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A note on generalized Bessel functions and Fourier series

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RIASSUNTO: *In questo articolo si dimostra che le serie di Fourier di funzioni di* $C^{(i)}(-\pi,\pi)$, $i = 1, 2$ *sono la base naturale per ottenere le funzioni generatrici di particolari funzioni di Bessel a infinite variabili di interesse in problemi di emissione di radiazione.*

ABSTRACT: *In this note it is shown that functions of* $C^{(i)}(-\pi,\pi)$, $i = 1, 2$ *as well as their relevant Fourier series are the natural basis to get generating functions of particular infinite-variable generalized Bessel functions which can be exploited in radiation emission problems.*

1 – Introduction and background

We start from an infinite-dimensional generalization of the ordinary Bessel function which reads as follows

$$J_n(\{\alpha_m\}) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - \alpha_1 \sin\theta - \alpha_2 \sin 2\theta - \cdots - \alpha_m \sin m\theta - \cdots)\, d\theta, \quad n = 0, \pm 1, \pm 2, \ldots \tag{1.1}$$

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with $\{\alpha_m\}$ real and satisfying the condition that the series

$$\sum_{m=1}^{\infty} m|\alpha_m| \tag{1.2}$$

be convergent.

The introduction of this new type of Bessel functions is due to PÉRÈS [1] who gave also the relevant recurrences.

These functions, after a long period of quiescence, have been recently [2] rediscovered, essentially in connection with applications in emission radiation problems and studied in ref. [3] together with the corresponding modified version

$$I_n(\{\alpha_m\}) = \frac{1}{\pi}\int_0^{\pi} \cos n\theta \exp\Big(\sum_{m=1}^{\infty} \alpha_m \cos m\theta\Big) d\theta, \quad n = 0, \pm 1, \pm 2, \dots \tag{1.3}$$

with $\{\alpha_m\}$ satisfying the same requirements as in the J-case.

The related results have evidenced the analogy with the corresponding ordinary cases and this fact confirms the validity of the considered infinite-dimensional extension.

In addition, an interesting connection has been found between these functions and Fourier series of proper smooth functions. To this concern, it is worth stressing that the link with Fourier series can be established also for another class of infinite-variable Bessel functions, as it will be shown in the present treatment. The main results here obtained are essentially based on known properties of functions of $C^l(-\pi, \pi)$, $l = 1, 2$, hereafter reported from ref. [4], and on some propositions of ref. [3], also reported in the following for later use.

THEOREM 1.1. *If a function $f(\theta)$ is of $C^1(-\pi, \pi)$ and satisfies the end-point conditions*

$$f(-\pi) = f(\pi)\,, \tag{1.4}$$

then the following Fourier expansion holds

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta} = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos m\theta + \sum_{m=1}^{\infty} b_m \sin m\theta\,, \tag{1.5}$$

(*where $\{c_n\}, \{a_m\}, \{b_m\}$, are the usual complex, cosine and sine Fourier coefficients*) *and the relevant series are uniformly convergent on $[-\pi, \pi]$.*

Theorem 1.2. *If a function $f(\theta)$ is of $C^2(-\pi,\pi)$ and satisfies the conditions*

$$f^{(l)}(-\pi) = f^{(l)}(\pi), \qquad l = 0, 1 \tag{1.6}$$

then the series

$$\sum_{m=1}^{\infty} m^l(|a_m| + |b_m|), \qquad l = 0, 1 \tag{1.7}$$

where $\{a_m\}$ and $\{b_m\}$ are the Fourier coefficients of $f(\theta)$, are convergent.

In addition to the above results, in this note we make use also of a basic property of $I_n(\{\alpha_m\})$ functions, hereafter reported from ref. [3] in order to make the present treatment self-contained.

Proposition 1.1. *The infinite-dimensional modified Bessel functions, $I_n(\{\alpha_m\})$, admit the following expansion of the Jacobi-Anger type*

$$\exp\Big(\sum_{m=1}^{\infty} \alpha_m \cos m\theta\Big) = \sum_{n=-\infty}^{\infty} e^{in\theta} I_n(\{\alpha_m\})\,, \tag{1.8}$$

and the involved series is uniformly convergent on $[-\pi,\pi]$.

The aforementioned results will be used in Section 2 for the study of further generalized forms of the Bessel functions which will be shown to be naturally joined with the Fourier expansions of functions of $C^2(-\pi,\pi)$ satisfying conditions (1.6).

2 – Generalized Bessel functions of mixed type and Fourier series

In addition to the above multi-variable generalized Bessel functions we now consider another type of infinite-dimensional Bessel functions which, as it will be seen in the following, can be called of mixed type.

Definition 2.1. The infinite-variable Bessel functions of mixed type are defined as follows

$$\begin{aligned} II_n(\{\alpha_m\}/\{\beta_m\}) = \frac{1}{\pi}\int_0^{\pi} \cos\Big(n\theta - \sum_{m=1}^{\infty} \alpha_m \sin m\theta\Big)\times \\ \times \exp\Big(\sum_{m=1}^{\infty} \beta_m \cos m\theta\Big) d\theta, \quad n = 0, \pm 1, \pm 2, \ldots \end{aligned} \tag{2.1}$$

with $\{\alpha_m\}$ and $\{\beta_m\}$ real and satisfying the condition that the series

$$\sum_{m=1}^{\infty} m(|\alpha_m| + |\beta_m|)\,, \tag{2.2}$$

be convergent

REMARK 2.1. It is to be observed that for functions of the form (2.1) one easily obtains the particular values

$$II_n(\{0\}/\{0\}) = \delta_{n,0}\,, \tag{2.3}$$

where $\delta_{m,l}$ is the Kronecker symbol.

So-defined generalized Bessel functions are called of mixed type since they can be expressed in terms of the infinite-variable Bessel functions, $J_n\{.\}$ and $I_n\{.\}$, as it is shown in the following statement.

LEMMA 2.1. *The infinite-dimensional Bessel functions of mixed type, $II_n(\{\alpha_m\}/\{\beta_m\})$, can be expanded as follows*

$$II_n(\{\alpha_m\}/\{\beta_m\}) = \sum_{l=-\infty}^{\infty} J_{n-l}(\{\alpha_m\})I_l(\{\beta_m\})\,. \tag{2.4}$$

PROOF. The assertion can be easily obtained starting from the definition relation (2.1) and expanding the exponential term in the integrand in a uniformly convergent Fourier series according to the result, eq. (1.8), of proposition 1.1.

At this point, taking into account relations (2.1) and (2.2) and the result of theorem 1.2, one can make the following assertion.

REMARK 2.2. The Fourier cosine and sine coefficients, $\{a_m\}$ and $\{b_m\}$, $m = 1, 2\ldots$, of any function of theorem 1.2, ensure the existence of the corresponding infinite-variable Bessel functions, $II_n(\{b_m\}/\{a_m\})$.

Furthermore, it is worth stressing that the link with Fourier series of functions of theorem 1.2 can be obtained in a more explicit form if one considers, in analogy to the ordinary case, the extension of functions of the form (1.1) to the case of purely imaginary arguments according to the following definition

$$J_n(i\alpha_m) = \frac{1}{\pi}\int_0^{\pi} \cos\Big(n\theta - i\sum_{m=1}^{\infty} \alpha_m \sin m\theta\Big)d\theta\,, \tag{2.5}$$

with $\{\alpha_m\}$ real and satisfying the conditions that the series (1.2) be convergent. At this point, one has to consider also for the infinite-variable Bessel functions of the form (2.1) a proper extension to a purely imaginary case. More precisely, one needs to introduce the following function

$$II_n(\{i\alpha_m\}/\{\beta_m\}) = \frac{1}{\pi}\int_0^\pi \cos\Big(n\theta - i\sum_{m=1}^\infty \alpha_m \sin m\theta\Big)\times \times \exp\Big(\sum_{m=1}^\infty \beta_m \cos m\theta\Big)d\theta\,, \tag{2.6}$$

with $\{\alpha_m\}$ and $\{\beta_m\}$ real and satisfying the restriction that the series (2.2) be convergent.

For these generalized Bessel functions one can easily get an expansion analogous to that obtained for the corresponding real case, as described in the following statement.

Lemma 2.2. *The infinite-dimensional Bessel functions of the form* (2.6) *admit the following representation*

$$II_n(\{i\alpha_m\}/\{\beta_m\}) = \sum_{l=-\infty}^\infty J_{n-l}(\{i\alpha_m\})I_l(\{\beta_m\})\,. \tag{2.7}$$

Proof. The proof is omitted since it is the same as that of lemma 2.1. Finally, the forthcoming statement shows the connection existing between functions of theorem 1.2 and the class of Bessel functions defined by eq. (2.6).

Theorem 2.1. *Let $f(\theta)$, $-\pi \le \theta \le \pi$, be an arbitrary real function of theorem 1.2 and let $\{a_m\}$ and $\{b_m\}$ be the relevant Fourier cosine and sine coefficients, then the function* $\exp[f(\theta) - a_0/2]$ *is the generating function of the corresponding $II_n(\{-ib_m\}/\{a_m\})$ since the following expansion holds*

$$\exp[f(\theta) - a_0/2] = \sum_{n=-\infty}^\infty e^{in\theta} II_n(\{-ib_m\}/\{a_m\})\,, \tag{2.8}$$

with the involved series uniformly convergent on $[-\pi, \pi]$.

PROOF. Since $f(\theta)$ is a function of theorem 1.2 and hence of theorem 1.1, the same holds for the corresponding function $\exp[f(\theta)-a_0/2]$, so that, considering the relevant uniformly convergent Fourier expansions in standard and complex forms, respectively, one can write

$$\begin{aligned}\exp[f(\theta)-a_0/2] &= \exp\Big(\sum_{m=1}^{\infty} a_m \cos m\theta + \sum_{m=1}^{\infty} b_m \sin m\theta\Big) = \\ &= \sum_{n=-\infty}^{\infty} e^{in\theta}\bar{c}_n(\{a_m\}/\{b_m\})\,,\end{aligned} \tag{2.9}$$

with Fourier coefficients, $\bar{c}_n(\{.\}/\{.\})$, expressed, as usual, as follows

$$\begin{aligned}\bar{c}_n(\{a_m\}/\{b_m\}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}\exp[f(\theta)-a_0/2]d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}\times \\ \times \exp\Big(\sum_{m=1}^{\infty} a_m\cos m\theta + i\sum_{m=1}^{\infty}(-ib_m)\sin m\theta\Big)d\theta.\end{aligned} \tag{2.10}$$

At this point, considering eqs. (2.10) and (2.6) and taking into account the known Euler formula and the parities of the circular functions, one easily obtains that

$$\bar{c}_n(\{a_m\}/\{b_m\}) \equiv II_n(\{-ib_m\}/\{a_m\})\,, \tag{2.11}$$

where the existence of the infinite-variable Bessel functions, $II_n(\{.\}/\{.\})$, follows from the fact that the $\{a_m\}$ and $\{b_m\}$, as Fourier coefficients of a function, $f(\theta)$, of theorem 1.2, have the property that the series (1.7) converges and hence the existence condition for the relevant $II_n(\{-ib_m\}/\{a_m\})$ is satisfied. The proof is thus completed.

3 – Concluding remarks

The class of infinite-dimensional generalized Bessel functions of mixed type (GBF) here considered and defined by eq. (2.6), have been found to be strictly connected with the Fourier series of functions of $C^2(-\pi,\pi)$ with proper endpoint conditions. To this concern, it is to be mentioned that the link with Fourier series is extendible to the corresponding class of C^1-functions if one assumes as existence condition for the above GBF the convergence of the series

$$\sum_{m=1}^{\infty} (|\alpha_m| + |\beta_m|) , \tag{3.1}$$

instead of that of the series (2.2) proposed in this work.

As a matter of fact, condition (3.1) is weaker than that relevant to eq. (2.2) and hence, in principle, it should be preferred, but the choice of the assumption, eq. (2.2), has been made according to the results of ref. [3] where it has allowed to obtain for the class of infinite-variable Bessel functions, $J_n(\{.\})$ and $I_n(\{.\})$, a full analogy with the ordinary case.

Moreover, it is to be pointed out that, though the stronger existence condition is more satisfactory from a mathematical point of view (see ref. [3] for more details) the weaker one, as mentioned above, allows to extend the connection between GBF and Fourier series to the C^1-set and hence to enlarge the range of possible applications.

This assertion is easily verified considering that a proposition analogous to theorem 1.2 holds [4] for an arbitrary function of $C^1(-\pi,\pi)$ satisfying the first of conditions (1.6), since for the related Fourier coefficients, $\{a_m\}$ and $\{b_m\}$, one has that the first of the series (1.7) is convergent.

Summing up, one has that functions of $C^l(-\pi,\pi), l = 1, 2$, with proper endpoint conditions, are the natural basis to get generating functions of the considered GBF and this seems an interesting result since it allows to perform a rigorous treatment of physical problems like the radiation emission by charged particles moving in magnetic undulators which is hardly tractable with ordinary tools [2], [5]-[7].

As for the related determination, it is to be pointed out that, in practice, the relevant formalism can be simplified. In fact, when one can neglect higher-order harmonics, as it happens is many physical cases, the problem of determining the above GBF reduces to that of the corresponding GBF with few variables, which can be easily computed in regions of practical interest [8].

REFERENCES

[1] J. Pérès: C.R. Acad. Sci., **161** (1915), 168.

[2] G. Dattoli – C. Chiccoli – S. Lorenzutta – G. Maino – M. Richetta – A. Torre: Radiat. Phys. Chem., **47** (1996), 183.

[3] S. Lorenzutta – G. Maino – G. Dattoli – A. Torre – C. Chiccoli: Rend. Mat., **15** (1995), 405.

[4] B.M. Budak – S.V. Fomin: *Multiple Integrals, Field Theory and Series*, (MIR Publ., Moscow, 1973.)

[5] G. Dattoli – L. Giannessi – L. Mezi – M. Richetta – A. Torre: Phys. Rev., **45** A (1992), 4023.

[6] G. Dattoli – G. Voykov: Phys. Rev., **48** E (1993), 3030.

[7] G. Dattoli – S. Lorenzutta – G. Maino – A. Torre – G. Voykov – C. Chiccoli: J. Math. Phys., **35** (1994), 3636.

[8] G. Dattoli – C. Mari – A. Torre – C. Chiccoli – S. Lorenzutta – G. Maino: J. Scient. Comput., **7** (1992), 175.

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