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# Pseudo-symmetric spaces of constant type in dimension three — elliptic spaces

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RIASSUNTO: *Diamo una classificazione quasiesplicita delle varietà riemanniane 3-dimensionali cosiddette "ellittiche", cioè con autovalori di Ricci* $\rho_1 = \rho_2 \neq \rho_3$, $\rho_3 =$ costante $\neq 0$. (*Nel caso non ellittico il problema é stato risolto esplicitamente in* [14]). *Le classi locali di isometria delle metriche di tali varietà dipendono essenzialmente da tre funzioni arbitrarie di due variabili. Diamo anche un esempio di famiglie esplicite di metriche che dipendono da due funzioni arbitrarie di due variabili.*

ABSTRACT: *We give a quasiexplicit classification of three-dimensional Riemannian manifolds with Ricci eigenvalues* $\rho_1 = \rho_2 \neq \rho_3$, $\rho_3 =$ constant $\neq 0$*; which are called "elliptic".* (*In the nonelliptic cases the problem was solved explicitly in* [14]). *The local isometry classes of metrics of such manifolds depend on essentially three arbitrary functions of two variables. We also give an example of an explicit family of metrics depending on two arbitrary functions of two variables.*

## – Introduction

According to [5], a Riemannian manifold $(M, g)$ is said to be *pseudo-symmetric* if the following formula holds for arbitrary vector fields $X$ and

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$Y$ on $M$:

$$R(X,Y)\cdot R = F((X\wedge Y)\cdot R)\,, \tag{0.1}$$

where

a) $R$ denotes the Riemannian curvature tensor of type (1,3) on $(M,g)$ and
$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$
denote the corresponding curvature transformations,

b) $X\wedge Y$ denotes the endomorphism of the tangent bundle $TM$ defined by

$$(X\wedge Y)Z = g(Y,Z)X - g(X,Z)Y\,, \tag{0.2}$$

c) $F$ is a smooth function on $M$,

d) the dot in each side of the formula (0.1) denotes the derivation on the tensor algebra of $TM$ induced by an endomorphism of this tangent bundle.

We call a pseudo-symmetric space $(M,g)$ *of constant type* if $F = \tilde{c} =$ constant. According to [4] we have the following characterization in dimension three (see [14] for more details):

PROPOSITION 0.1. *A three-dimensional Riemannian manifold* $(M,g)$ *is pseudo-symmetric of constant type* $F=\tilde{c}$ *if and only if its principal Ricci curvatures* $\rho_1$, $\rho_2$ *and* $\rho_3$ *locally satisfy the following conditions* (*up to a numeration*):
(i) $\rho_3 = 2\tilde{c}$,
(ii) $\rho_1 = \rho_2$ *everywhere.*

We are not interested in the case when $(M,g)$ is a space of constant curvature and therefore we assume always $\rho_1 = \rho_2 \neq \rho_3$.

If $\tilde{c} = 0$, and hence $F = 0$, we obtain a definition of *semi-symmetric space*. The theory of semi-symmetric spaces has been developed in [17], [18], [19], [9], [6], [1], [2] and especially in the book [3]. For the three-dimensional case, see the explicit classification in [9], [6] and [3, Chapter 6].

For $\tilde{c} \neq 0$, the present authors made an explicit classification in [14] for the so-called "asymptotically foliated" (or "non-elliptic") spaces in dimension three. (See section 4 for the terminology.) The aim of this paper is to treat the more complicated "elliptic" spaces in the full generality.

Let us mention that the first author in [10] solved the special case when $\rho_1 = \rho_2$ is a constant(*), and the present authors treated in [13] a more general case-here $\rho_1 = \rho_2$ is supposed to be constant along each trajectory of the principal Ricci curvature $\rho_3$. The basic methods of [9], [10] and [13] are used also here but the corresponding calculations became more complicated. A computer check (the software "*Mathematica*" by Wolfram Research Inc.) was also used during this work.

## 1 – The basic system of partial differential equations for the problem

Let $(M, g)$ be a three-dimensional Riemannian manifold whose Ricci tensor $\hat{R}$ has eigenvalues $\rho_1 = \rho_2 \neq \rho_3$ with nonzero constant $\rho_3$. Choose a neighborhood $\tilde{U}$ of a fixed point $m \in M$ and a smooth vector field $E_3$ of unit eigenvectors corresponding to the Ricci eigenvalue $\rho_3$ in $\tilde{U}$. Let $S : D^2 \to \tilde{U}$ be a surface through $m$ which is transversal with respect to all trajectories generated by $E_3$ at all cross-points and not orthogonal to such a trajectory at $m$. (The vector field $E_3$ determines an orientation of $S$.) Then there is a normal neighborhood $U$ of $m$, $U \subset \tilde{U}$, with the property that each point $p \in U$ is projected to exactly one point $\pi(p) \in S$ via some trajectory. We fix any local coordinate system $(w, x)$ on $S$ and then a local coordinate system $(w, x, y)$ on $U$ such that the values $w(p)$ and $x(p)$ are defined as $w(\pi(p))$ and $x(\pi(p))$, respectively, for each point $p \in U$, $y(p)$ is the oriented length $d^+(\pi(p), p)$ of the trajectory joining $p$ with $\pi(p)$. Then $E_3 = \partial/\partial y$ can be extended in $U$ to an orthonormal moving frame $\{E_1, E_2, E_3\}$. Let $\{\omega^1, \omega^2, \omega^3\}$ be the corresponding dual coframe. Then the $\omega^i$ are of the form

$$(1.1) \qquad \omega^i = a^i dw + b^i dx, \quad i = 1, 2, \qquad \omega^3 = dy + H dw + G dx\,.$$

The Ricci tensor $\hat{R}$ expressed with respect to $\{E_1, E_2, E_3\}$ has the form $\hat{R}_{ij} = \rho_i \delta_{ij}$. Because each $\rho_i$ is expressed through the sectional curvature

(*) See [11], [12], [15], [16] and [3], Chapter 12, for the related topics.

$K_{ij}$ by the formula $\rho_i = \hat{R}_{ii} = \sum_{j\neq i} K_{ij}$, there exist a function $k = k(w,x,y)$ of the variables $w$, $x$ and $y$, and a constant $\tilde{c}$ such that

$$K_{12} = k, \quad K_{13} = K_{23} = \tilde{c}, \quad \rho_1 = \rho_2 = k + \tilde{c}, \quad \rho_3 = 2\tilde{c}\,. \tag{1.2}$$

Define now the components $\omega^i_j$ of the connection form by the standard formulas

$$d\omega^i - \sum_j \omega^j \wedge \omega^i_j = 0, \quad \omega^i_j + \omega^j_i = 0, \quad i,j = 1,2,3\,. \tag{1.3}$$

Because the Riemannian curvature tensor satisfies $R_{ijkl} = 0$ whenever at least three of the indices $i$, $j$, $k$ and $l$ are distinct, the formulas (1.2) are equivalent to

$$\begin{cases} d\omega^1_2 + \omega^1_3 \wedge \omega^3_2 = k\,\omega^1 \wedge \omega^2\,, \\ d\omega^1_3 + \omega^1_2 \wedge \omega^2_3 = \tilde{c}\,\omega^1 \wedge \omega^3\,, \\ d\omega^2_3 + \omega^2_1 \wedge \omega^1_3 = \tilde{c}\,\omega^2 \wedge \omega^3\,. \end{cases} \tag{1.4}$$

Next, differentiate the equations (1.4) and substitute from (1.4). We obtain easily

$$\omega^1_3 \wedge \omega^1 \wedge \omega^2 = 0, \quad \omega^2_3 \wedge \omega^1 \wedge \omega^2 = 0 \tag{1.5}$$

and

$$d((k - \tilde{c})\,\omega^1 \wedge \omega^2) = 0\,. \tag{1.6}$$

The relations (1.5) mean that $\omega^1_3$ and $\omega^2_3$ are linear combinations of $\omega^1$ and $\omega^2$ only, and from the third equation of (1.3) it follows that $d\omega^3$ is a multiple of $\omega^1 \wedge \omega^2$, i.e., a multiple of $dw \wedge dx$. Then (1.1) implies that the functions $G$ and $H$ are independent of $y$.

Now, there is a local coordinate system $(\bar{w}, \bar{x}, y)$ (possibly in a smaller neighborhood of $m$) such that $\bar{w} = \bar{w}(w,x)$ and $\bar{x} = \bar{x}(w,x)$ are functions of $w$ and $x$, and

$$\omega^1 = P^1 d\bar{w} + Q^1 d\bar{x}, \quad \omega^2 = P^2 d\bar{w} + Q^2 d\bar{x}, \quad \omega^3 = dy + \bar{H}(\bar{w},\bar{x})d\bar{w}\,. \tag{1.7}$$

Indeed, because the surface $S$ is not orthogonal to the vector field $E_3$ at $m$, the Pfaffian form $Hdw+Gdx$ from (1.1) is nonzero in a neighborhood of $m$ in $M$. Then we define $\bar{w}=\bar{w}(w,x)$ as a potential function of the Pfaffian equation $Hdw+Gdx=0$, and the second function $\bar{x}=\bar{x}(w,x)$ can be defined as an arbitrary smooth function which is functionally independent of $\bar{w}$. In addition, there are new Pfaffian forms $\tilde{\omega}^1$ and $\tilde{\omega}^2$ such that $(\tilde{\omega}^1)^2+(\tilde{\omega}^2)^2=(\omega^1)^2+(\omega^2)^2$ and $\tilde{\omega}^1$ does not involve the differential $d\bar{x}$. We can summarize:

Proposition 1.1. *In a normal neighborhood of any point $m\in M$ there exist an orthonormal coframe $\{\omega^1,\omega^2,\omega^3\}$ and a local coordinate system $(w,x,y)$ such that*

$$\omega^1=fdw,\quad \omega^2=Adx+Cdw,\quad \omega^3=dy+Hdw\,. \tag{1.8}$$

*Here $f$, $A$ and $C$ are smooth functions of the variables $w$, $x$ and $y$, $fA\neq 0$, and $H$ is a smooth function of the variables $w$ and $x$.*

The formula (1.6) can be now written in the form

$$((k-\tilde{c})fA)'_y=0,\quad \text{i.e.,}\quad k-\tilde{c}=\frac{\sigma}{fA} \tag{1.9}$$

for some function $\sigma=\sigma(w,x)\neq 0$.

Now, define the function $\chi=\chi(w,x,y)$ of the variables $w$, $x$ and $y$ by

$$\chi=\frac{1}{fA}=\frac{k-\tilde{c}}{\sigma}\,. \tag{1.10}$$

Then, using (1.8) and (1.10), we obtain easily the following expression for the components of the connection form:

$$\begin{cases} \omega_2^1=-A\alpha dx+Rdw+\beta dy\,,\\ \omega_3^1=A\beta dx+Sdw\\ \omega_3^2=A'_y dx+Tdw\,, \end{cases} \tag{1.11}$$

where

$$\text{(1.12)} \qquad \begin{cases} \alpha = \chi(A'_w - C'_x - HA'_y)\,, \\ \beta = \dfrac{\chi}{2}(H'_x + AC'_y - CA'_y) \end{cases}$$

and

$$\text{(1.13)} \qquad \begin{cases} R = \chi f f'_x - C\alpha + H\beta\,, \\ S = f'_y + C\beta\,, \\ T = C'_y - f\beta\,. \end{cases}$$

The curvature conditions (1.4) then give a system of nine partial differential equations for our problem:

$$\text{(A1)} \qquad (A\alpha)'_y + \beta'_x = 0\,,$$

$$\text{(A2)} \qquad R'_y - \beta'_w = 0\,,$$

$$\text{(A3)} \qquad (A\alpha)'_w + R'_x + SA'_y - A\beta T = -fAk\,,$$

$$\text{(B1)} \qquad A''_{yy} - A\beta^2 = -\tilde{c}A\,,$$

$$\text{(B2)} \qquad -A''_{yw} + T'_x + A(\beta R + \alpha S) = \tilde{c}AH\,,$$

$$\text{(B3)} \qquad T'_y - S\beta = -\tilde{c}C\,,$$

$$\text{(C1)} \qquad (A\beta)'_y + A'_y\beta = 0\,,$$

$$\text{(C2)} \qquad S'_x - (A\beta)'_w - (A\alpha T + A'_y R) = 0\,,$$

$$\text{(C3)} \qquad S'_y + T\beta = -\tilde{c}f\,.$$

## 2 – The first integrals and the reduction of the basic system of partial differential equations

The aim of this section is to replace the partial differential equations of the series (B) and (C) by a system of algebraic equations for the new functions depending only on the variables $w$ and $x$.

First of all, we can eliminate (B2) and (C2) by the same procedure as in[10]: the equation (B2) is a consequence of (A1) and (B1); the equation (C2) is a consequence of (A1), (A2) and (C1). Moreover, proposition 2.3 from [10] still holds (with a slight change of the notation). We have

Proposition 2.1. *The equations* (B3) *and* (C3) *are satisfied if and only if*

$$fT - CS = \varphi_0 \,, \tag{2.1}$$

*where* $\varphi_0 = \varphi_0(w, x)$ *is an arbitrary function of the variables* $w$ *and* $x$*. Moreover, we have, in the* **hyperbolic case** $\tilde{c} = -\lambda^2$,

$$S^2 + T^2 = \lambda[\varphi_1 \cosh(2\lambda y) + \varphi_2 \sinh(2\lambda y) - \varphi_3] \,, \tag{2.2h}$$

$$fS + CT = \varphi_2 \cosh(2\lambda y) + \varphi_1 \sinh(2\lambda y) \,, \tag{2.3h}$$

$$f^2 + C^2 = \frac{1}{\lambda}[\varphi_1 \cosh(2\lambda y) + \varphi_2 \sinh(2\lambda y) + \varphi_3] \,, \tag{2.4h}$$

*where the functions* $\varphi_i = \varphi_i(w, x)$, $i = 1, 2, 3$, *of the variables* $w$ *and* $x$ *satisfy the single relation*

$${\varphi_0}^2 + {\varphi_2}^2 - ({\varphi_1}^2 - {\varphi_3}^2) = 0 \tag{2.5h}$$

*and in the* **elliptic case** $\tilde{c} = \lambda^2$,

$$S^2 + T^2 = \lambda[\varphi_1 \cos(2\lambda y) - \varphi_2 \sin(2\lambda y) + \varphi_3] \,, \tag{2.2e}$$

$$fS + CT = \varphi_2 \cos(2\lambda y) + \varphi_1 \sin(2\lambda y) \,, \tag{2.3e}$$

$$f^2 + C^2 = \frac{1}{\lambda}[-\varphi_1 \cos(2\lambda y) + \varphi_2 \sin(2\lambda y) + \varphi_3] \,, \tag{2.4e}$$

*where the functions* $\varphi_i = \varphi_i(w, x)$, $i = 1, 2, 3$, *of the variables* $w$ *and* $x$ *satisfy the single relation*

$${\varphi_0}^2 + {\varphi_2}^2 + {\varphi_1}^2 - {\varphi_3}^2 = 0 \,. \tag{2.5e}$$

Proposition 2.2. *From the equations* (A1), (A2), (B1), (C1) *and* (C3), *we have, in the hyperbolic case,*

$$fA = f_1\cosh(2\lambda y) + f_2\sinh(2\lambda y) + f_3 \tag{2.6h}$$

*and, in the elliptic case,*

$$fA = f_1\cos(2\lambda y) + f_2\sin(2\lambda y) + f_3\,, \tag{2.6e}$$

*where* $f_i = f_i(w,x)$, $i = 1,2,3$, *are some functions of the variables* $w$ *and* $x$.

*There is a function* $\varphi_4 = \varphi_4(w,x)$ *of the variables* $w$ *and* $x$ *such that, in the hyperbolic case,*

$$SA = \lambda f_2\cosh(2\lambda y) + \lambda f_1\sinh(2\lambda y) + \varphi_4 \tag{2.7h}$$

*and, in the elliptic case,*

$$SA = \lambda f_2\cos(2\lambda y) - \lambda f_1\sin(2\lambda y) + \varphi_4\,. \tag{2.7e}$$

*Further, the equation* (A3) *is reduced to the equation*

$$(A\alpha)'_w + R'_x + \tau = 0\,, \tag{2.8}$$

*where*

$$\tau = (SA)'_y + fA\,\rho_1 \tag{2.9}$$

*is a function of the variables* $w$ *and* $x$.

Proof. From (C3) we obtain, using also (1.13),

$$\begin{aligned}(SA)'_y &= SA'_y - A\beta T - \tilde{c}fA =\\ &= f'_yA'_y + \beta(CA'_y - AC'_y) + f(A\beta^2 - \tilde{c}A)\,.\end{aligned} \tag{2.10}$$

Due to (B1) we obtain

$$\begin{aligned}(SA)'_y &= f'_yA'_y + A''_{yy}f + \beta(CA'_y - AC'_y) =\\ &= (A'_yf)'_y + \beta(CA'_y - AC'_y)\,.\end{aligned} \tag{2.11}$$

On the other hand, using (1.13) first and (C1) later, we get

$$(SA)'_y = [f'_y A + (A\beta)C]'_y = (f'_y A)'_y - \beta(CA'_y - AC'_y)\,. \tag{2.12}$$

As the sum of (2.11) and (2.12) we obtain

$$2(SA)'_y = (fA'_y)'_y + (f'_y A)'_y = (fA)''_{yy}\,. \tag{2.13}$$

Using (A1) and (A2), we obtain

$$[(A\alpha)'_w + R'_x]'_y = 0\,. \tag{2.14}$$

Due to (2.10), (1.10) and $\rho_1 = k + \tilde{c}$, the equation (A3) takes in the form

$$(A\alpha)'_w + R'_x + (SA)'_y + fA\,\rho_1 = 0\,. \tag{2.15}$$

According to (2.14), the function $\tau$ defined by (2.9) does not depend on $y$. This together with (2.15) implies (2.8). Also, the equations (2.13) and (2.9) imply

$$(fA)''_{yy} + 2fA\,\rho_1 = 2\tau\,. \tag{2.16}$$

Substituting (1.10) and $\rho_1 = k + \tilde{c}$ into (2.16), we obtain

$$\left(\frac{\sigma}{k - \tilde{c}}\right)''_{yy} + \frac{2(k + \tilde{c})\sigma}{k - \tilde{c}} - 2\tau = 0\,. \tag{2.17}$$

Because $\sigma$ does not depend on $y$, putting

$$F = \frac{1}{k - \tilde{c}} - \frac{\tau - \sigma}{2\tilde{c}\sigma}\,, \tag{2.18}$$

we obtain

$$F''_{yy} + 4\tilde{c}F = 0\,. \tag{2.19}$$

Moreover we get, from (2.18) and (1.10),

$$fA = F\sigma + f_3\,, \tag{2.20}$$

where $f_3 = f_3(w,x)$ is an arbitrary function of the variables $w$ and $x$.

The general solution of the partial differential equation (2.19) is, in the hyperbolic case,

$$F = F_1 \cosh(2\lambda y) + F_2 \sinh(2\lambda y) \tag{2.21h}$$

and, in the elliptic case,

$$F = F_1 \cos(2\lambda y) + F_2 \sin(2\lambda y)\,, \tag{2.21e}$$

where $F_1 = F_1(w,x)$ and $F_2 = F_2(w,x)$ are arbitrary functions of the variables $w$ and $x$. This together with (2.20) implies (2.6h) and (2.6e).

From (2.6he) and (2.13) we obtain (2.7he), respectively. □

Proposition 2.3. *The equations* (B1) *and* (C1) *are satisfied if and only if*

$$\beta A^2 = \lambda a_0\,, \tag{2.22}$$

*where* $a_0 = a_0(w,x)$ *is an arbitrary function and, moreover, we have*
**(a)** *in the hyperbolic case,*

$$A^2 = a_1 \cosh(2\lambda y) + a_2 \sinh(2\lambda y) + a_3\,, \tag{2.23h}$$

*where* $a_i = a_i(w,x)$, $i = 1,2,3$, *are functions of the variables* $w$ *and* $x$ *satisfying*

$${a_0}^2 + {a_2}^2 - ({a_1}^2 - {a_3}^2) = 0\,; \tag{2.24h}$$

**(b)** *in the elliptic case,*

$$A^2 = a_1 \cos(2\lambda y) + a_2 \sin(2\lambda y) + a_3\,, \tag{2.23e}$$

*where* $a_i = a_i(w,x)$, $i = 1,2,3$, *are functions of the variables* $w$ *and* $x$ *satisfying*

$${a_0}^2 + {a_2}^2 + {a_1}^2 - {a_3}^2 = 0\,. \tag{2.24e}$$

The *proof* is the same as for proposition 2.5 in [10] (with a slight change of the notation).

Proposition 2.4. *We have, in the hyperbolic case,*

$$\begin{aligned}2\lambda a_0 AC = {} & [a_1\varphi_5 + 2\lambda(a_2 f_3 - a_3 f_2)]\cosh(2\lambda y) + \\ & + [a_2\varphi_5 - 2\lambda(a_3 f_1 - a_1 f_3)]\sinh(2\lambda y) + \\ & + a_3\varphi_5 - 2\lambda(a_2 f_1 - a_1 f_2)\end{aligned} \tag{2.25h}$$

*and, in the elliptic case,*

$$\begin{aligned}2\lambda a_0 AC = {} & [a_1\varphi_5 + 2\lambda(a_2 f_3 - a_3 f_2)]\cos(2\lambda y) + \\ & + [a_2\varphi_5 + 2\lambda(a_3 f_1 - a_1 f_3)]\sin(2\lambda y) + \\ & + a_3\varphi_5 + 2\lambda(a_2 f_1 - a_1 f_2)\,,\end{aligned} \tag{2.25e}$$

*where* $\varphi_5 = \varphi_5(w,x)$ *is an arbitrary function of the variables* $w$ *and* $x$.

Proof. Subtracting equations (2.11) and (2.12), we get

$$(fA'_y - f'_y A)'_y + 2\beta(A'_y C - AC'_y) = 0\,,$$

that is,

$$(fA'_y - f'_y A)'_y = 2\beta A^2 \frac{AC'_y - A'_y C}{A^2}\,.$$

Using (2.22), we get

$$(fA'_y - f'_y A)'_y = 2\lambda a_0 \left(\frac{C}{A}\right)'_y\,. \tag{2.26}$$

Integrating (2.26) with respect to $y$ and multiplying by $A^3$, we get

$$2\lambda a_0 AC = \varphi_5 A^2 + (fA)(A^2)'_y - A^2(fA)'_y\,, \tag{2.27}$$

where $\varphi_5 = \varphi_5(w,x)$ is an arbitrary function of the variables $w$ and $x$. Substituting (2.6he) and (2.23he) into (2.27), we obtain our assertions, respectively. ∎

The following proposition is more explicit.

PROPOSITION 2.5. *We have, in the hyperbolic case,*

$$AC = b_1 \cosh(2\lambda y) + b_2 \sinh(2\lambda y) + b_3 \tag{2.28h}$$

*and, in the elliptic case,*

$$AC = b_1 \cos(2\lambda y) + b_2 \sin(2\lambda y) + b_3 \,, \tag{2.28e}$$

*where* $b_i = b_i(w, x)$, $i = 1, 2, 3$, *are functions of the variables* $w$ *and* $x$.

PROOF. For $a_0 \neq 0$, the assertion (2.28he) is a direct consequence of (2.25he), respectively.

Suppose now $\tilde{c} = \epsilon\lambda^2$, $\epsilon = \pm 1$, and $a_0 = 0$. Then $\beta = 0$ by (2.22) and we get from $(1.13)_3$ and (B3) that

$$C''_{yy} = -\tilde{c}\, C = -\epsilon\lambda^2 C \,.$$

Hence we get, in the hyperbolic case,

$$C = r \cosh(\lambda y) + s \sinh(\lambda y) \tag{2.29h}$$

and, in the elliptic case,

$$C = r \cos(\lambda y) + s \sin(\lambda y) \,, \tag{2.29e}$$

where $r = r(w, x)$ and $s = s(w, x)$ are arbitrary functions of the variables $w$ and $x$. On the other hand, (2.23he) and (2.24he) with $a_0 = 0$ imply, in the hyperbolic case,

$$A = p \cosh(\lambda y) + q \sinh(\lambda y) \tag{2.30h}$$

and, in the elliptic case,

$$A = p \cos(\lambda y) + q \sin(\lambda y) \tag{2.30e}$$

with some functions $p = p(w, x)$ and $q = q(w, x)$ of the variables $w$ and $x$. Hence (2.28he) follows. ∎

Remark. We denote $\operatorname{sgn}\tilde{c}$ by $\epsilon$ in the sequel. This notation will be used later to unify many formulas for the hyperbolic and the elliptic case.

Now we introduce the function $h = h(w,x)$ by

$$h = H'_x \,. \tag{2.31}$$

Proposition 2.6. *We have*

$$\begin{cases} ha_1 = 2\lambda[a_0 f_1 + a_2 b_3 - a_3 b_2]\,, \\ ha_2 = 2\lambda[a_0 f_2 + \epsilon(a_3 b_1 - a_1 b_3)]\,, \\ ha_3 = 2\lambda[a_0 f_3 - (a_1 b_2 - a_2 b_1)]\,. \end{cases} \tag{2.32}$$

Proof. From $(1.12)_2$ we get

$$h = 2fA\,\beta - (AC)'_y + 2A'_y C\,.$$

Then (2.22) and (1.10) imply

$$hA^2 = 2\lambda a_0 fA - A^2(AC)'_y + (AC)(A^2)'_y\,. \tag{2.33}$$

Now we use (2.6he), (2.23he) and (2.28he) to get (2.32he). ∎

From (2.21he), (1.10) and (2.1) we obtain

$$S = f\chi Q, \qquad T = C\chi Q + \varphi_0\chi A\,, \tag{2.34}$$

where, in the hyperbolic case,

$$Q = \lambda f_2 \cosh(2\lambda y) + \lambda f_1 \sinh(2\lambda y) + \varphi_4 \tag{2.35h}$$

and, in the elliptic case,

$$Q = \lambda f_2 \cos(2\lambda y) - \lambda f_1 \sin(2\lambda y) + \varphi_4\,. \tag{2.35e}$$

Substituting from (2.34) into the partial differential equation (C3), we obtain, using also (2.22),

$$\left(f\chi Q'_y - \frac{A'_y}{A^2}Q\right)A^2 + \lambda a_0 C\chi Q + \lambda a_0\varphi_0\chi A = -\tilde{c}fA^2\,.$$

Multiplying this equation by $A$ and using (2.27) and (1.10), we get

$$2fAQ'_y + \varphi_5 Q - Q(fA)'_y + 2\lambda a_0\varphi_0 + 2\tilde{c}(fA)^2 = 0\,. \tag{2.36}$$

Substituting from (2.6he) and (2.35he) into (2.36), we obtain

$$\begin{cases} f_1(\varphi_5 - 2\varphi_4) = 0, \quad f_2(\varphi_5 - 2\varphi_4) = 0\,, \\ \varphi_4\varphi_5 + 2\lambda a_0\varphi_0 - 2\lambda^2[{f_2}^2 + \epsilon({f_1}^2 - {f_3}^2)] = 0\,. \end{cases} \tag{2.37}$$

Substituting (2.35he) into (2.34), we obtain, in the hyperbolic case,

$$S = f\chi[\lambda f_2\cosh(2\lambda y) + \lambda f_1\sinh(2\lambda y) + \varphi_4]\,, \tag{2.38h}$$

$$T = C\chi[\lambda f_2\cosh(2\lambda y) + \lambda f_1\sinh(2\lambda y) + \varphi_4] + \varphi_0\chi A \tag{2.39h}$$

and, in the elliptic case,

$$S = f\chi[\lambda f_2\cos(2\lambda y) - \lambda f_1\sin(2\lambda y) + \varphi_4]\,, \tag{2.38e}$$

$$T = C\chi[\lambda f_2\cos(2\lambda y) - \lambda f_1\sin(2\lambda y) + \varphi_4] + \varphi_0\chi A\,. \tag{2.39e}$$

Hence we obtain, in the hyperbolic case,

$$\begin{aligned} &fA(CT + fS) = \\ &= \varphi_0 AC + [\lambda f_2\cosh(2\lambda y) + \lambda f_1\sinh(2\lambda y) + \varphi_4](f^2 + C^2) \end{aligned} \tag{2.40h}$$

and, in the elliptic case,

$$\begin{aligned} &fA(CT + fS) = \\ &= \varphi_0 AC + [\lambda f_2\cos(2\lambda y) - \lambda f_1\sin(2\lambda y) + \varphi_4](f^2 + C^2)\,. \end{aligned} \tag{2.40e}$$

Substituting (2.3he), (2.4he) and (2.6he) into (2.40he), we get in the hyperbolic case,

$$\begin{aligned} \varphi_0 AC = {} & (f_3\varphi_2 - f_2\varphi_3 - \frac{1}{\lambda}\varphi_1\varphi_4)\cosh(2\lambda y) + \\ & + (f_3\varphi_1 - f_1\varphi_3 - \frac{1}{\lambda}\varphi_2\varphi_4)\sinh(2\lambda y) + \\ & + f_1\varphi_2 - f_2\varphi_1 - \frac{1}{\lambda}\varphi_3\varphi_4 \end{aligned} \tag{2.41h}$$

and, in the elliptic case,

$$
\begin{aligned}
\varphi_0 AC = &(f_3\varphi_2 - f_2\varphi_3 + \frac{1}{\lambda}\varphi_1\varphi_4)\cos(2\lambda y)+ \\
&+ (f_3\varphi_1 + f_1\varphi_3 - \frac{1}{\lambda}\varphi_2\varphi_4)\sin(2\lambda y)+ \\
&+ f_1\varphi_2 + f_2\varphi_1 - \frac{1}{\lambda}\varphi_3\varphi_4\,.
\end{aligned}
\tag{2.41e}
$$

Another consequence of (2.38he) and (2.39he) is, in the hyperbolic case,

$$
\begin{aligned}
&(fA)^2(S^2+T^2) = [\lambda^2{f_2}^2\cosh^2(2\lambda y)+ \\
&\quad + \lambda^2{f_1}^2\sinh^2(2\lambda y) + 2\lambda^2 f_1 f_2\cosh(2\lambda y)\sinh(2\lambda y)+ \\
&\quad + 2\lambda f_2\varphi_4\cosh(2\lambda y) + 2\lambda f_1\varphi_4\sinh(2\lambda y) + {\varphi_4}^2](f^2+C^2)+ \\
&\quad + 2\varphi_0 AC[\lambda f_2\cosh(2\lambda y) + \lambda f_1\sinh(2\lambda y) + \varphi_4] + {\varphi_0}^2A^2
\end{aligned}
\tag{2.42h}
$$

and, in the elliptic case,

$$
\begin{aligned}
&(fA)^2(S^2+T^2) = [\lambda^2{f_2}^2\cos^2(2\lambda y)+ \\
&\quad + \lambda^2{f_1}^2\sin^2(2\lambda y) - 2\lambda^2 f_1 f_2\cos(2\lambda y)\sin(2\lambda y)+ \\
&\quad + 2\lambda f_2\varphi_4\cos(2\lambda y) - 2\lambda f_1\varphi_4\sin(2\lambda y) + {\varphi_4}^2](f^2+C^2)+ \\
&\quad + 2\varphi_0 AC[\lambda f_2\cos(2\lambda y) - \lambda f_1\sin(2\lambda y) + \varphi_4] + {\varphi_0}^2A^2\,.
\end{aligned}
\tag{2.42e}
$$

Using the formulas (2.2he), (2.4he), (2.6he), (2.23he) and (2.41he), we obtain from (2.42he)

$$
\begin{cases}
\lambda{\varphi_0}^2a_1 = \varphi_1[\lambda^2({f_1}^2 - \epsilon{f_2}^2 + {f_3}^2) - \epsilon{\varphi_4}^2]+ \\
\qquad\qquad +2\lambda^2 f_1(\epsilon f_3\varphi_3 - f_2\varphi_2) + 2\lambda\varphi_4(f_2\varphi_3 - f_3\varphi_2)\,, \\
\lambda{\varphi_0}^2a_2 = \epsilon\varphi_2[\lambda^2({f_1}^2 - \epsilon{f_2}^2 - {f_3}^2) + \epsilon{\varphi_4}^2]+ \\
\qquad\qquad +2\lambda^2 f_2(f_1\varphi_1 + \epsilon f_3\varphi_3) - 2\lambda\varphi_4(f_3\varphi_1 + \epsilon f_1\varphi_3)\,, \\
\lambda{\varphi_0}^2a_3 = \epsilon\varphi_3[\lambda^2({f_1}^2 + \epsilon{f_2}^2 + {f_3}^2) + \epsilon{\varphi_4}^2]+ \\
\qquad\qquad +2\lambda^2 f_3(f_1\varphi_1 - f_2\varphi_2) - 2\lambda\varphi_4(f_1\varphi_2 + \epsilon f_2\varphi_1)\,.
\end{cases}
\tag{2.43}
$$

Consider now the identity $(AC)^2 = A^2(f^2+C^2) - (Af)^2$. Substituting from (2.4he), (2.6he), (2.23he) and (2.28he), we get a system of quadratic equations

$$(2.44)\quad \begin{cases} \lambda({b_1}^2 - \epsilon {b_2}^2 + {f_1}^2 - \epsilon {f_2}^2) = -\epsilon(a_1\varphi_1 + a_2\varphi_2)\,, \\ \lambda({b_1}^2 + \epsilon {b_2}^2 + 2{b_3}^2 + {f_1}^2 + \epsilon {f_2}^2 + 2{f_3}^2) = -\epsilon(a_1\varphi_1 - a_2\varphi_2) + 2a_3\varphi_3\,, \\ 2\lambda(b_1 b_2 + f_1 f_2) = a_1\varphi_2 - \epsilon a_2\varphi_1\,, \\ 2\lambda(b_1 b_3 + f_1 f_3) = a_1\varphi_3 - \epsilon a_3\varphi_1\,, \\ 2\lambda(b_2 b_3 + f_2 f_3) = a_2\varphi_3 + a_3\varphi_2\,. \end{cases}$$

In the notation (2.28he) we can rewrite (2.25he) in the form

$$(2.45)\quad \begin{cases} 2\lambda a_0 b_1 = a_1\varphi_5 + 2\lambda(a_2 f_3 - a_3 f_2)\,, \\ 2\lambda a_0 b_2 = a_2\varphi_5 + 2\epsilon\lambda(a_3 f_1 - a_1 f_3)\,, \\ 2\lambda a_0 b_3 = a_3\varphi_5 - 2\lambda(a_1 f_2 - a_2 f_1)\,. \end{cases}$$

Also, we can rewrite (2.41he) in the form

$$(2.46)\quad \begin{cases} \lambda\varphi_0 b_1 = -\lambda(f_2\varphi_3 - f_3\varphi_2) + \epsilon\varphi_1\varphi_4\,, \\ \lambda\varphi_0 b_2 = \lambda(f_3\varphi_1 + \epsilon f_1\varphi_3) - \varphi_2\varphi_4\,, \\ \lambda\varphi_0 b_3 = \lambda(f_1\varphi_2 + \epsilon f_2\varphi_1) - \varphi_3\varphi_4\,. \end{cases}$$

Proposition 2.7. *If $a_0 \neq 0$, then we have*

$$(2.47)\quad h = -\frac{2\lambda[\epsilon(a_1 f_1 - a_3 f_3) + a_2 f_2]}{a_0}\,.$$

Proof. The assertion follows from (2.32), (2.45) and (2.24he). ∎

Now we have the main results of this section.

Theorem 2.8. *Let $\lambda$ be a nonzero constant. Let $\varphi_0, \varphi_1, \dots, \varphi_5, a_0, a_1, a_2, a_3, b_1, b_2, b_3, f_1, f_2, f_3$ and $h$ be functions of two variables $w$ and $x$ defined in some domain $V \subset \mathbb{R}^2(w,x)$, satisfying eight collections of algebraic equations* (2.5), (2.24), (2.32), $(2.37)_2$, (2.43), (2.44), (2.45) *and* (2.46) (*either of hyperbolic type, or of elliptic type*) *with the corresponding parameter $\lambda$, and such that ${a_1}^2 + {a_2}^2 + {a_3}^2 > 0$ in $V$.*

*Let $A$, $f$, $C$ and $H$ be functions defined in a domain $U \subset \mathbb{R}^3(w,x,y)$, where $A \neq 0$, by the formulas* (2.23), (2.6), (2.28) *and* (2.31) *of the corresponding type, and let the metric $g$ be defined on $U$ by* (1.8). *Further, let $\alpha$, $\beta$ and $R$ be defined as in* $(1.12)_1$, (2.22), $(1.13)_1$. *Then the curvature conditions* (1.4) *are satisfied for some function $k = k(w,x,y)$ of the variables $w$, $x$ and $y$, and for the corresponding constant $\tilde{c} = \pm\lambda^2$ if and only if the system of partial differential equations* (A1) *and* (A2) *is satisfied.*

Proof. The assertion follows from the whole series of propositions and formulas given in this section. The only point here is to show that, if we *do not prescribe* the function $k = k(w,x,y)$ in advance, then the equation (A3) (or, equivalently, (2.8)) does not give any additional condition. But, due to (2.9) and (1.2), the equation (2.8) can be considered just as a formula for calculating the Ricci eigenvalue $\rho_1$ or the scalar curvature $\mathrm{Sc}(g) = 2k + 4\tilde{c}$ of $(M,g)$. □

Remark. The algebraic conditions mentioned above are, of course, far from being independent, but they are all useful.

We conclude this section by proving additional algebraic equations between our basic functions.

Proposition 2.9. *We have*

$$\varphi_5 = 2\varphi_4\,, \tag{2.48}$$

$$\varphi_0 A^2 - \lambda a_0(f^2 + C^2) + \varphi_5 AC + hfA = 0\,. \tag{2.49}$$

Proof. If ${f_1}^2 + {f_2}^2 \neq 0$, then (2.48) follows from $(2.37)_{1,2}$. If $f_1 = f_2 = 0$, then we proceed as in the proof of proposition 4.1 in [10].

To derive (2.49), we rewrite (2.37) using (2.48) in the form

$$\lambda a_0 \varphi_0 = \lambda^2[{f_2}^2 + \epsilon({f_1}^2 - {f_3}^2)] - {\varphi_4}^2\,. \tag{2.50}$$

Suppose $a_0 \neq 0$. Then (2.45) and (2.48) imply

$$\left\{\begin{aligned} b_1 &= \frac{a_1\varphi_4 + \lambda(a_2 f_3 - a_3 f_2)}{\lambda a_0}\,, \\ b_2 &= \frac{a_2\varphi_4 + \epsilon\lambda(a_3 f_1 - a_1 f_3)}{\lambda a_0}\,, \\ b_3 &= \frac{a_3\varphi_4 - \lambda(a_1 f_2 - a_2 f_1)}{\lambda a_0}\,. \end{aligned}\right. \tag{2.51}$$

Now we substitute for $A^2$, $f^2+C^2$, $AC$, $\varphi_5$, $h$ and $fA$ of the left hand side of (2.49) from (2.23he), (2.4he), (2.28he), (2.48) and (2.6he), respectively. Then the identity (2.49) follows. If $a_0 = 0$, we use the direct check as in [10]. ∎

Proposition 2.10. *The following algebraic formulas hold*

$$2\lambda(a_1 f_1 + \epsilon a_2 f_2 - a_3 f_3) = -\,\epsilon a_0 h\,, \tag{2.52}$$

$$4\lambda^2(b_1 f_1 + \epsilon b_2 f_2 - b_3 f_3) = -\,\epsilon\varphi_5 h\,, \tag{2.53}$$

$$2\lambda(\varphi_1 f_1 - \varphi_2 f_2 - \varphi_3 f_3) = \epsilon\varphi_0 h\,, \tag{2.54}$$

$$2\lambda(a_1 b_1 + \epsilon a_2 b_2 - a_3 b_3) = -\,\epsilon a_0\varphi_5\,. \tag{2.55}$$

Proof. From (2.24he) and (2.32) we obtain

$$2\lambda a_0(a_1 f_1 + \epsilon a_2 f_2 - a_3 f_3) = -\epsilon {a_0}^2 h\,.$$

Hence we obtain (2.52) if $a_0 \neq 0$. From (2.45) and (2.24he) we obtain

$$2\lambda a_0(b_1 f_1 + \epsilon b_2 f_2 - b_3 f_3) = \varphi_5(a_1 f_1 + \epsilon a_2 f_2 - a_3 f_3)\,,$$

which together with (2.52) implies (2.53) when $a_0 \neq 0$. From (2.46) we obtain

$$\varphi_4(\varphi_1 f_1 - \varphi_2 f_2 - \varphi_3 f_3) = -\lambda\varphi_0(b_1 f_1 + b_2 f_2 - b_3 f_3)\,,$$

hence, if $a_0\varphi_4 \neq 0$, we obtain (2.54) using (2.53) and (2.48). Finally from (2.45) we obtain

$$2\lambda a_0(a_1b_1 + \epsilon a_2b_2 - a_3b_3) = -\epsilon {a_0}^2\varphi_5\,.$$

Thus we obtain (2.55) when $a_0\varphi_4 \neq 0$.

For $a_0\varphi_4 = 0$ we use the continuity argument or a rather lengthy direct check (cf. proposition 4.10 in [9]). ∎

## 3 – The Riemannian invariants

Let $(M, g)$ be given locally as in proposition 1.1. We rewrite the formulas (1.11) using the forms $\omega^1$, $\omega^2$ and $\omega^3$ as a basis. It follows

$$(3.1)\qquad \begin{cases} \omega_2^1 = \chi f'_x\,\omega^1 - \alpha\,\omega^2 + \beta\,\omega^3\,, \\ \omega_3^1 = \frac{f'_y}{f}\,\omega^1 + \beta\,\omega^2\,, \\ \omega_3^2 = (\beta - h\chi)\,\omega^1 + \frac{A'_y}{A}\,\omega^2, \qquad h = H'_x\,. \end{cases}$$

We also write, for brevity,

$$(3.2)\qquad \omega_3^1 = a\,\omega^1 + b\,\omega^2, \quad \omega_3^2 = c\,\omega^1 + e\,\omega^2\,,$$

where

$$(3.3)\qquad a = \frac{f'_y}{f}, \quad b = \beta, \quad c = \beta - h\chi, \quad e = \frac{A'_y}{A}\,.$$

Using the standard formula $\nabla_{E_j}E_i = \sum_k \omega_i^k(E_j)E_k$, $i, j = 1, 2, 3$, from [7], we obtain

$$(3.4)\qquad \begin{cases} \nabla_{E_1}E_1 = -\chi f'_x E_2 - aE_3, & \nabla_{E_1}E_2 = \chi f'_x E_1 - cE_3\,, \\ \nabla_{E_2}E_1 = \alpha E_2 - bE_3, & \nabla_{E_2}E_2 = -\alpha E_1 - eE_3\,, \\ \nabla_{E_1}E_3 = aE_1 + cE_2, & \nabla_{E_2}E_3 = bE_1 + eE_2\,, \\ \nabla_{E_3}E_1 = -bE_2, & \nabla_{E_3}E_2 = bE_1\,, \quad \nabla_{E_3}E_3 = 0\,. \end{cases}$$

The last formula shows that the trajectories of the unit vector field $E_3$ (consisting of the eigenvectors of the Ricci tensor $\hat{R}$ corresponding to $\rho_3 = 2\tilde{c}$) are geodesics.

For the Ricci tensor $\hat{R}$ we get, using the notation (1.2) and the adapted local orthonormal coframe $\{\omega^1, \omega^2, \omega^3\}$,

$$\hat{R} = (k+\tilde{c})(\omega^1\otimes\omega^1+\omega^2\otimes\omega^2)+2\tilde{c}\,(\omega^3\otimes\omega^3)\,. \tag{3.5}$$

Using (3.1), (3.2) and the standard formula $\nabla_X\omega^i = -\sum_j \omega^i_j(X)\omega^j$, we obtain

$$\begin{aligned}\nabla\hat{R} =& dk\otimes(\omega^1\otimes\omega^1+\omega^2\otimes\omega^2)+\\ &+(\tilde{c}-k)\{(a\,\omega^1+b\,\omega^2)\otimes(\omega^1\otimes\omega^3+\omega^3\otimes\omega^1)+\\ &+(c\,\omega^1+e\,\omega^2)\otimes(\omega^2\otimes\omega^3+\omega^3\otimes\omega^2)\}\,,\end{aligned} \tag{3.6}$$

where $a$, $b$, $c$ and $e$ are given by (3.3). Hence we also get

$$\begin{aligned}||\nabla\hat{R}||^2 =& 2||dk||^2+2(\tilde{c}-k)^2(a^2+b^2+c^2+e^2)=\\ =& 2||d\rho_1||^2+2(\rho_1-\rho_3)^2(a^2+b^2+c^2+e^2)\,.\end{aligned} \tag{3.7}$$

Because $\hat{R}$ is a Riemannian invariant tensor, $\nabla\hat{R}$ is an invariant tensor. Also, because $E_3 = \partial/\partial y$ is uniquely determined by the geometry of $(M,g)$ up to sign, $\omega^3\otimes\omega^3$ is an invariant tensor. Hence we see from (3.5) and (3.6) that the tensor

$$\begin{aligned}Q =& (a\,\omega^1+b\,\omega^2)\otimes(\omega^1\otimes\omega^3+\omega^3\otimes\omega^1)+\\ &+(c\,\omega^1+e\,\omega^2)\otimes(\omega^2\otimes\omega^3+\omega^3\otimes^2)\end{aligned} \tag{3.8}$$

is also invariant. Now because $E_1$ and $E_2$ are determined up to an orthogonal transformation (with functional coefficients), the functions

$$\begin{cases} Q(E_1,E_1,E_3)+Q(E_2,E_2,E_3) = a+e\,,\\ Q(E_2,E_1,E_3)-Q(E_1,E_2,E_3) = b-c\end{cases} \tag{3.9}$$

are Riemannian invariants up to sign.

The square of the norm $||Q||^2 = 2(a^2+b^2+c^2+e^2)$ is a Riemannian invariant and hence (equivalently) $ae-bc$ is a Riemannian invariant. We summarize:

PROPOSITION 3.1. *The function $ae-bc$ is a Riemannian invariant, and $a+e$ and $b-c$ are Riemannian invariants up to sign* (i.e., *depending on the orientation of the principal geodesics*). *Further, the partial derivative of any Riemannian invariant with respect to $y$ is a Riemannian invariant up to sign.*

Using (1.10), we get

$$\begin{cases} a+e = (\ln(fA))'_y = -(\ln(k-\epsilon\lambda^2))'_y\,, \\ b-c = h\chi = \dfrac{h(k-\epsilon\lambda^2)}{\sigma}\,. \end{cases} \tag{3.10}$$

Further we have

$$ae-bc = \epsilon(2\lambda^2 f_3\chi - \lambda^2)\,. \tag{3.11}$$

The last formula is obtained by lengthy calculations using (2.52) and the obvious identities

$$(AA'_y)^2 + \lambda^2 {a_0}^2 = -\,\epsilon\lambda^2[(A^2-a_3)^2 - a_3^2]\,, \tag{3.12}$$

$$A^3 f'_y = (fA)'_y A^2 - (fA)(AA'_y)\,. \tag{3.13}$$

Using (3.11) we see that, in the hyperbolic case,

$$\frac{fA}{f_3} = \frac{f_1\cosh(2\lambda y) + f_2\sinh(2\lambda y) + f_3}{f_3} \tag{3.14h}$$

is a Riemannian invariant and, in the elliptic case,

$$\frac{fA}{f_3} = \frac{f_1\cos(2\lambda y) + f_2\sin(2\lambda y) + f_3}{f_3} \tag{3.14e}$$

is a Riemannian invariant (assuming $f_3 \neq 0$ everywhere). (According to $(3.10)_2$, $fA/h$ and $f_3/h$ are then Riemannian invariants *up to sign* assuming $h \neq 0$ everywhere.)

Next, we give some simple results concerning isometry of Riemannian manifolds with the Ricci eigenvalues $\rho_1 = \rho_2$ and nonzero constant $\rho_3$ to be used later. Let $(M, g)$ be such a manifold with the metric $g$ given by (1.8) and let $(\bar{M}, \bar{g})$ be another such manifold with the metric $\bar{g}$ given by the orthonormal coframe

$$\bar{\omega}^1 = \bar{f} d\bar{w}, \quad \bar{\omega}^2 = \bar{A} d\bar{x} + \bar{C} d\bar{w}, \quad \bar{\omega}^3 = d\bar{y} + \bar{H} d\bar{w} \,. \tag{3.15}$$

Suppose that there is an isometry $\Phi : (M, g) \longrightarrow (\bar{M}, \bar{g})$ given by

$$\bar{w} = \bar{w}(w, x, y), \quad \bar{x} = \bar{x}(w, x, y), \quad \bar{y} = \bar{y}(w, x, y) \,. \tag{3.16}$$

Here we identify $\bar{w}$, $\bar{x}$ and $\bar{y}$ with $\bar{w} \circ \Phi$, $\bar{x} \circ \Phi$ and $\bar{y} \circ \Phi$, respectively.

Propositions 5.2 and 5.3 from [9] still hold without change. We have:

PROPOSITION 3.2. *The equation* (3.16) *can be reduced to the form*

$$\bar{w} = \bar{w}(w, x), \quad \bar{x} = \bar{x}(w, x), \quad \bar{y} = \varepsilon y + \phi(w, x), \quad \varepsilon = \pm 1 \,, \tag{3.17}$$

*where $\phi = \phi(w, x)$ is an arbitrary function of the variables $w$ and $x$.*

## 4 – The asymptotic foliations and four types of spaces

Recall that the principal geodesics are trajectories of the vector field $E_3$. We introduce two basic definitions.

DEFINITION 4.1. A smooth surface $N \subset (M, g)$ is called an *asymptotic leaf* if it is generated by the principal geodesics and its tangent planes are parallel along these principal geodesics with respect to the Riemannian connection $\nabla$ of $(M, g)$.

DEFINITION 4.2. An *asymptotic distribution* on $M$ is a two-dimensional distribution which is integrable and whose integral manifolds are asymptotic leaves. The integral manifolds of an asymptotic distribution determine a foliation of $M$, which is called an *asymptotic foliation*.

Let $N$ be an asymptotic leaf. Then we see as in [10] that the tangent distribution of $N$ satisfies the quadratic equation

$$c(\omega^1)^2 + (e - a)\omega^1\omega^2 - b(\omega^2)^2 = 0 \,, \tag{4.1}$$

where $a$, $b$, $c$ and $e$ are given by (3.3). Of course, an asymptotic distribution must satisfy these equations locally on the whole of $M$. Conversely, *any smooth distribution satisfying* (4.1) *is an asymptotic distribution.*

The following proposition is almost obvious.

PROPOSITION 4.3. *Let* $\Delta = (e-a)^2 + 4bc$ *be the discriminant of the quadratic equation* (4.1). *Then we have:*

(E) *If* $\Delta < 0$ *on* $(M, g)$*, then there is no real asymptotic distribution on* $M$.

(H) *If* $\Delta > 0$ *on* $(M, g)$*, then there are exactly two different asymptotic distributions on* $M$.

(P) *If* $\Delta = 0$ *on* $(M, g)$ *and some of the functions* $a_0$*,* $\varphi_0$ *and* $\varphi_5$ *are nonzero at each point, then there is a unique asymptotic distribution on* $M$.

(P$\ell$) *If* $a_0 = \varphi_0 = \varphi_5 = 0$ *on* $M$*, then any* $\pi$*-projectable smooth two-dimensional distribution on* $M$ *is asymptotic.*

(For the last part we only have to prove $b = c = e - a = 0$, which follows from the formulas (2.22), (2.27) and (2.49).)

DEFINITION 4.4. A space $(M, g)$ is said to be of *subtype* (E), (H), (P) or (P$\ell$), respectively, if the corresponding case of proposition 4.3 holds on the whole of $M$.

Let us remark that the above symbols are abbreviations for "elliptic", "hyperbolic", "parabolic" and "planar" (cf. [9]). Yet, the reader should keep in mind that, e.g., "elliptic subtype" from proposition 4.3 and "elliptic case" from proposition 2.1 are completely different geometric notions, which can be combined.

We add some more details:

PROPOSITION 4.5. *The equation* (4.1) *is equivalent to the equation*

$$\lambda a_0 dx^2 + \varphi_5 dx dw - \varphi_0 dw^2 = 0\,. \tag{4.2}$$

PROOF. We can apply the same procedure as in [9] (proof of theorem 6.5). Here we use formulas (2.49) and (2.27) for this purpose. ∎

Hence we can decide about the subtype of the space $(M, g)$ according to the following

Proposition 4.6. *Let* $\Delta' = {\varphi_5}^2 + 4\lambda a_0 \varphi_0$ *be the discriminant of the quadratic equation* (4.2). *Then the analogue of proposition* 4.3 *holds if* $\Delta$ *is replaced by* $\Delta'$.

Proof. One can show easily that $\Delta' = (fA)^2\Delta$. ∎

Also, notice that $\Delta'$ is given alternatively by the formula

$$\Delta' = 4\lambda^2[{f_2}^2 + \epsilon({f_1}^2 - {f_3}^2)]\,. \tag{4.3}$$

Indeed, combining $(2.37)_3$ with (2.48), we obtain at once

$${\varphi_5}^2 + 4\lambda a_0\varphi_0 - 4\lambda^2[{f_2}^2 + \epsilon({f_1}^2 - {f_3}^2)] = 0\,. \tag{4.4}$$

Here we stop our consideration about *asymptotically foliated* spaces, i.e., those of the non-elliptic subtype. These spaces have been treated in [3, Chapter 11 (see [14]). The present authors showed that, for each asymptotically foliated space, there is an adapted system of local coordinates such that $a_0 = 0$ identically. Then the expression for $A$, $C$ and $f$ are "linearized" (cf. (2.29he) and (2.30he) above) and the solution of the problem is dramatically simplified. In [14] we have shown that (generically), for each subtype (H), (P) and (P$\ell$), the general solution can be expressed in a *closed form*, i.e., in the form involving only arbitrary functions, algebraic operations, elementary functions, differentiations and integrations.

In the rest of this paper we concentrate ourselves only on the more complicated elliptic subtype (E).

## 5 – The quasiexplicit classification of spaces of elliptic subtype

The spaces of elliptic subtype are much more difficult to deal with because the coefficients $A$, $f$ and $C$ in (1.8) cannot be expressed in general in the form of linear combinations of $\cosh(\lambda y)$ and $\sinh(\lambda y)$; or of $\cos(\lambda y)$ and $\sin(\lambda y)$. We are not able to solve the classification problem explicitly, but we can still prove the local isometry classes of metrics depend on essentially three arbitrary functions of two variables. Also, we give an example of an explicit family of metrics depending on two arbitrary functions of two variables.

We see first that the functions $a_0$ and $\varphi_0$ are always non-zero on a space of subtype (E) (cf. (4.2)). Also, we must have $h \neq 0$. (If $h = 0$, then $b = c$ in (3.3) and hence $\Delta \geq 0$ in proposition 4.3.) From (2.24he) and (2.5he) we see that

$$\epsilon({a_1}^2 - {a_3}^2) + {a_2}^2 < 0, \quad \epsilon({\varphi_1}^2 - {\varphi_3}^2) + {\varphi_2}^2 < 0\,, \tag{5.1}$$

and, from (4.3), we have

$$\epsilon({f_1}^2 - {f_3}^2) + {f_2}^2 < 0\,. \tag{5.2}$$

We start with the following simplification:

Proposition 5.1. *Every metric $g$ of subtype* (E) *can be expressed locally, using the convenient coordinates and convenient coframe, in the form* (1.8), *where $f_2 = 0$, $a_2 \neq 0$ and $b_2 = 0$.*

The *proof* is a modification of that of proposition 8.1 from [9] using the fact that $fA/f_3$ is a Riemannian invariant ( cf. (3.14he)). Notice that the two cases in proposition 8.1 from [9] are reduced to one case only. In fact, if $f_2 = a_2 = 0$, then, making substitution $\bar{y} = y + 1$, we have $\bar{a}_2 \neq 0$.

Now we study the "fine structure" of the partial differential equations

$$(A\alpha)'_y + \beta'_x = 0 \tag{A1}$$

and

$$R'_y - \beta'_w = 0 \tag{A2}$$

with $\beta \neq 0$, that is, we shall write (A1) and (A2) as a system of partial differential equations for functions of two variables only.

We substitute into (A1) the function $A\alpha$ in the form

$$A\alpha = \frac{1}{2fA}\left[(A^2)'_w - 2(AC)'_x + \frac{AC}{A^2}(A^2)'_x - H(A^2)'_y\right],$$

which follows from (1.12) and (1.10), and the function $\beta$ in the form $\beta = \lambda a_0/A^2$ (see (2.22)). Taking the common denominator $2A^4(fA)^2$ and using (2.6he), (2.23he) and (2.28he), respectively, we obtain the nominator of the left-hand side of the equation (A1) as a linear combination of $c^3$, $c^2s$, $c^2$, $cs$, $c$, $s$ and 1, where $c = \cosh(2\lambda y)$ and $s = \sinh(2\lambda y)$ in the hyperbolic case; $c = \cos(2\lambda y)$ and $s = \sin(2\lambda y)$ in the elliptic case. Each coefficient of this linear combination depends on $w$ and $x$ only, and thus it must vanish if (A1) is satisfied. This gives seven partial differential equations which are linear with respect to $a'_{0x}$, $a'_{1x}$, $a'_{2x}$, $a'_{3x}$, $V_1$, $V_2$ and $V_3$, where

$$(5.3)\qquad \begin{cases} V_1 = a'_{1w} - 2b'_{1x} - 2\lambda H a_2\,, \\ V_2 = a'_{2w} - 2b'_{2x} + \epsilon 2\lambda H a_1\,, \\ V_3 = a'_{3w} - 2b'_{3x}\,. \end{cases}$$

Using the formula (2.24he) in the form

$$(5.4)\qquad {a_0}^2 = -\epsilon({a_1}^2 - {a_3}^2) - {a_2}^2$$

and its derivative

$$(5.5)\qquad a_0 a'_{0x} = -\epsilon(a_1 a'_{1x} - a_3 a'_{3x}) - a_2 a'_{2x}\,,$$

we can eliminate the derivative $a'_{0x}$ in all equations. We obtain the final form of the equation (A1) as the system of partial differential equations

$$(5.6)\qquad \sum_{i=1}^{3} a_0 P^i_\alpha V_i + \sum_{i=1}^{3} Q^i_\alpha a'_{ix} = 0, \quad \alpha = 1, 2, \ldots, 7\,,$$

where

$$\begin{aligned}
&P^1_1 = 2a_1a_2f_3, && P^2_1 = ({a_1}^2 - \epsilon {a_2}^2)f_3, && P^3_1 = -2a_1a_2f_1,\\
&P^1_2 = ({a_1}^2 - \epsilon {a_2}^2)f_3, && P^2_2 = -2\epsilon a_1a_2f_3, && P^3_2 = -({a_1}^2 - \epsilon {a_2}^2)f_1,\\
&P^1_3 = 2a_2a_3f_3, && P^2_3 = ({a_1}^2 - \epsilon {a_2}^2)f_1 + 2a_1a_3f_3, && P^3_3 = -2a_2a_3f_1,\\
&P^1_4 = a_1a_3f_3, && P^2_4 = -\epsilon a_2(a_1f_1 + a_3f_3), && P^3_4 = -a_1a_3f_1,\\
&P^1_5 = 2a_1a_2f_3, && P^2_5 = -2a_1a_3f_1 - (\epsilon {a_2}^2 + {a_3}^2)f_3, && P^3_5 = -2a_1a_2f_1,\\
&P^1_6 = ({a_2}^2 + \epsilon {a_3}^2)f_3, && P^2_6 = -2a_2a_3f_1, && P^3_6 = -({a_2}^2 + \epsilon {a_3}^2)f_1,\\
&P^1_7 = 2a_2a_3f_3, && P^2_3 = -(\epsilon {a_2}^2 + {a_3}^2)f_1, && P^3_7 = -2a_2a_3f_1,
\end{aligned}$$

$$\begin{aligned}
Q^1_1 &= -\,a_0a_2b_3f_1 + a_0a_2b_1f_3 + ({a_2}^2 - \epsilon {a_3}^2){f_1}^2\,,\\
Q^1_2 &= -\,a_0(a_1b_3 - a_3b_1)f_1 + a_0a_1b_1f_3 + a_1a_2{f_1}^2\,,\\
Q^1_3 &= -\,a_0a_2b_1f_1 - \epsilon a_1a_3{f_1}^2 + 2({a_2}^2 - \epsilon {a_3}^2)f_1f_3\,,\\
Q^1_4 &= -\,a_0a_3b_1f_3 + a_1a_2f_1f_3\,,\\
Q^1_5 &= \;\; 2a_0a_2b_1f_3 - ({a_2}^2 - \epsilon {a_3}^2){f_3}^2 + 2\epsilon a_1a_3f_1f_3\,,\\
Q^1_6 &= \;\; \epsilon a_0a_3b_3f_3 + \epsilon a_1a_2{f_3}^2\,,\\
Q^1_7 &= \;\; a_0a_2b_3f_3 + \epsilon a_1a_3{f_3}^2\,,\\
Q^2_1 &= -\,a_0(a_1b_3 - a_3b_1)f_1 + a_0a_1b_1f_3 - a_1a_2{f_1}^2\,,\\
Q^2_2 &= \;\; \epsilon a_0a_2b_3f_1 - \epsilon a_0a_2b_1f_3 - ({a_1}^2 - {a_3}^2){f_1}^2\,,\\
Q^2_3 &= \;\; a_0a_1b_1f_1 + 2a_0a_3b_1f_3 - a_2a_3{f_1}^2 - 2a_1a_2f_1f_3\,,\\
Q^2_4 &= -\,({a_1}^2 - {a_3}^2)f_1f_3\,,\\
Q^2_5 &= -\,2a_0a_1b_3f_1 - a_0a_3b_3f_3 + a_1a_2{f_3}^2 + 2\epsilon a_2a_3f_1f_3\,,\\
Q^2_6 &= -\,a_0a_2b_3f_1 + a_0a_2b_1f_3 - ({a_1}^2 - {a_3}^2){f_3}^2\,,\\
Q^2_7 &= -\,a_0a_3b_3f_1 - a_0(a_1b_3 - a_3b_1)f_3 + \epsilon a_2a_3{f_3}^2\,,\\
Q^3_1 &= -\,a_0a_2b_1f_1 + \epsilon a_1a_3{f_1}^2\,,\\
Q^3_2 &= -\,a_0a_1b_1f_1 - a_2a_3{f_1}^2\,,\\
Q^3_3 &= -\,2a_0a_2b_3f_1 + (\epsilon {a_1}^2 - {a_3}^2){f_1}^2 + 2\epsilon a_1a_3f_1f_3\,,\\
Q^3_4 &= -\,a_0a_1b_3f_1 - a_2a_3f_1f_3\,,\\
Q^3_5 &= \;\; a_0a_2b_3f_3 - \epsilon a_1a_3{f_3}^2 + 2(\epsilon {a_1}^2 + {a_2}^2)f_1f_3\,,\\
Q^3_6 &= -\,a_0a_3b_3f_1 - \epsilon a_0(a_1b_3 - a_3b_1)f_3 - \epsilon a_2a_3f_1{f_3}^2\,,\\
Q^3_7 &= -\,a_0a_2b_3f_1 + a_0a_2b_1f_3 - (\epsilon {a_1}^2 + {a_2}^2){f_3}^2\,.
\end{aligned}$$

Next, we substitute into (A2) the function $R$ in the form

$$R = \frac{1}{2fA}\left[(f^2 + C^2)'_x + H(h + (AC)'_y) - \frac{AC}{A^2}(A^2)'_w\right]\,,$$

which we obtain from $(1.13)_1$, (1.12) and (1.10), and the function $\beta$ in the form $\beta = \lambda a_0/A^2$. By the same argument as that for the previous equation (A1), we obtain once more seven partial differential equations.

They are now linear with respect to $a'_{0w}$, $a'_{1w}$, $a'_{2w}$, $a'_{3w}$, $W_1$, $W_2$ and $W_3$, where

$$(5.7)\qquad \begin{cases} W_1 = -\epsilon\,\dfrac{1}{\lambda}\varphi'_{1x} + 2\lambda H b_2\,, \\[2mm] W_2 = \dfrac{1}{\lambda}\varphi'_{2x} + 2\epsilon\lambda H b_1\,, \\[2mm] W_3 = \dfrac{1}{\lambda}\varphi'_{3x} + Hh\,. \end{cases}$$

Using (5.4) and the formula for $a'_{0w}$ similar to (5.5), we can also eliminate the derivative $a'_{0w}$ in all equations. We obtain the final form of the equation (A2) as a system of partial differential equations analogous to (5.6):

$$(5.8)\qquad \sum_{i=1}^{3} a_0 P^i_\alpha W_i - \sum_{i=1}^{3} Q^i_\alpha a'_{iw} = 0\,,\quad \alpha = 1, 2, \ldots, 7\,.$$

The following proposition will be crucial for reducing our partial differential equations to essentially independent ones.

Proposition 5.2. *The rank of the matrix* $[P^i_\alpha,\ Q^i_\alpha]$ *is at most two.*

Proof. Since $a_2 \neq 0$ and $b_2 = f_2 = 0$, we have from (2.51)

$$(5.9)\qquad \varphi_4 = \epsilon\,\frac{\lambda(a_1 f_3 - a_3 f_1)}{a_2}\,,$$

and hence we have

$$(5.10)\qquad \begin{cases} b_1 = \dfrac{{a_2}^2 f_3 + \epsilon a_1(a_1 f_3 - a_3 f_1)}{a_0 a_2}\,, \\[2mm] b_3 = \dfrac{{a_2}^2 f_1 + \epsilon a_3(a_1 f_3 - a_3 f_1)}{a_0 a_2}\,. \end{cases}$$

Substituting from (5.10) for $b_1$ and $b_3$ in the entries of the matrix $[Q^i_\alpha]$, we see that

$$[P^3_\alpha] = -\frac{f_1}{f_3}\,[P^1_\alpha]\ ,$$

$$[Q^1_\alpha] = \epsilon\,\frac{a_1 f_3 - a_3 f_1}{a_2}\,[P^1_\alpha] - \epsilon f_3\,[P^2_\alpha]\ ,$$

$$[Q^2_\alpha] = -\frac{{f_1}^2 - {f_3}^2}{f_3}\,[P^1_\alpha] + \epsilon\,\frac{a_1 f_3 - a_3 f_1}{a_2}\,[P^2_\alpha]\ ,$$

$$[Q^3_\alpha] = -\epsilon\,\frac{f_1(a_1 f_3 - a_3 f_1)}{a_2 f_3}\,[P^1_\alpha] + \epsilon f_1\,[P^2_\alpha]\ ,$$

which prove the assertion. □

Corollary 5.3. *Each system of partial differential equations* (5.6) *or* (5.8) *contains at most two linearly independent equations.*

Thus, the equations (A1) and (A2) are essentially reduced to four partial differential equations in two variables. We shall see later that, as in [9], we can make an additional reduction to only two equations (one of the form (5.6) and one of the form (5.8)).

Proposition 5.4. *The following algebraic formulas are consequences of the algebraic equations from theorem* 2.8 *and of the assumptions of proposition* 5.1:

(5.11)
$$\varphi_1 = \nu a_1, \qquad \varphi_2 = \epsilon\nu a_2, \qquad \varphi_3 = -\epsilon\nu a_3\,,$$

*where*

(5.12)
$$\nu = \frac{\lambda[{a_2}^2({f_1}^2 - {f_3}^2) - \epsilon(a_1 f_3 - a_3 f_1)^2]}{{a_0}^2{a_2}^2}, \quad \nu = \epsilon\,\frac{\varphi_0}{a_0}\,,$$

(5.13)
$${a_0}^2 = -\epsilon({a_1}^2 - {a_3}^2) - {a_2}^2\,.$$

*Further,* $f_2 = 0$ *and*

(5.14)
$$\begin{cases} b_1 = \dfrac{{a_2}^2 f_3 + \epsilon a_1(a_1 f_3 - a_3 f_1)}{a_0 a_2}, \\ b_2 = 0\,, \\ b_3 = \dfrac{{a_2}^2 f_1 + \epsilon a_3(a_1 f_3 - a_3 f_1)}{a_0 a_2}\,, \end{cases}$$

$$(5.15)\quad h = -\epsilon\,\frac{2\lambda(a_1f_1 - a_3f_3)}{a_0}, \quad \varphi_4 = \epsilon\,\frac{\lambda(a_1f_3 - a_3f_1)}{a_2}, \quad \varphi_5 = 2\varphi_4\,.$$

*Conversely, if $a_1$, $a_2$, $a_3$, $f_1$ and $f_3$ are arbitrary functions, and if the other basic functions are defined as above, then all algebraic equations of theorem* 2.8 *hold.*

PROOF. We show only the necessity of (5.11)-(5.15). The sufficiency will be proved by the direct check. The equations $(2.44)_3$ and $(2.44)_5$ imply $a_1\varphi_2 - \epsilon a_2\varphi_1 = 0$ and $a_2\varphi_3 + a_3\varphi_2 = 0$. Hence the formulas (5.11) hold with some function $\nu = \nu(w, x)$ of the variables $w$ and $x$. Substituting (5.11) and $(5.10)_1$ into $(2.44)_1$, and using (2.24he), we obtain $(5.12)_1$. The formula (5.13) is a direct consequence of (2.24he). The formulas $(5.14)_{1,3}$ and $(5.15)_2$ follow from $b_2 = f_2 = 0$ as shown in the proof of proposition 5.2. Next, from (5.11), (2.5he) and (2.24he), we have ${\varphi_0}^2 = \nu^2{a_0}^2$. Here, the relation (4.3) implies that $\epsilon({f_1}^2 - {f_3}^2)$ is negative because the discriminant $\Delta'$ is negative, hence $\epsilon\nu$ is negative. On the other hand, (4.4) together with (4.3) implies that $a_0\varphi_0$ is negative. Hence we obtain $(5.12)_2$. We obtain $(5.15)_1$ from (2.47) and $f_2 = 0$. Finally, $(5.15)_3$ is the same as (2.48). ∎

We need later the relation

$$(5.16)\qquad \nu = \frac{\lambda(f_1b_3 - f_3b_1)}{a_0a_1}\,,$$

which follows from (5.14) and (5.12).

Now let us return to the system of partial differential equations (5.6) and (5.8). Specifying corollary 5.3, we see easily that the system (5.6) reduces to two partial differential equations

$$(5.17)\qquad a_0V_2 - \epsilon f_3a'_{1x} + \epsilon\,\frac{a_1f_3 - a_3f_1}{a_2}a'_{2x} - f_1a'_{3x} = 0$$

and

$$(5.18)\qquad \begin{aligned} &a_0f_3V_1 + \frac{a_0(a_1f_3 - a_3f_1)}{a_2}V_2 - a_0f_1V_3 + \\ &\qquad + \left[\epsilon\left(\frac{a_1f_3 - a_3f_1}{a_2}\right)^2 - ({f_1}^2 - {f_3}^2)\right]a'_{2x} = 0\,. \end{aligned}$$

The system (5.8) reduces to two analogous equations

$$a_0 W_2 + \epsilon f_3 a'_{1w} - \epsilon \frac{a_1 f_3 - a_3 f_1}{a_2} a'_{2w} + f_1 a'_{3w} = 0 \tag{5.19}$$

and

$$\begin{aligned} a_0 f_3 W_1 + \frac{a_0(a_1 f_3 - a_3 f_1)}{a_2} W_2 - a_0 f_1 W_3 + \\ - \left[\epsilon \left(\frac{a_1 f_3 - a_3 f_1}{a_2}\right)^2 - ({f_1}^2 - {f_3}^2)\right] a'_{2w} = 0\,. \end{aligned} \tag{5.20}$$

Using (5.3), (5.7), (5.11), (5.14) and (5.16), we see, after lengthy but routine calculations, that (5.18) and (5.20) *are consequences* of (5.17) and (5.19).

Substituting $(5.3)_2$ and $(5.7)_2$ into (5.17) and (5.19), respectively, and using $(5.11)_2$, we have

$$\begin{cases} a_0 a'_{2w} + \epsilon 2\lambda H a_0 a_1 - \epsilon a_2 f_3 \left(\dfrac{a_1}{a_2}\right)'_x + \epsilon a_2 f_1 \left(\dfrac{a_3}{a_2}\right)'_x = 0\,, \\ a_0(\nu a_2)'_x - 2\lambda^2 H a_0 b_1 + \lambda a_2 f_3 \left(\dfrac{a_1}{a_2}\right)'_w - \lambda a_2 f_1 \left(\dfrac{a_3}{a_2}\right)'_w = 0\,. \end{cases} \tag{5.21}$$

Further, due to $(5.15)_1$, we have the relation

$$2\lambda(a_1 f_1 - a_3 f_3) = -\epsilon a_0 H'_x\,. \tag{5.22}$$

Introducing new functions $u = u(w,x)$ and $v = v(w,x)$ of the variables $w$ and $x$ such that

$$a_1 = u a_2, \quad a_3 = v a_2, \quad -\epsilon(u^2 - v^2) > 0\,, \tag{5.23}$$

we rewrite (5.21) in the form

$$\begin{cases} a_0 a'_{2w} + \epsilon 2\lambda H a_0 a_1 - \epsilon a_2 f_3 u'_x + \epsilon a_2 f_1 v'_x = 0\,, \\ a_0(\nu a_2)'_x - 2\lambda^2 H a_0 b_1 + \lambda a_2 f_3 u'_w - \lambda a_2 f_1 v'_w = 0\,. \end{cases} \tag{5.24}$$

Here, from (5.12)-(5.14), we get

$$(5.25)\qquad \begin{cases} a_0 = \sqrt{-\epsilon(u^2 - v^2) - 1}\, a_2 \,, \\ b_1 = \dfrac{f_3 + \epsilon u(uf_3 - vf_1)}{\sqrt{u^2 - v^2 - 1}} \,, \\ \nu = \dfrac{\lambda \left[ {f_1}^2 - {f_3}^2 - \epsilon \left( uf_3 - vf_1 \right)^2 \right]}{\left( u^2 - v^2 - 1 \right) {a_2}^2} \,, \end{cases}$$

where we normalize the signs of $a_2$ and $a_0$ to make them positive.

Let now $u$, $v$ and $H$ be arbitrary analytic functions. Substituting for $a_0$ from $(5.25)_1$ into (5.22) and into $(5.24)_1$, and solving them with respect to $f_1$ and $f_3$, we can express $f_1$ and $f_3$ in the form

$$(5.26)\qquad \begin{cases} f_1 = g_1 a'_{2w} + g_2 a_2 + g_3 \,, \\ f_3 = h_1 a'_{2w} + h_2 a_2 + h_3 \,, \end{cases}$$

where $g_i$'s and $h_i$'s are known functions. Substituting (5.26) into $(5.24)_2$ which has been transformed by (5.25), we obtain a partial differential equation of the form

$$(5.27)\qquad a''_{2wx} = \Psi(a'_{2w}, a'_{2x}, a_2, w, x) \,,$$

where $\Psi$ is a fixed analytic function of five variables. The general solution of (5.27) depends on two arbitrary (analytic) functions of one variable. Thus, *the generic family of metrics of subtype* (E) *depends on three arbitrary functions of two variables,* namely, $u$, $v$ and $H$.

Now, we can go further and prove that even *the local isometry classes* of our metrics still depend essentially on three functions. The proof is a modification of that of theorem 8.5 from [9]. We use the fact that $fA/f_3$ is a Riemannian invariant (see (3.14he)) and that the hyperbolic cosine function and the cosine function are even functions.

THEOREM 5.5. *The local isometry classes of metrics of subtype* (E) *are parametrized by three arbitrary functions of two variables modulo two arbitrary functions of one variable.*

The equation (5.27) can not be solved explicitly, in general. Yet, we give here an explicit family of the metrics of subtype (E).

EXAMPLE 5.6. Consider the “singular” case $a_2 = 0$ of proposition 5.4. Then we have

$$\begin{cases} \varphi_1 = \nu a_1, \quad \varphi_2 = a_2 = 0, \quad \varphi_3 = -\epsilon\nu a_3\,, \\ \varphi_0 = \epsilon\nu a_0, \quad \varphi_5 = 2\varphi_4 \end{cases} \tag{5.28}$$

and

$$b_2 = f_2 = 0\,. \tag{5.29}$$

From $(2.45)_2$ we see that there is a function $\xi = \xi(w, x)$ of the variables $w$ and $x$ such that

$$f_1 = \xi a_1, \quad f_3 = \xi a_3\,. \tag{5.30}$$

Hence, using (5.22) and (2.51), we have

$$a_0 h = -\,\epsilon 2\lambda\xi({a_1}^2 - {a_3}^2)\,, \tag{5.31}$$

$$b_1 = \frac{a_1\varphi_4}{\lambda a_0}, \quad b_3 = \frac{a_3\varphi_4}{\lambda a_0}\,. \tag{5.32}$$

Finally, we have

$${a_0}^2 = -\epsilon({a_1}^2 - {a_3}^2)\,, \tag{5.33}$$

and, from $(2.43)_1$ or $(2.44)_1$, we deduce

$$\frac{{\varphi_4}^2}{\lambda {a_0}^2} + \lambda\xi^2 = -\epsilon\nu\,. \tag{5.34}$$

Here $a_1$, $a_3$, $\xi$ and $\varphi_4$ are arbitrary functions of the variables $w$ and $x$. Conversely, if $a_1$, $a_3$, $\xi$ and $\varphi_4$ are arbitrary functions of the variables $w$ and $x$, and if the other basic functions are given by (5.28)-(5.34), then all algebraic equations mentioned in theorem 2.8 are satisfied.

In addition, from (5.31) and (5.33), we get

$$h = 2\lambda\xi a_0, \quad h = H'_x\,. \tag{5.35}$$

Further, a careful check shows that the system of partial differential equations (5.6) and (5.8) can be now reduced, instead of the form (5.21), to the form

$$a_0 V_2 - \epsilon(f_3 a'_{1x} - f_1 a'_{3x}) = 0\,, \tag{5.36}$$

$$a_0 W_2 + \epsilon(f_3 a'_{1w} - f_1 a'_{3w}) = 0\,. \tag{5.37}$$

All other partial differential equations are consequences of (5.36) and (5.37). Putting $U = a_3/a_1$, we can rewrite (5.36) and (5.37) in the form

$$2\lambda H a_0 + \xi a_1 U'_x = 0\,, \tag{5.38}$$

$$2H\varphi_4 + \xi a_1 U'_w = 0\,. \tag{5.39}$$

Then we have the following explicit family of solutions satisfying the equations (5.38) and (5.39) and the condition (5.35). Choose $U$ and $H$ as arbitrary functions of the variables $w$ and $x$, and put

$$\begin{cases} a_1 = -\epsilon\,\dfrac{hU'_x}{4\lambda^2 H(U^2-1)}, \quad a_3 = a_1 U, \quad a_0 = a_1\sqrt{\epsilon(U^2-1)}\,, \\ \xi = -\dfrac{2\lambda H\sqrt{\epsilon(U^2-1)}}{U'_x}, \quad \varphi_4 = -\dfrac{hU'_w}{4\lambda H\sqrt{\epsilon(U^2-1)}}, \quad h = H'_x\,. \end{cases} \tag{5.40}$$

Here we always assume $U'_x \neq 0$ and $\epsilon(U^2-1) > 0$. (Also, we normalize the signs of $a_1$, $a_3$ and $a_0$ to make them all positive.) Then the function $\nu$ is calculated from (5.34) and remaining coefficients are given by (5.28)-(5.30) and (5.32). This defines the wanted class of metrics.

## REFERENCES


[1] E. Boeckx: *Foliated semi-symmetric spaces*, Doctoral Thesis, Leuven, 1994.

[2] E. Boeckx – O. Kowalski – L. Vanhecke: *Non-homogeneous relatives of symmetric spaces*, Diff. Geom. Appl., **4**(1994), 45-69.

[3] E. Boeckx – O. Kowalski – L. Vanhecke: *Riemannian Manifolds of Conullity Two*, World Scientific, Singapore, 1996.

[4] J. Deprez – R. Deszcz – L. Verstraelen: *Examples of pseudo-symmetric conformally flat warped products*, Chinese J. Math., **17**(1989), 51-65.

[5] R. Deszcz: *On pseudo-symmetric spaces*, Bull. Soc. Math. Belgium, Série A.,**44** (1992), 1-34.

[6] V. Hájková: *Foliated semi-symmetric spaces in dimension 3* (*in Czech*), Doctoral Thesis, Prague, 1995.

[7] S. Kobayashi – K. Nomizu: *Foundations of Differential Geometry I*, Interscience Publishers, New York-London-Sydney, 1963.

[8] S. Kobayashi – K.Nomizu: *Foundations of Differential Geometry II*, Interscience Publishers, New York-London-Sydney, 1969.

[9] O. Kowalski: *An explicit classification of 3-dimensional Riemannian spaces satisfying* $R(X,Y) \cdot R = 0$, Czechoslovak Math. J., **46** (121) (1996), 427-474.

[10] O. Kowalski: *A classification of Riemannian 3-manifolds with constant principal Ricci curvatures* $\rho_1 = \rho_2 \neq \rho_3$, Nagoya Math. J., **132** (1993), 1-36.

[11] O. Kowalski – S. Ž.Nikčević: *On Ricci eigenvalues of locally homogeneous Riemannian 3-manifolds*, Geometriæ Dedicata, **62** (1996), 65-72.

[12] O. Kowalski – M. Sekizawa: *Local isometry classes of Riemannian 3-manifolds with constant Ricci eigenvalues* $\rho_1 = \rho_3 \neq \rho_3$, Archivum Math., **32** (1996), 137-145.

[13] O. Kowalski – M. Sekizawa: *Riemannian 3-manifolds with c-conullity two*, Bolletino U.M.I., (7) **11**-B (1997), Suppl. fasc. 2, 161-184.

[14] O. Kowalski – M. Sekizawa: *Three-dimensional Riemannian manifolds of c-conullity two*, Chapter 11 of “Riemannian Manifolds of Conullity Two”, World Scientific, Singapore, 1996.

[15] O. Kowalski – F. Tricerri – L. Vanhecke: *Exemples nouveaux de variétés riemanniennes non homogènes dont le tenseur de courbure est celui d'un espace symétrique riemannien*, C. R. Acad. Sci. Paris Sér.I, **311** (1990), 355-360.

[16] O. Kowalski – F. Tricerri – L. Vanhecke: *Curvature homogeneous Riemannian manifolds*, J. Math. Pures Appl., **71** (1992), 471-501.

[17] Z. I. Szabó: *Structure theorems on Riemannian manifolds satisfying* $R(X,Y) \cdot R = 0$, I, Local version, J. Diff. Geom., **17** (1982), 531-582.

[18] Z. I. Szabó: *Structure theorems on Riemannian manifolds satisfying* $R(X,Y)\cdot R=0$, II, Global version, Geom. Dedicata, **19** (1985), 65-108.

[19] Z. I. Szabó: *Classification and construction of complete hypersurfaces satisfying* $R(X,Y)\cdot R=0$, Acta. Sci. Math. (Hung.), **47** (1984), 321-348.

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