{(r,r)}(a,a)}$ or $\lambda > 0$, that is, $\lambda d_{n-1} > 0$. Then, by proposition 4.4, $R_n(x)$ has n real zeros $\{z_{n,k}\}_{k=1}^{n}$ such that

$$x_{n,k} < z_{n,k} < x_{n,k+1} < z_{n,k+1}, \qquad 1 \le k \le n-1. \tag{4.7}$$

Let $Q(x) = R_n(x) - P_n(x)$ and $\pi_{n-1,k}(x) = Q(x)l_{n-1,k}(x)$. Then $\deg(Q) \le n-1$ and $\deg(\pi_{n-1,k}) \le 2n-3$. As in the proof of proposition 4.2, we have positive constants $A_{j,k}^{n-1}$, $1 \le j,k \le n-1$, such that

$$\begin{aligned}\langle \sigma, \pi_{n-1,k}\rangle =& A_{k,k}^{n-1} Q(x_{n-1,k}) P'_{n-1}(x_{n-1,k}) = \frac{-\lambda}{d_{n-1}} l_{n-1,k}^{(r)}(a) P_n^{(r)}(a),\\ & 1 \le k \le n-1,\end{aligned}$$

so that $\operatorname{sgn} Q(x_{n-1,k}) = (-1)^{n-k}$, $1 \le k \le n-1$. Since $\operatorname{sgn} P_n(x_{n-1,k}) = (-1)^{n-k}$,

$$\begin{aligned}\operatorname{sgn} R_n(x_{n-1,k}) =& \operatorname{sgn}(P_n(x_{n-1,k}) + Q(x_{n-1,k})) = (-1)^{n-k},\\ & 1 \le k \le n-1,\end{aligned} \tag{4.8}$$

so that $R_n(x)$ has at least one and at most three zeros in each interval $(x_{n-1,k},\ x_{n-1,k+1})$, $1 \le k \le n-2$. If a certain $(x_{n-1,k}, x_{n-1,k+1})$ has three zeros of $R_n(x)$, then they must be in $(x_{n,k}, x_{n,k+2})$, which is impossible by (4.7). Hence, $R_n(x)$ has exactly one zero in each interval $(x_{n-1,k}, x_{n-1,k+1})$, $1 \le k \le n-2$.

Since $x_{n,n} < z_{n,n}$, $R_n(x_{n,n}) < 0$ by (4.3), and $R_n(x_{n-1,n-1}) < 0$ by (4.8), $R_n(x)$ has no zero in $(x_{n-1,n-1}, x_{n,n})$. Hence, $x_{n,n-1} < z_{n,n-1} < x_{n-1,n-1}$ from which (4.6) follows inductively on $k = n-1, n-2, \cdots, 1$. □

When $\mu = r = 0$ and $a \notin \overset{\circ}{I}$

$$R_n(a) = \frac{1}{d_{n-1}} P_n(a) \ne 0, \qquad n \ge 0\,,$$

so that we can modify proposition 4.4 as:

PROPOSITION 4.6. *Assume $\mu = r = 0$ and $a \ge \eta$.*

(i) *If* $\frac{-1}{K_{n-1}(a,a)} < \lambda < 0$ *and* $(-1)^{n+1}R_n(\xi) \geq 0$, *then*

$$z_{n,1} \leq \xi < x_{n,1} < z_{n,2} < \cdots < z_{n,n} < x_{n,n} < \eta.$$

(ii) *If* $\frac{-1}{K_{n-1}(a,a)} < \lambda < 0$ *and* $(-1)^{n+1}R_n(\xi) < 0$, *then*

$$\xi < z_{n,1} < x_{n,1} < z_{n,2} < \cdots < z_{n,n} < x_{n,n} < \eta.$$

(iii) *If* $\lambda > 0$ *and* $R_n(\eta) > 0$, *then* $R_n(a) > 0$ *and*

$$\xi < x_{n,1} < z_{n,1} < \cdots < x_{n,n} < z_{n,n} < \eta \leq a.$$

(iv) *If* $\lambda > 0$ *and* $R_n(\eta) \leq 0$, *then* $R_n(a) > 0$ *and*

$$\xi < x_{n,1} < z_{n,1} < \cdots < x_{n,n} < \eta \leq z_{n,n} \leq a.$$

(v) *If* $\lambda < \frac{-1}{K_{n-1}(a,a)}$, *then* $R_n(a) > 0$ *and*

$$\xi < x_{n,1} < z_{n,1} < \cdots < x_{n,n} < \eta \leq a < z_{n,n}.$$

Proof. (i) and (ii) are the same as in proposition 4.4.

(iii) and (iv): Assume $\lambda > 0$. Then $d_{n-1} > 0$ and $R_n(x_{n,n}) < 0$ so that $R_n(a) > 0$ since $P_n(a) > 0$. Hence, the conclusions follow.

(v): Assume $\lambda < \frac{-1}{K_{n-1}(a,a)}$. Then $\lambda < 0$, $d_{n-1} < 0$, and $R_n(x_{n,n}) < 0$ so that $R_n(a) < 0$. Hence, $a < z_{n,n}$. ∎

When $\mu = r = 0$,

$$\phi(p,q) = \langle \tau, pq \rangle,$$

where $\tau = \sigma + \lambda\delta(x-a)$ is a moment functional. Let

$$\lambda_0 := \frac{-1}{\lim\limits_{n\to\infty} K_n(a,a)}\,.$$

Then

$$-\langle \sigma, 1 \rangle < \lambda_0 \leq 0$$

since $0 < \langle \sigma, 1\rangle^{-1} = K_0(a,a) \le K_1(a,a) \le \cdots$ and $K_n(a,a) < K_{n+2}(a,a)$, $n \ge 0$.

Lemma 4.7. *For $\tau = \sigma + \lambda\delta(x-a)$, the following statements are equivalent:*

(i) *τ is positive-definite;*

(ii) *$e_n := 1 + \lambda K_n(a,a) > 0$, $n \ge 0$;*

(iii) *$\lambda \ge \lambda_0$.*

Proof. When $\mu = r = 0$, $d_n = e_n$, $n \ge 0$, so that τ is quasi-definite if and only if $e_n \ne 0$, $n \ge 0$ by theorem 2.1. Moreover, when τ is quasi-definite (cf. (2.3))

$$\langle \tau, Q_n^2\rangle = \frac{e_n}{e_{n-1}}\langle \sigma, P_n^2\rangle, \qquad n \ge 0,\ (e_{-1} = 1)\,,$$

where $\{Q_n(x)\}_{n=0}^{\infty}$ is the MOPS relative to τ. Hence, (i) $\Leftrightarrow$ (ii). Then (ii) $\Leftrightarrow$ (iii) since $\{\frac{-1}{K_n(a,a)}\}_{n=0}^{\infty}$ increases to λ_0. □

When $\lambda \ge \lambda_0$ so that τ is also positive-definite, let $I_1 = [\xi_1, \eta_1]$ be the true interval of orthogonality of I. How are I and I_1 related? If $\lambda > 0$ and $a \notin \mathring{I}$, then τ is positive-definite on $[\xi,\eta] \cup \{a\}$ so that

$$[\xi_1, \eta_1] \subseteq ch([\xi,\eta] \cup \{a\})\,,$$

where $ch(A)$ stands for the convex-hull of A. More precisely we have:

Corollary 4.8. *Assume $\mu = r = 0$, $a \ge \eta$, and $\lambda \ge \lambda_0$ so that τ is also positive-definite.*

(i) *If $\lambda_0 \le \lambda < 0$, then $\xi_1 \le \xi < \eta_1 \le \eta \le a$.*

(ii) *If $\lambda_0 \le \lambda < 0$ and $(-1)^{n+1}R_n(\xi) < 0$ for all n large enough, then $\xi_1 = \xi < \eta_1 \le \eta \le a$.*

(iii) *If $\lambda > 0$, then $\xi \le \xi_1 < \eta \le \eta_1 \le a$.*

(iv) *If $\lambda > 0$ and $R_n(\eta) > 0$ for all n large enough, then $\xi \le \xi_1 < \eta = \eta_1 \le a$.*

Proof (i) and (ii). Assume $\lambda_0 \leq \lambda < 0$. Then $\frac{-1}{K_{n-1}(a,a)} < \lambda_0 \leq \lambda < 0$, $n \geq 0$, so that by proposition 4.7 (i) and (ii).

$$z_{n,1} < x_{n,1} \quad \text{and} \quad z_{n,n} < x_{n,n} < \eta\,.$$

Hence, $\xi_1 = \lim_{n\to\infty} z_{n,1} \leq \xi = \lim_{n\to\infty} x_{n,1}$ and $\eta_1 = \lim_{n\to\infty} z_{n,n} \leq \eta = \lim_{n\to\infty} x_{n,n}$. If furthermore $(-1)^{n+1}R_n(\xi) < 0$ for $n \geq k$, then by proposition 4.7 (ii),

$$\xi < z_{n,1} < x_{n,1}, \qquad n \geq k$$

so that $\xi_1 = \xi$.

Proof (iii) and (iv). Assume $\lambda > 0$. Then by proposition 4.7 (iii), (iv), and (v)

$$\xi < x_{n,1} < z_{n,1} \quad \text{and} \quad x_{n,n} < z_{n,n} \leq a$$

so that $\xi \leq \xi_1$ and $\eta \leq \eta_1 \leq a$.

If furthermore $R_n(\eta) > 0$ for $n \geq k$, then by proposition 4.7 (iii)

$$x_{n,n} < z_{n,n} < \eta \leq a$$

so that $\eta = \eta_1 \leq a$. ∎

Remark. It's easy to see that $\tau = \sigma + \lambda\delta(x-a)$ cannot be negative-definite.

Case II: $a = b$, $\lambda \neq 0$, $\mu \neq 0$, and $0 \leq r < s$.

Then $\pi(x) = (x-a)^{s+1}$ so that $N(n) \geq n - s - 2$.

Proposition 4.9. *Assume $a = b$, $\lambda \neq 0$, $\mu \neq 0$, $0 \leq r < s$, and $a \in \mathring{I}$. If a is a nodal zero of $R_n(x)$ or s is odd, then $N(n) \geq n - s - 1$.*

Proof. The proof is the same as in proposition 4.1. ∎

Proposition 4.10. *Assume $a = b$, $\lambda \neq 0$, $\mu \neq 0$, $0 \leq r < s$, and $a \notin \mathring{I}$. Then $N(n) \geq n - 1$ for $r + 1 \leq n \leq s$ and $N(n) \geq n - 2$ for $n \geq s + 1$.*

Proof. See theorem 2.2 in [1]. ∎

More results on zeros of $R_n(x)$ can be found in [4] in case $\lambda > 0, \mu > 0$, and $1 \leq r < s$. When $r = 0$ and $\tau = \sigma + \lambda\delta(x-a)$ is also positive-definite, we can have more precise information on the zeros of $R_n(x)$.

Proposition 4.11. *Assume $a = b$, $\lambda \geq \lambda_0 (\lambda \neq 0)$, $\mu \neq 0$, and $r = 0 < s$.*

(i) *If $a \notin \overset{\circ}{I}_1$, then $R_n(x)$, $n \geq s+1$, has n real simple zeros, which interlace with the zeros of $\{Q_n(x)\}_{n=0}^{\infty}$.*

(ii) *If $\lambda > 0$ and $a \notin \overset{\circ}{I}$, then $R_n(x)$, $n \geq s+1$, has n simple real zeros of which at least $n-1$ lie in (ξ, a) when $a \geq \eta$ or in (a, η) when $a \leq \xi$. Furthermore, if $a = \xi$ or η, then $N(n) \geq n-1$.*

Proof. When $\lambda \geq \lambda_0$ and $r = 0$, $\phi(p,q) = \langle \tau, pq \rangle + \mu p^{(s)}(a) q^{(s)}(a)$ and τ is positive-definite. When $\lambda > 0$ and $a \notin \overset{\circ}{I}$, τ is positive-definite on $ch(I \cup \{a\})$. Hence, the conclusion follows from proposition 4.2 □

Proposition 4.11 (ii) was also proved in [2] for $\lambda > 0$, $\mu > 0$, $r = 0$, $s = 1$ and $a = \xi$ or η. See also [21] for further informations on zeros of $R_n(x)$ in case $\lambda > 0$, $\mu > 0$, $r = 0$, $s = 1$, and $a \notin \overset{\circ}{I}$.

Case III: $a \neq b$, $\lambda \neq 0$, $\mu \neq 0$, $0 \leq r \leq s$.

Then $\pi(x) = (x-a)^{r+1}(x-b)^{s+1}$ so that $N(n) \geq n - r - s - 4$. In this case, we have not much to say yet about zeros of $R_n(x)$ unless $r = 0$ and $\lambda \geq \lambda_0$ or $n \leq s+1$. When $r = 0$ and $\lambda \geq \lambda_0$,

$$\phi(p,q) = \langle \tau, pq \rangle + \mu p^{(s)}(b) q^{(s)}(b),$$

where $\tau = \sigma + \lambda\delta(x-a)$ is positive-definite, so that it can be handled as in Case I with σ replaced by τ. When $a \notin \overset{\circ}{I}$ and $r+1 \leq n \leq s+1$, we can show (see lemma 2.1 and theorem 2.2 in [1]) that

$$N(n) \geq \begin{cases} n-1 & \text{if} \quad n = r+1 \text{ or } \ r+2 \leq n \leq s\,, \\ n-2 & \text{if} \quad 0 \leq r < s \text{ and } \ n = s+1. \end{cases}$$

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