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Scattering theory: a possible approach to the homogenization problem for the Euler equations

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RIASSUNTO: *Si analizza il comportamento asintotico di un "vortex patch" che evolve secondo le equazioni di Eulero bidimensionali per un fluido incomprimibile. Più precisamente si studia l'esistenza di dati iniziali tali che, asintoticamente, la vorticità $\omega(x,t)$ converga debolmente, nel senso della misura, ad una soluzione stazionaria, sia $\omega_\infty(|x|)$, dell'equazione di Eulero: in altre parole, si studia se la vorticità sia o meno "omogeneizzata". Si dimostra che si può dare una caratterizzazione dell'omogeneizzazione formulando un problema di scattering per le equazioni di Eulero; inoltre, attraverso un approccio iterativo al problema di Eulero, si dimostra che le soluzioni del primo ordine non banale nell'iterazione omogeneizzano.*

ABSTRACT: *We are interested in the analysis of the asymptotic behavior of a vortex patch that evolves according to the two dimensional Euler equation for incompressible fluids. More precisely, we consider the problem of the existence of initial data, such that, as $t \to \infty$, the vorticity $\omega(x,t)$ weakly converges, in the sense of measures, to a stationary solution, say $\omega_\infty(|x|)$, of the Euler equations: in other words, we want to study if or not the vorticity is "homogenized". In this paper we show that a characterization of homogenization can be given in terms of a scattering problem for the Euler equations. Moreover, via an iterative approach to the Euler problem, we show that the solutions of the equations of the first non trivial order homogenize.*

KEY WORDS AND PHRASES: *Euler equations – Asymptotic behavior of vortex patches – Scattering theory*

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1 – Introduction

In this paper we study large time behavior of the solutions of the 2D (two dimensional) Euler equations describing the evolution of an inviscid incompressible fluid.

We are interested, in particular, in the asymptotic evolution of an initial vorticity field given by the characteristic function of a measurable bounded set D (vortex patch): $\omega(x,0) \equiv \chi_D(x)$. As it is well known, the Euler equations for a vortex patch can be interpreted as the hamiltonian evolution of the set D, the time dependent hamiltonian being the stream function Ψ, which is a linear functional of the characteristic function of the set.

Time behavior of vortex patches has been widely investigated. It is known, first of all, that, due to the fact that the hamiltonian flow is measure preserving, the area of the domain D is constant and its evolution is global in time and unique, see [1]. Moreover it is also known, see [2], [3] and [4], that, if the boundary ∂D is initially regular, then also the boundary of the evoluted domain, ∂D_t, is regular and its lenght is bounded at any finite time.

Furthermore, from the conservation of the inertial momentum of the vorticity, $M_t \equiv \int_{D_t} |x|^2 \omega(x,t) dx$, it follows that D_t lives, for all times, in an essentially bounded domain. In particular it has been proved, see [5], [6], [7], that a circular patch is stable in the L_1 norm. On the other hand, these results do not exclude very complicated behaviors for D_t as, for example, long and thin filaments. (The existence of the filaments is, nowadays, numerically and experimentally proved, see, for example, [8].)

Then a first natural question is if or not the support of the vortex patch is definitively bounded. This problem is, till now, unsolved in its generality. Nevertheless partial results on this problem can be found in [9] and in [10].

A second well known question, strictly related to the previous one is if or not it is possible to predict the asymptotic behavior of filaments. More precisely, one can ask, for example, if there exist initial conditions for the vorticity such that, as $t \to \infty$, $\omega(x,t)$ converges, in some suitable sense, to an equilibrium state of the Euler equations. This is exactly the question we are concerned with in this paper. We investigate, in fact, if the vorticity $\omega(x,t)$ converges, as $t \to \infty$, to a stationary profile of vorticity, say $\omega_\infty(|x|)$, depending only on the distance. This hard question, known as the problem of "the homogenization of vorticity", has been widely investigated, see for example [11], [12], [13], [14] and, in our opinion, very interesting related problems are still open.

Before illustrating in some detail the result we prove, we only recall that the problem of the description of organization and asymptotic behavior of "vortex blobs" in terms of solutions of the 2D Euler equations - which is actually strictly related to the question we study - is also of great interest in understanding turbulence, and has been approached also from the point of view of the statistical mechanics. But the aim of this paper is not a detailed discussion on this subject, therefore we refer to [14]–[22] for some of the results on the argument.

Coming back to this paper, we prove here that, at least for a particular model, the foresaid homogenization problem for the vorticity can be solved. More precisely, we consider a circular vortex patch $\mathcal{C}$, encircled by an annulus $\mathcal{A}$ which has a regular boundary. We assume, as particular model of evolution, that $\mathcal{A}$ is passive (that is it evolves only under the action of the velocity field due to $\mathcal{C}$), then we show that the homogenization is realized for the characteristic function of the set $D = \mathcal{C} \cup \mathcal{A}$. Let us remark that this passive model can be regarded, equivalently, as the first non trivial step of a hierarchy of linear evolutive problems approximating, in a suitable sense, the Euler equations (see sections 4 and 5 below).

The ingredients we use to prove the result are essentially two.
We remark, first of all, that the solutions of an autonomous one degree of freedom hamiltonian system homogenize (for the proof see the "ergodic lemma" of section 2).

It is not difficult, in fact, to see that if one consider an hamiltonian system in action-angle variables, say (I, ϕ), and an hamiltonian given by $H(I, \phi) \equiv h(I) = \frac{I^2}{2}$, then the evolution, at time t, of a patch D transported by the flow related to $h(I)$ is simply given by $D_t = \{(I, \phi) : (I, \phi - It) \in D\}$. Then, as time goes on, the domain is stretched and stretched. To have an idea, in the figure below one can see the evolution of the set $D_0 = \{(\phi, I) : \phi \in [0, 2\pi),\ \ I \in (a, b)\}$ under the flow given by $h(I)$.

If one considers, moreover, the characteristic function associated to D_t, one may note that it converges, weakly, to a constant in $[0, 2\pi) \times (a, b)$, and 0 elsewhere.

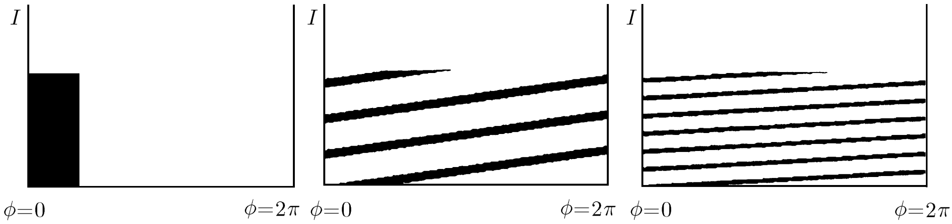

Fig. 1

We make here two more remarks: the first one is that the result can be obtained also if the hamiltonian $h(I)$ does not have the previous simple form. It is sufficient, in fact, that the anisochronicity condition $\frac{\partial^2 h}{\partial I^2} \neq 0$ is satisfied to transform the hamiltonian in the previous form. Moreover the anisochronicity condition is necessary: in fact homogenization does not occur if, for example, all the orbits are covered with the same period (as in the harmonic oscillator case). It is important, finally, to stress that the previous behavior is realized because the flow is linear.

The second ingredient that allows us to prove our result is the approach to the asymptotic problem of homogenization via the scattering theory.

In general, the scattering theory characterizes the asymptotic behavior of a solution of an evolutive, often non linear, equation in terms of an asymptotically linear evolution. An example of application of the theory is, as it is well known, the scattering of a particle in a potential field, [23]. Under suitable assumptions on the potential (behavior at infinity, regularity,...) it is possible to find initial conditions for position and velocity such that the corresponding trajectory asymptotically behaves as $x^* + v^* t, v^*$, i.e. satisfies "the scattering problem"

$$\lim_{t\to\infty} [|x(t) - (x_* + v_* t)| + |v(t) - v_*|] = 0 .$$

The case we deal with here is more complicated than the previous one, but the idea is essentially the same. One can ask if there exist solutions of the 2D Euler equation (which is nonlinear), that, as $t \to \infty$, behave as the solutions of a suitable linear evolutive problem (the solutions of the Euler problem which have this asymptotic behavior are said to satisfy a scattering condition, see section 3).

It is not difficult to see, moreover, that solutions which satisfy a scattering condition homogenize. Therefore the problem of homogenization reduces to the one of the existence of solutions of the Euler equation which satisfy the scattering condition.

In this paper we show, via an iterative approach to the Euler problem, that, at least for the first non trivial step of the procedure (which corresponds to the passive model), the previous problem admits a solution, see section 4, 5 and 6.

We anticipate that the main tool in proving this result is the study of the asymptotic expansion, with respect to t, of the stream function related with our model. In particular it is possible to prove that, as $t \to \infty$, this function converges to the "free" hamiltonian of the scattering problem, and, moreover the correction, which is of order $t^{-3/2}$, can be explicitely computed and depends, in a simple way, on the critical points of the boundary of the patch one is evolving. To conclude we note that this result reminds, as it is natural, the stationary phase method for integrals of fast oscillating functions, [24], [25], but it is obtained with a more direct approach (see the Appendix).

To conclude, a final remark. It is well known that there is an analogy between the 2D Euler equations and the 1D Vlasov-Poisson equation for an uncollisional charged plasma, see, for example [26]. For the 1D Vlasov-Poisson problem, in [27], [28], Landau predicts, in particular, that, due to the spatial dispersion of the plasma, an uncollisional dissipation is realized. Then the plasma, as $t \to \infty$, relaxes towards an homogeneous equilibrium state. This phenomenon is known as "the Landau damping".

For a complete discussion on this phenomenon and its implications one can see the papers quoted above, we only recall here that the rigorous proof of the previous behavior is given only for the linear problem, obtained by considering the linear part of Vlasov-Poisson equation around the homogeneous equilibrium.

In a forthcoming paper, [29], it is proved, in the same spirit of the results obtained here, that the homogenization phenomenon and the Landau damping are related and the mechanism of the Landau damping is realized for a suitable class of solutions of the full 1D Vlasov-Poisson equation.

2 – General definitions, notations and an "ergodic lemma"

In this section we briefly recall some well known notions, and prove a lemma which will play an important role in what follows.

Let us start, in particular, with the definition of a 2D hamiltonian system and the corresponding Liouville equation.

Given a C^2 hamiltonian $H(x,t)$, and a vector $x=(x_1,x_2)$, the differential system:

$$\dot{x_1} = \partial_{x_2} H$$
$$\dot{x_2} = -\partial_{x_1} H$$

is a 2D hamiltonian system in the phase space.

Setting $\nabla^\perp = (\partial_{x_2}, -\partial_{x_1})$, the previous system can be rewritten, more compactly, as

$$\dot{x} = \nabla^\perp H(x,t)\,. \tag{2.1}$$

Definition 2.1 (evolution operator in the phase space). The evolution operator ${T_H}^{t_1,t_2} : \Lambda \to \Lambda$, $\Lambda \subset \mathbb{R}^2$, from time t_1 to time t_2, is the solution of the problem

$$\begin{aligned}\partial_{t_1} {T_H}^{t_1,t_2} x &= -\nabla^\perp H({T_H}^{t_1,t_2} x, t_1)\,,\\ \partial_{t_2} {T_H}^{t_1,t_2} x &= \nabla^\perp H({T_H}^{t_1,t_2} x, t_2)\,,\\ {T_H}^{t,t} x &= x\,.\end{aligned}$$

In other words, by $y={T_H}^{t_1,t_2}x$, we mean that y is the evoluted, along (2.1), at time t_2, of the point that, at time t_1, is in x. In the particular case $t_1=0$, $t_2=t$, we set, from now on

$$T_H^{0,t} \equiv T_H^t\,.$$

Furthermore it is useful to notice that, by construction, ${T_H}^{t_1,t_2}$ is a canonical trasformation of Λ in itself, and therefore, ${T_H}^{t_1,t_2}$ is a bijective, measure preserving map.

Given a C^1 function $f : \Lambda \times \mathbb{R} \to \mathbb{R}$, $f(x,t) = f_t(x)$, the Liouville equation, associated with the hamiltonian $H(x,t)$ is, as it is well known, given by

$$\begin{aligned}&\partial_t f + \nabla^{\perp} H \cdot \nabla f = 0,\\ &f(x,0) = f_0(x)\,.\end{aligned} \tag{2.2}$$

If $f(x,t)$ is a solution of (2.2), then we say that the density f is "constant along the solutions of (2.1)" (or that f is "a first integral"). In this case, for any t_1, t_2 one has

$$f(x,t_2) = f({T_H}^{t_2,t_1} x, t_1). \tag{2.3}$$

Definition 2.2 (evolution operator for the Liouville equation). Given a function $g(x,t) \equiv g_t(x)$, $g_t : \Lambda \to \mathbb{R}$, the operator ${U_H}^{t_1,t_2} : \mathbb{R} \to \mathbb{R}$ is defined, for all t_1, t_2, by

$$ {U_H}^{t_1,t_2} g_{t_1}(x) = g_{t_1}({T_H}^{t_2,t_1} x). \tag{2.4}$$

This definition is consistent with the definition of first integral, and allows us to rewrite (2.3) in the following form:

$$f(x,t_2) = {U_H}^{t_1,t_2} f(x,t_1)\,.$$

If $t_1 = 0, t_2 = t$, we write $f(x,t) = {U_H}^{t} f(x,0) = {U_H}^{t} f_0(x)$.

We recall, moreover, that the 2D Euler equation for a vorticity field $\omega(x,t)$, in a domain $\Lambda \in \mathbb{R}^2$, is the following problem

$$\begin{aligned}&\partial_t \omega(x,t) + (u \cdot \nabla)\omega(x,t) = 0\\ &u(x,t) = \nabla^{\perp}\Psi(x,t),\end{aligned} \tag{2.5}$$

where $u(x,t)$ is the divergence-less velocity vector field, and the stream function $\Psi(x,t)$ satisfies the Poisson problem

$$\begin{aligned}-\Delta\Psi(x,t) &= \omega(x,t) \quad \text{on } \Lambda\\ \Psi(x,t) &= 0 \qquad\quad\;\; \text{on } \partial\Lambda.\end{aligned} \tag{2.6}$$

Because of $(2.6)_1$, from now on, we use the compact notation $\Psi(x,t) = (-\Delta)^{-1}\omega(x,t)$.

Let us notice that 2D Euler equation (2.5) may be seen, because of $(2.5)_2$, as a nonlinear Liouville equation for the density $\omega(x,t)$ associated with the hamiltonian $\Psi(x,t)$.

We introduce also an evolution operator associated with the problem (2.5), (2.6).

Definition 2.3 (evolution operator for the Euler equation). Given $\omega(x,t)$, solution of (2.5), (2.6), the evolution operator E^{t_1,t_2} is defined, for all t_1, t_2, by

$$\omega(x,t_2) = E^{t_1,t_2}\omega(x,t_1)\,.$$

We conclude the section with a lemma that, as we already pointed out, will play an important role in the proof of our result.

Let us consider an hamiltonian, depending only on one of the two components of x. To be more specific, assume that $H(x,t) = h(x_2)$, $h \in C^2(\mathbb{R})$, $h''(x_2) \neq 0$. Assume, furthermore, that $x = (x_1, x_2)$, $x_1 \in [0, 2\pi]$, $x_2 \in \mathbb{R}$, and that the function $f(x,t) = f_t(x)$, is conserved along the solutions of the hamiltonian system with hamiltonian $h(x_2)$, i.e.

$$f_t(x_1,x_2) = f_0(x_1 - h'(x_2)t, x_2)\,. \tag{2.7}$$

The following result holds.

Lemma 2.1 (ergodic lemma). *If $f_t \in L_1 \cap L_\infty([0,2\pi]\times\mathbb{R})$, then for $t \to \infty$, $f_t(x)$ weakly converges, in the sense of measure, to $f_\infty(x_2)$, where $f_\infty(x_2) = \frac{1}{2\pi}\int_0^{2\pi} f_0(x_1,x_2)dx_1$.*

Proof. The lemma is proved in the particular case $h(x_2) = \frac{x_2^2}{2}$. In the general case $h(x_2) = F(x_2^2)$, the result can be showed following essentially the same path, taking into account the fact that the hamiltonian system $\dot{x}_1 = h'(x_2)$, $\dot{x}_2 = 0$ can be transformed, because of the hypothesis $h''(x_2) \neq 0$, into system $\dot{y}_1 = y_2$, $\dot{y}_2 = 0$.

In the case we deal with, the (2.7) becomes

$$f_t(x_1,x_2) = f_0(x_1 - x_2 t, x_2)\,. \tag{2.8}$$

Given a test function $\phi \in C^0$, let us consider the scalar product in L_2 $\langle f_t, \phi\rangle$. From (2.8) one has

$$\langle f_t, \phi\rangle = \langle f_0, \phi_t\rangle\,, \tag{2.9}$$

where $\phi_t(x_1, x_2) = \phi(x_1 + x_2 t, x_2)$.

Let us call now $\hat{\phi}(k_1, k_2)$ the Fourier transform of ϕ_t, that is

$$\hat{\phi}(k_1, k_2) = \int_{-\infty}^{\infty} dx_2 \int_0^{2\pi} e^{-i(k_1 x_1 + k_2 x_2)} \phi_t(x_1, x_2) dx_1 .$$

It is not difficult to show that it is

$$\hat{\phi}(k_1, k_2) = \hat{\phi}(k_1, k_2 + k_1 t) . \tag{2.10}$$

Now, from (2.8), (2.9), (2.10) and the Parceval formula, it follows that

$$\begin{aligned}\langle f_t(x_1, x_2), \phi(x_1, x_2)\rangle &= \langle \hat{f}(k_1, k_2)\hat{\phi}(k_1, k_2)\rangle = \langle \hat{f}(k_1, k_2)\hat{\phi}(k_1, k_2 + k_1 t)\rangle = \\ &= \sum_{k_1 \geq 0} \int_{-\infty}^{\infty} [\hat{f}(k_1, k_2)\hat{\phi}(k_1, k_2 + k_1 t)] dk_2 = \\ &= \int_{-\infty}^{\infty} [\hat{f}(0, k_2)\hat{\phi}(0, k_2)] dk_2 + \\ &\quad + \sum_{k_1 > 0} \int_{-\infty}^{\infty} [\hat{f}(k_1, k_2)\hat{\phi}(k_1, k_2 + k_1 t)] dk_2 .\end{aligned}$$

The first integral in the previous sum gives exactly $f_\infty(x_2)$.

If one applies to the remainder terms the Lebesgue's dominated convergence theorem, having in mind that $\hat{f}(k_1, k_2)\hat{\phi}(k_1, k_2 + k_1 t) \to 0$ as $t \to \infty$, the lemma is proved.

3 – Homogenization and scattering theory for 2D Euler equations

Let us start the section with an important definition.

Definition 3.1 (homogenization). Given an initial vorticity $\omega_0(x)$, we say that homogenization occurs if there exists a function $\omega_\infty(|x|)$ such that

$$\begin{gathered}\omega(x, t) \to \omega_\infty(r), \\ r = |x|,\end{gathered} \tag{3.1}$$

where $\omega(x,t)$ is the solution of the Euler equation with initial condition $\omega_0(x)$, and the convergence is weak, in the sense of measure.

Let us remark that lemma 2.1 states that the solutions f_t of the linear Liouville equation, associated with $h(x_2)$, homogenize.

It is not difficult to see, moreover, that an immediate consequence of (3.1) is that, if $\omega_0 \in L_1 \cap L_\infty$, then for $t \to \infty$,

$$u(x,t) \to u_\infty(r)$$
$$\Psi(x,t) \to \Psi_\infty(r),$$

where both u and Ψ converge pointwise. Moreover u_∞ and Ψ_∞ are respectively $\mathcal{C}^{1-\epsilon}$ and $\mathcal{C}^{2-\epsilon}$ functions. (For the proof, and more details on the subject one can refer to [20] .)

We are ready now to define the "scattering problem" for the Euler equations.

Set, first of all, $x = (r\cos\theta, r\sin\theta)$. Suppose now that $H(r,\theta,t) \equiv \Psi_\infty(r)$ is a given hamiltonian and call $\Omega_t(x)$ a solution of the linear Liouville equation associated with $\Psi_\infty(r)$.

If one wants to study the asymptotic behavior of $E^t\omega_0(x)$, solution of the Euler problem with initial datum $\omega_0(x)$, it seems natural to compare it, in L_1, with the solution $U^t_{\Psi_\infty}\Omega_0(x)$ of the linear Liouville equation associated with the "free" hamiltonian $\Psi_\infty(r)$.

More precisely, it is quite natural to assume the following definition.

Definition 3.2 (scattering). Given $\Omega_0(x)$ and $\Psi_\infty(r)$, if there exists an initial condition $\omega_0(x)$ of the Euler equation such that

$$||U^t_{\Psi_\infty}\Omega_0(x) - E^t\omega_0(x)||_{L_1} \to 0 \text{ as } t \to \infty, \tag{3.2}$$

we say that $E^t\omega_0(x) = \omega(x,t)$ satisfies a scattering condition.

Remark. Notice that we are not interested here in discussing the scattering theory for the Euler equation in its generality. But rather, we discuss rigorously the connection between the problem (3.2) and the homogenization.

For an exstensive, clear and complete discussion of the scattering problems and relative techniques, one can see [23].

As usual, in the scattering theory, we introduce a scattering operator.

Definition 3.3 (scattering operator). Given $\Psi_\infty(r)$, the operator $S : L_1 \cap L_\infty(\mathbb{R}^2) \to L_1 \cap L_\infty(\mathbb{R}^2)$ such that

$$S : \Omega_0 \to \omega_0, \tag{3.3}$$

is said the scattering operator.

We want to show, now, that there is a strict connection between the scattering problem and the occurrence of homogenization.

Lemma 3.2. *Given the two functions $\omega_0(x)$, and $\Omega_0(x)$, and a function $\Psi_\infty(r)$, $r = |x|$, if the triple $\omega_0(x), \Omega_0(r)$ and $\Psi_\infty(r)$ solves the the scattering problem* (3.2), *then there exists a function $\omega_\infty(r)$ such that $E^t\omega_0(x)$ weakly converges, in the sense of measure, to $\omega_\infty(r)$, that is homogenization is realized for $\omega(x,t)$.*

Proof. The proof easily follows from definitions 3.2 and 3.1, taking into account the ergodic lemmma.

Remark. Assuming that it is possible to solve the scattering problem for a vorticity profile ω_0, that is assuming that homogenization occurs for the correspondent solution of the 2D Euler equation, nothing is known on the set of vorticity profiles satisfying these hypotheses. Of course it can be empty, except for the trivial radial solutions. Nevertheless, we think that it may be fruitful to approach the homogenization as a scattering problem, mainly if one is interested in studying the question from a numerical point of view.

We end the section by noting that, because of the nonlinearity of the Euler equation, the scattering problem seems very hard to be approached directly. Then, as usual in the theory, one introduces a perturbative, iterative approach.

More precisely, call $x = (r\cos\theta, r\sin\theta)$, $|x| = r$. Given a function $\Omega_0(x)$, set $\Psi_\infty(r) = (-\Delta)^{-1}\langle\Omega_0(r)\rangle$, where $\langle\Omega_0(r)\rangle = (1/2\pi)\int_0^{2\pi}\Omega_0(r,\theta)d\theta$. For $n = 0, 1, 2, \dots$, let us consider the following sequence of Liouville equations

$$\partial_t\omega^{(n+1)}(x,t) + (u^{(n)} \cdot \nabla)\omega^{(n+1)}(x,t) = 0\,, \tag{3.7}$$

where, at any step, the velocity field is given by the solution of the previous step by

$$u^{(n)}(x,t) = \nabla^{\perp}\Psi_n(x,t) = \nabla^{\perp}(-\Delta)^{-1}\omega^{(n)}(x,t), \tag{3.8}$$

and

$$u^{(0)}(x,t) = \nabla^{\perp}\Psi_{\infty}(r)\,.$$

Assume, furthermore, that every equation has to be solved with the scattering condition

$$||U^t_{\Psi_\infty}\Omega_0(x) - U^t_{\Psi_n}\omega_0^{(n+1)}(x)||_{L_1} \to 0 \quad \text{as} \quad t \to \infty\,. \tag{3.9}$$

In other words, we have to solve, for all n, a linear (in fact the hamiltonian pertaining to the (n+1)-th order equation is given in terms of the solution of the n-th order problem) Liouville equation, satisfying the condition (3.9) (which is indeed an equation).

Remark, moreover, that one can define a scattering operator at every step.

Assume now to be able to solve the problem (3.7), (3.8), (3.9) at every step. To conclude with the homogenization for the full Euler problem, one would have to show that there is an Ω_0 (that is a "free" hamiltonian Ψ_∞) such that the solutions of previous hierarchy of evolutive linear problems converge to the solutions of the Euler equations, in some suitable norm. These problems, which look very hard, are not studied in this context, and will be the subject of further investigations. In this paper we confine ourselves to study the previous iterate problems up to the first non trivial order. To be more specific, in what follows we show that:

(a) for any choice of $\Omega_0(x)$, first step of the iteration is trivially solved;

(b) for a particular choice of $\Omega_0(x)$ and $\omega_0^{(2)}(x)$, also the second step can be studied in all details and the homogenization can be proved for the related problem.

4 – The model: second order perturbation theory

Let us consider the first step in the iterative procedure of previous section. Assume that a function $\Psi_\infty(r)$ is given. One has to solve the problem

$$\begin{aligned} &\partial_t \omega^{(1)}(x,t) + (u^{(0)} \cdot \nabla)\omega^{(1)}(x,t) = 0, \\ &u^{(0)}(x,t) = \nabla^{\perp}\Psi_{\infty}(r), \end{aligned} \tag{4.1}$$

with the scattering condition

$$||U^t_{\Psi_\infty}\Omega_0(x) - U^t_{\Psi_\infty}\omega_0^{(1)}(x)||_{L_1} \to 0 \quad as \quad t \to \infty\,. \tag{4.2}$$

But it is immediate to see that (4.2) is trivially satisfied if and only if $\omega_0^{(1)}(x) = \Omega_0(x)$. Remark that the scattering operator is, in this case, the identity.

We approach now the second order in the iteration. To do this, as a particular model, we assume from now on, that $\Omega_0(x) = \omega_0^{(1)}(x) = \chi_D(x)$, where χ_D is the characteristic function of a domain $D = \{x \in \mathbb{R}^2, x = (r\cos\theta, r\sin\theta), \theta \in [0, 2\pi], r \in [0, f(\theta)]\}$, where f is a non negative 2π-periodic function.

Let us write $(3.7), (3.8)$ for $n = 1$. One has

$$\begin{aligned} &\partial_t \omega^{(2)}(x,t) + (u^{(1)} \cdot \nabla)\omega^{(2)}(x,t) = 0, \\ &u^{(1)}(x,t) = \nabla^{\perp}\Psi_1(x,t) = \nabla^{\perp}(-\Delta)^{-1}\langle\omega^{(1)}(x,t)\rangle\,, \end{aligned} \tag{4.3}$$

and (3.9) reads

$$||U^t_{\Psi_\infty}\Omega_0(x) - U^t_{\Psi_1}\omega_0^{(2)}(x)||_{L_1} \to 0 \quad as \quad t \to \infty\,. \tag{4.4}$$

We want to prove, in what follows, that we are able to solve completely the problem $(4.3), (4.4)$. Taking into account the fact that lemma 3.2 holds, this will imply the homogenization for the solutions of second order approximation.

5 – The stream function Ψ_1

In order to study the problem $(4.3), (4.4)$, we need to know the function $\Psi_1(x,t)$. In this section we give, in particular, a representation of the stream function as a function of t.

Let us recall, first of all, a well known definition.

Definition 5.1 (Morse function). A $\mathcal{C}^2$ function $h(x)$, $x \in I \subset \mathbb{R}$ is said to be a Morse function if, for all $x_1, ..., x_n$, critical points of h in I , one has $h'(x_i) = 0$, and $h''(x_i) \neq 0$ for $i = 1, ..., n$.

Let us give, now, a preliminary lemma.

Lemma 5.1. *Assume that $g(x)$ is a $\mathcal{C}^2$ function and $h(x)$ is a $\mathcal{C}^3$ Morse function. Call $x_1, ..., x_n$ the critical points of $h(x)$ belonging to an interval $[a, b]$. Then the following expansion holds*

$$\begin{aligned}\int_a^b g(x)e^{ith(x)}dx = \frac{1}{t^{1/2}}\Big(\frac{\pi^{1/2}}{2}\Big)\sum_{k=1}^{n} C_k e^{ith(x_k)} + \\ - \frac{i}{t}\Big[\frac{g(b)}{h'(b)}e^{ith(b)} - \frac{g(a)}{h'(a)}e^{ith(a)}\Big] + O\Big(\frac{1}{t^{3/2}}\Big),\end{aligned} \tag{5.1}$$

where $C_k = C_k(x_k) = \frac{g(x_k)}{\sqrt{|h''(x_k)|}}$ *for* $k = 1, ..., n$

Remark. Previous (5.1) holds also if $a = x_1$, or $b = x_n$ or, eventually if $a = x_1, b = x_n$. More precisely one has, respectively

$$\begin{aligned}\int_{x_1}^b g(x)e^{ith(x)}dx = \frac{1}{t^{1/2}}\Big(\frac{\pi^{1/2}}{2}\Big)\Big\{C_1 i e^{ith(x_1)} + \sum_{k=1}^{n} C_k e^{ith(x_k)}\Big\} + \\ - \frac{i}{t}\frac{g(b)}{h'(b)}e^{ith(b)} + O\Big(\frac{1}{t^{3/2}}\Big),\end{aligned} \tag{5.1$_1$}$$

or

$$\begin{aligned}\int_a^{x_n} g(x)e^{ith(x)}dx = \frac{1}{t^{1/2}}\Big(\frac{\pi^{1/2}}{2}\Big)\Big\{\sum_{k=1}^{n} C_k e^{ith(x_k)} e^{ith(x_k)} - C_n i e^{ith(x_n)}\Big\} + \\ + \frac{i}{t}\frac{g(a)}{h'(a)}e^{ith(a)} + O\Big(\frac{1}{t^{3/2}}\Big),\end{aligned} \tag{5.1$_2$}$$

or, finally

$$\begin{aligned}\int_{x_1}^{x_n} g(x)e^{ith(x)}dx = \frac{1}{t^{1/2}}\Big(\frac{\pi^{1/2}}{2}\Big)\Big\{\sum_{k=1}^{n} C_k e^{ith(x_k)} + \\ + i\Big[C_1 e^{ith(x_1)} - C_n e^{ith(x_n)}\Big]\Big\} + O\Big(\frac{1}{t^{3/2}}\Big),\end{aligned} \tag{5.1$_3$}$$

Proof. See Appendix.

Let us come back to the stream function Ψ_1.
As it is well known, from $(4.3)_2$ we can write

$$\Psi_1(y,t) = \int_{\mathbb{R}^2} G(x,y)\omega^{(1)}(x,t)dx, \tag{5.2}$$

where $G(x,y) = -\frac{1}{2\pi}\ln|x-y|$ is the Green function of $\mathbb{R}^2$.

Assume that $y = (\rho\cos\phi, \rho\sin\phi)$ and $x = (r\cos\theta, r\sin\theta)$, that $\omega_0^{(1)}(x) = \chi_D(r,\theta)$, where χ_D is the characteristic function of D (which is the same set of the previous section 4), and assume, furthermore, that the function $f(\theta)$, defining the boundary of D, is a non negative, 2π-periodic, Morse function. Call θ_α, $\alpha = 1,...,n$ its critical points, $\theta_\alpha \in [0,2\pi]$, and let be $M = \max(f(\theta),\ \theta \in [0,2\pi])$.

For all $\rho \le M$, call, moreover $i_j = i_j(\rho)$, $j = 1,...,p$, the values of θ such that $f(i_j) = \rho$. Let finally be $\Psi_\infty(r) = (-\Delta)^{(-1)}\langle\omega_0^{(1)}(r)\rangle$, where $\langle\omega_0^{(1)}(r)\rangle = (1/2\pi)\int_0^{2\pi}\omega_0^{(1)}(r,\theta)d\theta$.

The "free" evolution law is given by $r\dot\theta = \partial_r\Psi_\infty(r)$, $\dot r = 0$. Set, by definition, $\frac{1}{r}\partial_r\Psi_\infty(r) \equiv v(r)$. (Let us remark that, by definition, $v(r) > 0$ for all $r > 0$).
The following result can be proved.

Theorem 5.1. *Let $v(r)$ be a monotone function of r. The stream function $\Psi_1(y,t)$ can be expanded, as function of t, as follows*

$$\Psi_1(y,t) = \Psi_\infty(\rho^2) - \frac{1}{t^{3/2}}\sum_{\alpha=1}^{n} c_\alpha F_\alpha(y,t) - \frac{1}{t^2}\sum_{j=1}^{p} L_j(y,t) + O\Big(\frac{1}{t^{5/2}}\Big) \tag{5.3}$$

where the F_α's $\alpha = 1,...,n$ are given by the formula

$$F_\alpha(y,t) = \frac{\sqrt{\pi}}{2}\sum_{k\ge 1}\frac{1}{k^{5/2}}R_\alpha^k \sin[k(\theta_\alpha - \phi + tv_\alpha)],$$

with $v_\alpha \equiv v(f(\theta_\alpha))$, the c_α's are suitable constants depending on θ_α,

$$R_\alpha = R_\alpha(\theta_\alpha,\rho) = \begin{cases} \dfrac{\rho}{f(\theta_\alpha)} & \text{if } \ \rho < f(\theta_\alpha) \\[2ex] \dfrac{f(\theta_\alpha)}{\rho} & \text{if } \ \rho > f(\theta_\alpha)\,, \end{cases}$$

and finally

$$L_j(y,t) = \sum_{k\geq 1} \frac{1}{k^3(v'(\rho))^2} l(i_j)\sin[k(i_j - \phi + tv(\rho))]$$

where

$$l(i_j) = \begin{cases} \mathrm{sign} f'(i_j) & \text{if} \quad \rho \leq M \\ 0 & \text{if} \quad \rho > M. \end{cases}$$

PROOF. See Appendix.

REMARK. Before concluding the section it is interesting to remark that the functions $F_\alpha(y,t)$ of formula (5.3) can be suitably interpreted. More precisely, for $\alpha = 1, ..., n$, let us introduce the functions

$$g_\alpha(y,t) = \delta(\rho - f(\theta_\alpha)) \sum_{h\geq 1} \frac{1}{h^{3/2}} \sin k(\phi - \theta_\alpha + tv_\alpha)\,, \tag{5.4}$$

where $\delta(*)$ is the usual delta-function, and v_α are the same as in the previous theorem 5.1. Then it is not difficult to see that

$$F_\alpha(y,t) = (-\Delta)^{-1} g_\alpha(y,t)\,, \tag{5.5}$$

that is, the F_α's are, for all α, the "stream functions" of a Poisson problem for which the "vorticity" is given by (5.4).

The proof of (5.5) can be obtained, by a direct computation, from the fact that $F_\alpha(y,t) = \int_{\mathbb{R}^2} G(x,y) g_\alpha(x,t) dx$, and from the usual representation of the Green function (see the proof of theorem 5.1 in the Appendix).

6 – The scattering problem

Assuming that the stream function Ψ_1 is given by (5.3), we want to show, in this section, that the scattering problem (4.3), (4.4) can be completely solved.

As first step, we perform a variables transformation. More precisely let us set $\frac{\rho^2}{2} = \mathcal{I}$. The function Ψ_1, in terms of the new coordinates, becomes

$$\Psi_1(\mathcal{I},\phi,t)=\Psi_\infty(\mathcal{I})-\frac{1}{t^{3/2}}\sum_{\alpha=1}^{n}c_\alpha F_\alpha(\mathcal{I},\phi,t)-\frac{1}{t^2}\sum_{j=1}^{p}L_j(\mathcal{I},\phi,t)+O\Big(\frac{1}{t^{5/2}}\Big) \tag{6.1}$$

where

$$F_\alpha(\mathcal{I},\phi,t)=\frac{\sqrt{\pi}}{2}\sum_{k\geq 1}\frac{1}{k^{5/2}}R_\alpha^k(\mathcal{I})\sin[k(\theta_\alpha-\phi+tv_\alpha)]\,,$$

$$R_\alpha=R_\alpha(\mathcal{I})=\begin{cases}\dfrac{\sqrt{2\mathcal{I}}}{f(\theta_\alpha)} & \text{if} \quad \mathcal{I}<f^2(\theta_\alpha)/2\\[2ex] \dfrac{f(\theta_\alpha)}{\sqrt{2\mathcal{I}}} & \text{if} \quad \mathcal{I}>f^2(\theta_\alpha)/2\,,\end{cases}$$

and

$$L_j(\mathcal{I},\phi,t)=\sum_{k\geq 1}\frac{1}{k^3(v'(\mathcal{I}))^2}l(i_j)\sin[k(i_j-\phi+tv(\mathcal{I}))]\,,$$

$\partial_{\mathcal{I}}\Psi_\infty\equiv v(\mathcal{I})$.

Let us remark that the previous transformation is not canonical, but the function $\Psi_1(\mathcal{I},\phi,t)$, given by (6.1) turns out to be hamiltonian.

We give now a preliminary lemma, which is important for our main result.

Lemma 6.1. *For all* $(\mathcal{I},\phi)\in[0,M^2/2]\times[0,2\pi]$, *except at most a set whose measure tends to zero as* $t\to\infty$, *there exists a* $T=T(\theta_\alpha,M)$ *such that, for* $t\geq T$, *the hamiltonian* Ψ_1 *given by* (6.1), *can be transformed, via two canonical transforms, into the new hamiltonian*

$$H(I,\Phi,t)=\frac{1}{t^2}\sum_{j=1}^{p}h_j(I,\Phi)+O\Big(\frac{1}{t^{5/2}}\Big)\,,$$

where the i_j*'s are the same as in theorem* 5.1, *and*

$$h_j(I,\Phi)=-\sum_{k\geq 1}\frac{1}{k^3(v'(I))^2}\sin k(i_j-\Phi)\,.$$

PROOF. Let us perform, as first step, a time-dependent canonical transform, say $\mathcal{C}_1 : (\mathcal{I}, \phi) \to (\mathcal{I}', \phi')$, such that the hamiltonian Ψ_1 is transformed into a new hamiltonian $\Psi_1'(\mathcal{I}', \phi', t) = \Psi_\infty(\mathcal{I}') + O(\frac{1}{t^2})$.

In order to do this, let us consider, as usual in the theory of canonical transforms, [30], a generating function

$$G_1(\mathcal{I}', \phi, t) = \mathcal{I}'\phi + \frac{1}{t^{3/2}} g_1(\mathcal{I}', \phi, t)\,, \tag{6.2}$$

where g_1 must be a suitable function. From (6.2) it follows that

$$\begin{aligned} \mathcal{I} &= \frac{\partial G_1}{\partial \phi} = \mathcal{I}' + \frac{1}{t^{3/2}} \frac{\partial g_1}{\partial \phi} \\ \phi' &= \frac{\partial G_1}{\partial \mathcal{I}'} = \phi + \frac{1}{t^{3/2}} \frac{\partial g_1}{\partial \mathcal{I}'}\,, \end{aligned}$$

and

$$\begin{aligned} \Psi_1'(\mathcal{I}', \phi', t) &= \Psi_1(\mathcal{I}, \phi, t)\Big|_{\mathcal{C}_1} + \frac{\partial G_1}{\partial t}\Big|_{\mathcal{C}_1} = \\ &= \Psi_1(\mathcal{I}, \phi, t)\Big|_{\mathcal{C}_1} + \frac{1}{t^{3/2}} \frac{\partial g_1}{\partial t} + o\Big(\frac{1}{t^{3/2}}\Big)\,. \end{aligned} \tag{6.3}$$

Moreover, in order to obtain ϕ' as a function of $\mathcal{I}, \phi$ and t, it must be, from (6.2),

$$\frac{1}{t^{3/2}} \frac{\partial^2 g_1}{\partial \phi \partial \mathcal{I}'} \neq 1\,. \tag{6.4}$$

It is not difficult to see that, setting

$$g_1(\mathcal{I}', \phi, t) = \sum_{\alpha=1}^{n} \frac{c_\alpha}{v(\mathcal{I}) - v_\alpha} \sum_{k \geq 1} \frac{1}{k^{7/2}} R_\alpha^k(\mathcal{I}) \cos[k(\theta_\alpha - \phi + t(v_\alpha))]\,,$$

(6.4) is satisfied for all $t \geq T$, $T = T(\theta_\alpha, M)$.

Then $\mathcal{C}_1$ is given by

$$\begin{aligned} \mathcal{I}' &= \mathcal{I} + O\Big(\frac{1}{t^{3/2}}\Big) \\ \phi' &= \phi + O\Big(\frac{1}{t^{3/2}}\Big). \end{aligned} \tag{6.5}$$

Moreover, from (6.3) one can obtain

$$\Psi_1'(\mathcal{I}',\phi',t)=\Psi_\infty(\mathcal{I}')-\frac{1}{t^2}\sum_{j=1}^{p}L_j(\mathcal{I}',\phi',t)+O\Big(\frac{1}{t^{5/2}}\Big) \tag{6.6}$$

where

$$L_j(\mathcal{I}',\phi',t)=\sum_{k\geq 1}\frac{1}{k^3(v'(\mathcal{I}'))^2}l(i_j)\sin[k(i_j-\phi'+tv(\mathcal{I}'))]\,.$$

Let us consider now a second time-dependent canonical transform. Call it $\mathcal{C}_2:(\mathcal{I}',\phi')\to(I,\Phi)$. Then Ψ_1' is transformed into the hamiltonian $H(I,\Phi,t)=O(\frac{1}{t^2})$.

This transformation can be constructed via the generating function

$$G_2(I,\phi',t)=I\phi'-\Psi_\infty(I)t\,. \tag{6.7}$$

As before, (6.7) implies that $\mathcal{I}'=\frac{\partial G_2}{\partial\phi'}$, $\Phi=\frac{\partial G_2}{\partial I}$, that is, taking into account that, by definition $\partial\Psi_\infty/\partial I=v(I)$, $\mathcal{C}_2$ is given by

$$\begin{aligned} I&=\mathcal{I}'\\ \Phi&=\phi'-v(I)t\,, \end{aligned} \tag{6.8}$$

and

$$\begin{aligned} H(I,\Phi,t)&=\Psi_1'(\mathcal{I}',\phi',t)\Big|_{\mathcal{C}_2}+\frac{\partial G_2}{\partial t}\Big|_{\mathcal{C}_2}=\\ &=\Psi_1'(\mathcal{I}',\phi',t)\Big|_{\mathcal{C}_2}-\Psi_\infty(I)=\frac{1}{t^2}\sum_{j=1}^{p}h_j(I,\Phi)+O\Big(\frac{1}{t^{5/2}}\Big)\,, \end{aligned}$$

where

$$h_j(I,\Phi)=\sum_{k\geq 1}\frac{1}{k^3(v'(I))^2}\sin k(i_j-\Phi)\,.$$

The proof is concluded by remarking that, for $v(I)=v_\alpha$, that is on n circles of radius $I_\alpha=v^{-1}(v_\alpha)$, $\alpha=1,...,n$, the transformation (6.4) is not defined. However, taking into account the explicit form of the transformation $\mathcal{C}_2$, one can say that for all $\epsilon>0$, there exists a $\tau=\tau(\epsilon)$, such that $|\rho(t) - I(t)| < \epsilon$ for all $t \geq \tau$. Call now $T^* = max(T, \tau)$. If one has $|I(T^*) - I_\alpha)| \geq 2\epsilon$, then the transformation is defined everywhere except, at most a set whose measure tends to zero for $t \to \infty$. And this concludes our proof.

We are ready now for the main result of the section. More precisely, we are ready to solve the scattering problem (4.4).

THEOREM 6.1 (homogenization). *Let us assume that* $\Omega_0(x) = \chi_D(x)$, *where* χ_D *is the characteristic function of the set* D, *and* D *is the same as in the theorem* 5.1. *Assume, furthermore, that* Ψ_1 *is given by* (6.1). *Then there exists a function* $\omega_0^{(2)} \in L_1 \cap L_\infty(\mathbb{R}^2)$ *such that*

$$||U^t_{\Psi_\infty}\Omega_0(x) - U^t_{\Psi_1}\omega_0^{(2)}(x)||_{L_1} \to 0\,, \tag{6.9}$$

as $t \to \infty$.

PROOF. Let us remark that the left hand side of (6.9) can be rewritten as

$$||U^t_{\Psi_\infty}\Omega_0(x) - U^t_{\Psi_1}\omega_0^{(2)}(x)||_{L_1} = ||\Omega_0(T^{t,0}_{\Psi_\infty}x) - \omega_0^{(2)}(T^{t,0}_{\Psi_1}x)||_{L_1}\,. \tag{6.10}$$

But $T^{t,0}_{\Psi_1}x = T^{T,0}_{\Psi_1}T^{T,t}_H\mathcal{C}_2\mathcal{C}_1x$, therefore (6.10) is equivalent to

$$||\Omega_0(T^{t,0}_{\Psi_\infty}\mathcal{C}_1^{-1}\mathcal{C}_2^{-1}T^{T,t}_H T^{0,T}_{\Psi_1}x) - \omega_0^{(2)}(x)||_{L_1}\,. \tag{6.11}$$

If we show that $T^{t,0}_{\Psi_\infty}\mathcal{C}_1^{-1}\mathcal{C}_2^{-1}T^{T,t}_H T^{0,T}_{\Psi_1}x$ converges, as $t \to \infty$, taking into account the fact that Ω_0 is a characteristic function, we have proved that (6.9) holds.

But, due to the explicit form of $\mathcal{C}_1$, $\mathcal{C}_2$, H, Ψ_∞ it is not difficult to verify that

$$\lim_{t\to\infty} T^{t,0}_{\Psi_\infty}\mathcal{C}_1^{-1}\mathcal{C}_2^{-1}T^{T,t}_H T^{0,T}_{\Psi_1}x = \Big(c_1 + O\Big(\frac{1}{t^{1/2}}\Big), c_2 + O\Big(\frac{1}{t^{1/2}}\Big)\Big)\,,$$

where the c_i's, $i = 1, 2$, depend on the initial conditions (I, Φ) and c_2 depends also on T. This concludes the theorem.

– Appendix

Proof of lemma 5.1. We prove, as a particular case, (5.1). The other cases can be handled analogously.

We remark, first of all, that

$$\int_a^b g(x)e^{ith(x)}dx = \int_a^{x_1} g(x)e^{ith(x)}dx + \\ + \sum_{j=1}^{n-1} \int_{x_j}^{x_{j+1}} g(x)e^{ith(x)}dx + \int_{x_n}^b g(x)e^{ith(x)}dx\,.$$

Set $z = h(x)$ and $F(z) = \frac{g(h^{-1}(z))}{h'(h^{-1}(z))}$. Call $z_j = h(x_j)$, $j = 1, ...n-1$, and let us compute

$$I_j = \int_{x_j}^{x_{j+1}} g(x)e^{ith(x)}dx = \int_{z_j}^{z_{j+1}} F(z)e^{itz}dz = \\ = \int_{z_j}^{z_j+d_j} F(z)e^{itz}dz + \int_{z_j+d_j}^{z_{j+1}-d_{j+1}} F(z)e^{itz}dz + \int_{z_{j+1}-d_{j+1}}^{z_{j+1}} F(z)e^{itz}dz\,,$$

where $[z_j, z_j+d_j]$ (or $[z_{j+1}-d_{j+1}, z_{j+1}],$) $z_j < z_j+d_j < z_{j+1}-d_{j+1} < z_{j+1}$ is the neighborhood of z_j (resp. z_{j+1}) where, taking into account that h is a Morse function, we can write

$$F(z) = C_j|z-z_j|^{-(1/2)} + A_j + O(|z-z_j|^{1/2})\,,$$

where $C_j = C_j(x_j) = \frac{g(x_j)}{(2|h''(x_j)|)^{1/2}}$, $A_j = A_j(x_j) = \frac{g'(x_j)}{h''(x_j)}$ for $j = 1, ..., n-1$.

One can write

$$\int_{z_j}^{z_j+d_j} F(z)e^{itz}dz = \frac{C_j e^{itz_j}}{t} \int_{z_j}^{z_j+d_j} \Big[\frac{e^{it(z-z_j)}}{(z-z_j)^{1/2}} + o(|z-z_j|^{-(1/2)})\Big]d(tz) = \\ = \frac{C_j e^{itz_j}}{t^{1/2}}\Big\{\int_0^\infty \frac{e^{iu}}{u^{1/2}}du - \int_{td_j}^\infty \frac{e^{iu}}{u^{1/2}}du\Big\} + O\Big(\frac{1}{t^{3/2}}\Big)\,.$$

Because of the fact that the Fresnel's integral $\int_0^\infty \frac{e^{iu}}{u^{1/2}}du$ converges to $\sqrt{\frac{\pi}{2}}(1+i)$, integrating by parts the second integral, we conclude that

$$\text{(A.1)} \quad \int_{z_j}^{z_j+d_j} F(z)e^{itz}dz = C_j e^{itz_j}\Big\{\frac{1}{t^{1/2}}\sqrt{\frac{\pi}{2}}(1+i) + \frac{1}{t}\Big(\frac{e^{itd_j}}{id_j^{1/2}}\Big) + O\Big(\frac{1}{t^{3/2}}\Big).$$

Integrating again by parts, it is immediate to verify that

$$\int_{z_j+d_j}^{z_{j+1}-d_{j+1}} F(z)e^{itz}dz = \frac{1}{it}\Big[F(z_{j+1}-d_{j+1})e^{it(z_{j+1}-d_{j+1})}+ - F(z_j+d_j)e^{it(z_j+d_j)}\Big] + O\Big(\frac{1}{t^{3/2}}\Big). \tag{A.2}$$

Finally, the third integral of I_j can be calculated following the same path of the first integral. Then

$$\begin{aligned} I_j = {} & \frac{1}{t^{1/2}}\frac{\sqrt{\pi}}{2}\Big\{C_j e^{itz_j}(1+i) + C_{j+1}e^{it(z_{j+1}}(1-i)\Big\}+ \\ & + \frac{1}{it}\Big\{e^{it(z_j+d_j)}\Big[\frac{C-j}{d_j^{1/2}} - F(z_j+d_j)\Big]+ \\ & + e^{it(z_{j+1}-d_{j+1})}\Big[F(z_{j+1}-d_{j+1}) - \frac{C_{j+1}}{d_{j+1}^{1/2}}\Big]\Big\} + O\Big(\frac{1}{t^{3/2}}\Big)\,. \end{aligned}$$

The integrals all over $[a, x_1]$ and $[x_n, b]$ can be computed performing analogous calculations.

Summing all the contributions and taking into account the explicit expression of the function $F(z)$, the lemma is proved.

PROOF OF THEOREM 5.1. Let us remark, first of all, that $\omega^{(1)}$ is conserved along the solutions of $(4.1)_2$, that is $\omega^{(1)}(x,t) = \omega_0^{(1)}(T^{t,0}_{\Psi_\infty}x)$. Set, by definition, $z = T^{t,0}_{\Psi_\infty}x$. Then (5.2) can be rewritten as

$$\Psi_1(y,t) = \int_{\mathbb{R}^2} G(y, T^t_{\Psi_\infty}z)\omega_0^{(1)}(z)dz\,.$$

Taking into account the explicit formula for G, the fact that $\omega_0^{(1)}(z) = \chi_D(z)$, and the free evolution law, the previous formula becomes, in polar coordinates,

$$\Psi_1(y,t) = -\frac{1}{4\pi}\int_0^{2\pi} d\theta \int_0^{f(\theta)} \ln[r^2+\rho^2-2r\rho\cos(\theta-\phi+tv(r))]rdr\,.$$

Let us call M the (positive) maximum of $f(\theta)$, $\theta\in[0,2\pi]$.

We prove (5.3) in the case $\rho\in[0,M]$. If $\rho > M$, that is if $x\notin D$, the proof reduces to a particular case, and can be performed following the same ideas below.

Assume, without loss of generality, that the function $f(\theta)$ is given, for example, as in the figure

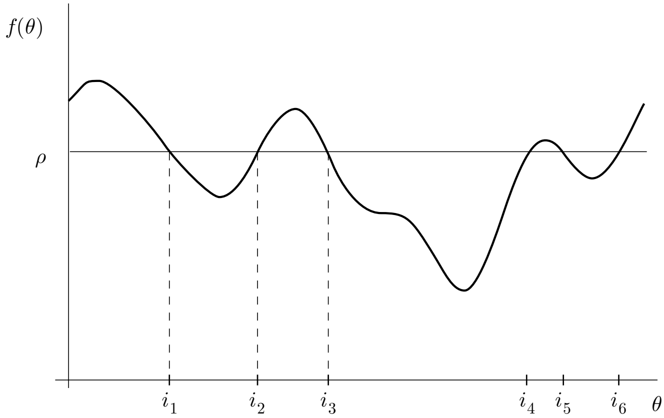

Fig. 2

Define $\theta = i_1, ..., \theta = i_m$ the intersections $f(\theta) = \rho$. Then it is $[0, 2\pi] = \cup_{j=1}^{m} I_j$ where $I_j = [i_j, i_{j+1}]$ and we set $i_{m+1} \equiv i_1$. Then the Ψ_1 can be rewritten as

$$\Psi_1(y,t) = -\frac{1}{4\pi} \sum_{\alpha=1}^{m} \int_{I_j} A(\theta, y, t) d\theta, \tag{A.3}$$

where one has, by definition,

$$A(\theta, y, t) = \int_0^{f(\theta)} \ln[r^2 + \rho^2 - 2r\rho\cos(\theta - \phi + tv(r))] r dr\,. \tag{A.4}$$

To estimate A, we assume, from now on, that, for all j, the critical points of $f(\theta)$ belong to the interior of I_j (as in the previous figure). If this is not the case, the proof can be performed analogously, by applying, when it is necessary, formula (5.1_1) (or (5.1_2) or (5.1_3)) instead of (5.1).

To estimate (A.3), let us remark, first of all, that from the definition of I_j, it follows that, for all $\theta \in I_j$, one has $\rho \leq f(\theta)$ or $\rho \geq f(\theta)$. To show our result, we study $\int_{I_j} A(\theta, y, t) d\theta$, and $\int_{I_{j+1}} A(\theta, y, t) d\theta$, and we assume, without loss of generality, that for $\theta \in I_j$ it is $\rho \leq f(\theta)$, (that is for $\theta \in I_{j+1}$ it is $\rho > f(\theta)$).

Let us note, first of all, that (A.4), can be rewritten as follows

$$
\begin{aligned}
A(\theta,y,t) = & \int_0^{\rho-\epsilon} \ln[r^2+\rho^2-2r\rho\cos(\theta-\phi+tv(r)]rdr+ \\
& + \int_{\rho+\epsilon}^{f(\theta)} \ln[r^2+\rho^2-2r\rho\cos(\theta-\phi+tv(r)]rdr+ \\
& + \int_{\rho-\epsilon}^{\rho+\epsilon} \ln[r^2+\rho^2-2r\rho\cos(\theta-\phi+tv(r)]rdr\,,
\end{aligned}
\tag{A.5}
$$

where ϵ is so small as we want.

We remark, first of all, that

$$
\int_{I_j}\int_{\rho-\epsilon}^{\rho+\epsilon} \ln[r^2+\rho^2-2r\rho\cos(\theta-\phi+tv(r))]rdrd\theta
$$

tends to zero as ϵ tends to zero. Then we have to study the first and the second term in (A.5).

Let us estimate the first integral in the previous sum.

Taking into account the fact that, whenever $\frac{r}{\rho} < 1$, the following formula holds

$$
-\ln(1+(r/\rho)^2-2(r/\rho)\cos x) = \sum_{k\geq 1} \frac{(r/\rho)^k}{k}\cos kx\,,
$$

it is

$$
\begin{aligned}
& \int_0^{\rho-\epsilon} \ln[r^2+\rho^2-2r\rho\cos(\theta-\phi+tv(r))]rdr = \\
& = \ln\rho^2\frac{(\rho-\epsilon)^2}{2} - \sum_{k\geq 1}\frac{1}{2k}\int_0^{\rho-\epsilon}\frac{r^{k+1}}{\rho^k}e^{ik(\theta-\phi+tv(r))}dr + c.c.t.,
\end{aligned}
$$

where, from now on, by "c.c.t.", we mean "complex conjugate terms". Integrating by parts twice, it is possible to verify that

$$
\begin{aligned}
\int_0^{\rho-\epsilon}\frac{r^{k+1}}{\rho^k}e^{ik(\theta-\phi+tv(r))}dr = & \frac{1}{ikt\rho^k}\Big[\frac{(\rho-\epsilon)^{k+1}}{v'(\rho-\epsilon)}e^{ik(\theta-\phi+tv(\rho-\epsilon))}+ \\
& -\int_0^{\rho-\epsilon} e^{ik(\theta-\phi+tv(r))}\Big(\frac{(k+1)r^k v'(r)-r^{k+1}v''(r)}{(v'(r))^2}\Big)\Big]dr =
\end{aligned}
$$

$$= \frac{e^{ik(\theta-\phi+tv(\rho-\epsilon))}}{k}\Big\{\frac{1}{t}\Big[\frac{(\rho-\epsilon)^{k+1}}{i\rho^k v'(\rho-\epsilon)}\Big]+\frac{1}{t^2}\Big[\frac{k+1}{k(v'(\rho-\epsilon))^2}\Big(\frac{\rho-\epsilon}{\rho}\Big)^k+$$
$$-\frac{v''(\rho-\epsilon)}{k(v'(\rho-\epsilon))^3}\frac{(\rho-\epsilon)^{k+1}}{\rho^k}\Big]+O\Big(\frac{1}{t^3}\Big)\Big\}+\text{c.c.t.}$$

Then

$$\int_0^{\rho-\epsilon}\ln[r^2+\rho^2-2r\rho\cos(\theta-\phi+tv(r))]rdr=$$
$$=\ln\rho^2\frac{(\rho-\epsilon)^2}{2}-\sum_{k\geq 1}\frac{e^{ik(\theta-\phi+tv(\rho-\epsilon))}}{2k^2}\Big\{\frac{1}{t}\Big[\frac{(\rho-\epsilon)^{k+1}}{i\rho^k v'(\rho-\epsilon)}\Big]+$$
$$+\frac{1}{t^2}\Big[\frac{k+1}{k}\Big(\frac{\rho-\epsilon}{\rho}\Big)^k\frac{1}{(v'(\rho-\epsilon))^2}-\frac{v''(\rho-\epsilon)(\rho-\epsilon)^{k+1}}{k(v'(\rho-\epsilon))^3\rho^k}\Big]+$$
$$+O\Big(\frac{1}{t^3}\Big)\}+\text{c.c.t.}$$

We study now the second integral of (A.5).

One has, taking into account that $\frac{\rho}{r}<1$, and integrating again by parts,

$$\int_{\rho+\epsilon}^{f(\theta)}\ln[r^2+\rho^2-2r\rho\cos(\theta-\phi+tv(r))]rdr=\frac{1}{2}[f^2(\theta)(\ln f^2(\theta)-1)+$$
$$-(\rho+\epsilon)^2(\ln(\rho+\epsilon)^2-1)]+$$
$$-\sum_{k\geq 1}\frac{\rho^k}{2k}\int_{\rho+\epsilon}^{f(\theta)}\frac{1}{r^{k-1}}e^{ik(\theta-\phi+tv(r))}dr+\text{c.c.t.}=$$
$$=\frac{1}{2}[f^2(\theta)(\ln f^2(\theta)-1)-(\rho+\epsilon)^2(\ln(\rho+\epsilon)^2-1)]+$$
$$-\sum_{k\geq 1}\frac{e^{ik(\theta-\phi+tv(f(\theta))}}{2k^2}\Big\{\frac{1}{t}\Big[\frac{\rho^k}{iv'(f(\theta))(f(\theta))^{k-1}}\Big]+$$
$$+\frac{1}{t^2}\Big[\frac{k-1}{k}\Big(\frac{\rho}{f(\theta)}\Big)^k\frac{1}{(v'(f(\theta))^2}-\frac{v''(f(\theta))}{k(v'(f(\theta))^3}\frac{(\rho)^k}{(f(\theta))^{k-1}}\Big]+$$
$$+\sum_{k\geq 1}\frac{e^{ik(\theta-\phi+tv(\rho+\epsilon))}}{2k^2}\Big\{\frac{1}{t}\Big[\frac{\rho^k}{i(v'(\rho+\epsilon))(\rho+\epsilon)^{k-1}}\Big]+$$
$$+\frac{1}{t^2}\Big[\frac{k-1}{k}\Big(\frac{\rho}{\rho+\epsilon}\Big)^k\frac{1}{(v'(\rho+\epsilon))^2}-\frac{v''(\rho+\epsilon)}{k(v'(\rho+\epsilon))^3}\frac{(\rho)^k}{(\rho+\epsilon)^{k-1}}\Big]\Big\}+$$
$$+O\Big(\frac{1}{t^3}\Big)+\text{c.c.t.}$$

Then, for $\epsilon \to 0$, (A.5) becomes

$$
\begin{aligned}
A(\theta, y, t) = \frac{1}{2}[f^2(\theta)(\ln f^2(\theta) - 1) + \rho^2] + \\
- \frac{1}{t} \sum_{k \geq 1} \frac{1}{2k^2} \frac{e^{ik(\theta - \phi + tv(f(\theta))}}{iv'(f(\theta))} \frac{\rho^k}{(f(\theta))^{k-1}} - \frac{1}{t^2} \sum_{k \geq 1} \frac{1}{2k^2} \times \\
\times \Big\{ \frac{e^{ik(\theta - \phi + tv(f(\theta))}}{(v'(f(\theta)))^2} \Big[\frac{k-1}{k} \Big(\frac{\rho}{f(\theta)} \Big)^k + \\
- \frac{\rho^k}{k(f(\theta))^{k-1}} \frac{v''(f(\theta))}{(v'(f(\theta))} \Big] + \frac{2e^{ik(\theta - \phi + tv(\rho))}}{(v'(\rho))^2} \Big\} + \\
+ O\Big(\frac{1}{t^3}\Big) + \text{c.c.t.}
\end{aligned}
$$

We have now to study $\int_{I_j} A(\theta, y, t) d\theta$. One has

$$
\begin{aligned}
\int_{I_j} A(\theta, y, t) d\theta = \frac{1}{2} \int_{I_j} [f^2(\theta)(\ln f^2(\theta) - 1) + \rho^2] d\theta + \\
- \frac{1}{t} \sum_{k \geq 1} \frac{\rho^k}{2k^2 i} \mathcal{A}_k(I_j) - \frac{1}{t^2} \sum_{k \geq 1} \frac{\rho^k}{2k^2} [\mathcal{B}_k(I_j) - \mathcal{C}_k(I_j)] + \\
- \frac{1}{t^2} \sum_{k \geq 1} \frac{1}{ik^3} \frac{1}{(v'(\rho))^2} (e^{ik(i_{j+1} - \phi + tv(\rho))} - e^{ik(i_j - \phi + tv(\rho))}) + O\Big(\frac{1}{t^3}\Big) + \text{c.c.t.}
\end{aligned}
$$

where we set, by definition,

$$
\text{(A.6)} \qquad \mathcal{A}_k(I_j) = \int_{i_j}^{i_{j+1}} \frac{1}{v'(f(\theta))(f(\theta))^{k-1}} e^{ik(\theta - \phi + tv(f(\theta))} d\theta \,,
$$

$$
\text{(A.7)} \qquad \mathcal{B}_k(I_j) = \frac{k-1}{k} \int_{i_j}^{i_{j+1}} \frac{e^{ik(\theta - \phi + tv(f(\theta))}}{(v'(f(\theta)))^2} \frac{1}{(f(\theta))^k} d\theta \,,
$$

and

$$
\text{(A.8)} \qquad \mathcal{C}_k(I_j) = \frac{1}{k} \int_{i_j}^{i_{j+1}} \frac{e^{ik(\theta - \phi + tv(f(\theta))}}{(v'(f(\theta)))^3} \frac{v''(f(\theta))}{(f(\theta))^{k-1}} d\theta \,.
$$

Let us apply now to (A.6), (A.7), and (A.8) lemma 5.1 assuming that, in the first case,

$$g = g(\theta) = \frac{e^{ik(\theta-\phi)}}{v'(f(\theta))(f(\theta))^{k-1}}, \qquad h = h(\theta) = kv(f(\theta))\,,$$

in the second case

$$g(\theta) = \frac{e^{ik(\theta-\phi)}}{(v'(f(\theta)))^2(f(\theta))^k}, \qquad h(\theta) = kv(f(\theta))\,,$$

and finally

$$g(\theta) = \frac{e^{ik(\theta-\phi)}v''(f(\theta)}{(v'(f(\theta)))^3(f(\theta))^{k-1}}, \qquad h(\theta) = kv(f(\theta))\,,$$

Call $\theta_h, \theta_{h+1}, ..., \theta_l$ the critical points of f that, following the hypothesis, are in the interior of the interval I_j. From formula (5.1), taking into account that, by definition, $f(i_j) = \rho$ for all j, one has

$$\begin{aligned}
&\int_{I_j} A(\theta,y,t)d\theta = A(\rho^2)+\\
&\quad -\frac{1}{t^{3/2}}\frac{\sqrt{\pi}}{4}\sum_{k\geq 1}\frac{1}{k^{5/2}i}\sum_{\alpha=h}^{l}\Big(\frac{\rho}{f(\theta_\alpha)}\Big)^k f(\theta_\alpha)\frac{e^{ik(\theta_\alpha-\phi+tv(f(\theta_\alpha)))}}{(|(v'(f(\theta_\alpha))^3 f''(\theta_\alpha)|)^{1/2}}+\\
&\quad -\frac{1}{t^2}\sum_{k\geq 1}\Big\{\frac{1}{k^3(v'(\rho))^2}\Big\{\frac{\rho}{2}\Big[\frac{e^{ik(i_{j+1}-\phi+tv(\rho))}}{f'(i_{j+1}))}-\frac{e^{ik(i_j-\phi+tv(\rho))}}{f'(i_j)}\Big]+\\
&\quad +\frac{1}{i}\Big[e^{ik(i_{j+1}-\phi+tv(\rho))}-e^{ik(i_j-\phi+tv(\rho))}\Big]\Big\}+\\
&\quad -\frac{1}{t^{5/2}}\frac{\sqrt{\pi}}{4}\sum_{k\geq 1}\frac{1}{k^{7/2}}\sum_{\alpha=h}^{l}\Big(\frac{\rho}{f(\theta_\alpha)}\Big)^k\frac{e^{ik(\theta_\alpha-\phi+tv(f(\theta_\alpha)))}}{(|(v'(f(\theta_\alpha))^5 f''(\theta_\alpha)|)^{1/2}}\times\\
&\quad \times\Big[k-1-\frac{v''(f(\theta_\alpha))f(\theta_\alpha)}{v'(f(\theta_\alpha))}\Big)\Big]+O\Big(\frac{1}{t^3}\Big)+\text{c.c.t.},
\end{aligned}$$

where we set, by definition, $A(\rho^2) = \frac{1}{2}\int_{I_j} f^2(\theta_j)(\ln f^2(\theta_j) - 1) + \rho^2)d\theta$.

Analogously we can study $\int_{I_{j+1}} A(\theta, x, t)d\theta$, where A is given by (A.4) and now $f(\theta) < \rho$ (that is $\frac{r}{\rho} < 1$).

Assuming, as before, that $\theta_{l+1}, ..., \theta_m$ are critical points in the interior of I_{j+1}, it is

$$\int_{I_{j+1}} A(\theta,y,t)d\theta = \int_{i_{j+1}}^{i_{j+2}} \frac{1}{2}\ln\rho^2 f^2(\theta)d\theta + \\ -\frac{1}{t}\sum_{k\geq 1}\frac{1}{2k^2\rho^k i}\mathcal{A}_k(I_{j+1}) - \frac{1}{t^2}\sum_{k\geq 1}\frac{1}{2k^2\rho^k}[\mathcal{B}_k(I_{j+1}) + \\ -\mathcal{C}_k(I_{j+1})] + O\Big(\frac{1}{t^3}\Big) + \text{c.c.t.},$$

where, as before,

$$\mathcal{A}_k(I_{j+1}) = \int_{I_{j+1}} \frac{(f(\theta))^{k+1}}{v'(f(\theta))} e^{ik(\theta-\phi+tv(f(\theta)))}d\theta\,,$$

$$\mathcal{B}_k(I_{j+1}) = \frac{k+1}{k}\int_{i_{j+1}}^{i_{j+2}} \frac{e^{ik(\theta-\phi+tv(f(\theta))}(f(\theta))^k}{(v'(f(\theta)))^2}d\theta\,,$$

and

$$\mathcal{C}_k(I_{j+1}) = \frac{1}{k}\int_{i_{j+1}}^{i_{j+2}} \frac{e^{ik(\theta-\phi+tv(f(\theta))}(f(\theta))^{k+1}v''(f(\theta))}{(v'(f(\theta)))^3}d\theta\,.$$

Applying lemma 5.1 again, one has

$$\int_{I_{j+1}} A(\theta,y,t)d\theta = B(\rho^2) + \\ -\frac{1}{t^{3/2}}\frac{\sqrt{\pi}}{4}\sum_{k\geq 1}\frac{1}{k^{5/2}i}\Big[\sum_{\alpha=l+1}^{m}\Big(\frac{f(\theta_\alpha)}{\rho}\Big)^k f(\theta_\alpha)\frac{e^{ik(\theta_\alpha-\phi+tv(f(\theta_\alpha)))}}{(|(v'(f(\theta_\alpha))^3 f''(\theta_\alpha)|)^{1/2}}\Big] + \\ -\frac{1}{t^2}\sum_{k\geq 1}\frac{\rho}{2k^3}\Big[\frac{e^{ik(i_{j+2}-\phi+tv(\rho))}}{(v'(\rho))^2 f'(i_{j+2})} - \frac{e^{ik(i_{j+1}-\phi+tv(\rho))}}{(v'(\rho))^2 f'(i_{j+1})}\Big] + \\ -\frac{1}{t^{5/2}}\frac{\sqrt{\pi}}{4}\sum_{k\geq 1}\frac{1}{k^{7/2}}\Big[\sum_{\alpha=l+1}^{m}\Big(\frac{f(\theta_\alpha)}{\rho}\Big)^k\frac{e^{ik(\theta_\alpha-\phi+tv(f(\theta_\alpha)))}}{(|(v'(f(\theta_\alpha))^5 f''(\theta_\alpha)|)^{1/2}}\times \\ \times\Big(-\frac{v''(\theta_\alpha)(f(\theta_\alpha))}{v'(f(\theta_\alpha))} + (k+1)\Big)\Big] + O\Big(\frac{1}{t^3}\Big) + \text{c.c.t.},$$

where we set, by definition, $B(\rho^2) = \frac{\ln\rho^2}{2}\int_{I_{j+1}} f^2(\theta)d\theta$.

Then, it follows that

$$\begin{aligned}
&\int_{I_j\cup I_{j+1}} A(\theta,y,t)d\theta = \\
&= C(\rho^2) - \frac{1}{t^{3/2}}\frac{\sqrt{\pi}}{4}\sum_{k\geq 1}\frac{1}{k^{5/2}}\sum_{\alpha=h}^{m} R_\alpha^k c_\alpha \frac{e^{ik(\theta_\alpha-\phi+tv(f(\theta_\alpha))}}{i} + \\
&- \frac{1}{t^2}\sum_{k\geq 1}\frac{1}{k^3(v'(\rho))^2}\Big\{\frac{\rho}{2}\Big[\frac{e^{ik(i_{j+2}-\phi+tv(\rho))}}{f'(i_{j+2})} + \\
&- \frac{e^{ik(i_j-\phi+tv(\rho))}}{f'(i_j)}\Big] + \\
&+ \frac{1}{i}[e^{ik(i_{j+1}-\phi+tv(\rho))} - e^{ik(i_j-\phi+tv(\rho))}]\Big\} + \\
&- \frac{1}{t^{5/2}}\frac{\sqrt{\pi}}{4}\sum_{k\geq 1}\frac{1}{k^{7/2}}\sum_{\alpha=h}^{m} R_\alpha^k f(\theta_\alpha)\frac{e^{ik(\theta_\alpha-\phi+tv(f(\theta_\alpha))}}{(|(v'(f(\theta_\alpha))^7 f''(\theta_\alpha)|)^{1/2}} \times \\
&\times [-2f(\theta_\alpha)v''(f(\theta_\alpha)) + 2v'(f(\theta_\alpha))] + O\Big(\frac{1}{t^3}\Big) + \text{c.c.t.}
\end{aligned}$$

where $C(\rho^2) = A(\rho^2) + B(\rho^2)$ is a suitable function of ρ^2, the R_α's are the same as in formula (5.3), and the c_α's are suitable constants.

Taking into account the periodicity of f, and summing all over the intervals I_j, the theorem is proved.

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