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A report on functorial connections and differential invariants

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Riassunto: *Si presentano alcuni recenti risultati sull'esistenza di connessioni funtoriali, relative alle G-strutture e sulla loro utilizzazione nella determinazione di invarianti differenziali e nel calcolo del numero degli invarianti differenziali indipendenti di un dato ordine.*

Abstract: *A report of several results on the existence of functorial connections attached to G-structures, its use in obtaining geometric differential invariants and the calculation of the number of functionally independent differential invariants of a given order is presented.*

1 – Introduction

We give a survey of results obtained by the authors in the last few years concerning functorial connections and differential invariants. Although these topics are different they are essentially related as the exis-

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tence of a functorial connection on the category of G-structures allows us to construct in a very natural way a number of differential invariants which in most cases are enough to generate the ring of differential invariants and also to solve the equivalence problem for G-structures. In addition, the non-existence of a functorial connection makes the construction of differential invariants extremely difficult: for example, think of the conformal or symplectic structures (cf. [5], [14], 15]), thus showing that there exists a deep relationship between the existence of a functorial connection on G-structures, the determination of all differential invariants on G-structures and solving the equivalence problem for G-structures.

2 – Linear connections functorially attached to G-structures

2.1 – Some examples

First of all, let us review some well-known examples of functorial connections attached to different G-structures.

(a): For the orthogonal group $G = O(n)$ there exists the Levi-Civita connection ∇, which is defined on the principal bundle of orthonormal linear frames $O(M)$ of a Riemannian manifold (M, g) (e.g., see [25, Chapter IV]).

(b): For the unitary group $G = U(n)$, there exists the reduction of the Levi-Cività connection to the bundle of unitary frames $U(M)$ on a Kähler manifold (M, J, g)(e.g., see [25, Chapter IX], [§(5·4.111·2)]).

(c): For the group $G = \mathrm{Sp}_1 \cdot Gl(n, \mathbb{R})$ we obtain the Obata connection ∇^H attached to a hypercomplex structure $H = (J_\alpha)$, $\alpha = 1, 2, 3$, over a manifold M (e.g., see [1], [35], [37]).

Remark 2.1. Let A be an n-dimensional $\mathbb{R}$-algebra (not necessarily commutative). For example: $A = \mathbb{R}[x]/(x^2 + 1)$(almost complex structures), $A = \mathbb{R}[x]/(x^3 + x)$(Yano structures), $A = \mathbb{R}[i, j, k]$ (almost hypercomplex structures), $A = \bigwedge^{\cdot}(\mathbb{R}^s)$, $n = 2^s$ (the Grassmann algebra of a vector space) or even an arbitrary Clifford algebra. An almost A -structure (resp. an A-structure) on a C^∞-manifold M, $\dim M = nr$, is a $Gl(r; A)$-structure (resp. an integrable $Gl(r; A)$-structure) on M (cf. [36]). Note that we have a natural representation $\lambda^r : Gl(r; A) \to Gl(rn; \mathbb{R})$, as $\mathfrak{gl}(r, A) =$ Lie algebra of $Gl(r; A)$ can be identified with the right A-linear mappings $\Lambda : A^r \to A^r$. By giving the right notions of A-differentiability and A-analyticity (cf. [18], [36], [40], [41]) conditions can be stated on an A-manifold in order to admit a functorial connection, thus generalizing the connections introduced in the previous three items.

(d): The Blaschke connection of a three-web is defined as follows. A three-web is given on a surface M by three foliations of smooth lines which are in general position, or equivalently (Chern [12]) it is defined by giving an $\mathbb{R}^*$-structure $P \subset F(M)$ on M, $\dim M = 2$. Blaschke's connection is the only symmetric linear connection defined on P (see [8], [22]).

(e): For the trivial subgroup $G = \{I\} \subset Gl(n, \mathbb{R})$, where I stands for the identity map, G-structures are the linear parallelisms. If $(X_1, \dots, X_n)$ is a linear frame on a parallelizable manifold, then there exists a unique linear connection ∇ such that $\nabla_{X_i} X_j = 0$, for all $1 \leq i \leq n$, $1 \leq j \leq n$.

(f): For the center of the full linear group, i.e., $G = \{\lambda I \,; \lambda \in \mathbb{R}^*\} \subset Gl(n, \mathbb{R})$, G-structures correspond with projective parallelisms; that is, with the fields of projective frames of the projective bundle associated to the tangent bundle. These are the $\mathbb{R}^*$-structures $P \subset F(M)$ on a manifold of arbitrary dimension M, $\dim M = n$. We thus obtain a generalization of the Blaschke notion of a web. In this case it is shown that there exists a unique linear connection on P such that $\mathrm{Trace}(\mathrm{Tor}_\nabla) = 0$ (see [39]). The above condition imposes the vanishing of a one-form. Also note that for $\dim M = n = 2$, the vanishing of the trace of the torsion tensor is equivalent to saying that ∇ is symmetric.

2.2 – Existence of functorial connections and obstructions to exist

In all previous examples the connection $\nabla(\sigma)$ attached to a G-structure σ on M satisfies a naturality condition which can be expressed as $f \cdot \nabla(\sigma) = \nabla(f \cdot \sigma)$, for every diffeomorphism f from M onto M', where the dot stands for either the natural action of a diffeomorphism on connections or on G-structures. More precisely,

Definition 2.1. Let $\pi : F(M) \to M$ be the bundle of linear frames. Two G -structures $\pi : P \to M$, $\pi' : P' \to M'$ are said to be *equivalent* if there exists a diffeomorphism $f : M \to M'$ such that $\tilde{f}(P) = P'$, where

$\tilde{f} : F(M) \to F(M')$ is the isomorphism of the bundles of linear frames induced from $\tilde{f}$; i.e., $\tilde{f}(X_1, \dots, X_n) = (f_* X_1, \dots, f_* X_n)$ (cf. [25, VI.1]).

Let $G \subseteq Gl(n, \mathbb{R})$ be a closed subgroup. As is well-known (cf. [7], [24]), G-structures $P \subseteq F(M)$ can be identified with the sections $\sigma_P : M \to F(M)/G$ of the quotient bundle $\bar{\pi} : F(M)/G \to M$ by setting:

$$\begin{aligned} P \mapsto \sigma_P :&\quad \sigma_P(x) = u_x \cdot G,\ u_x \in P_x \\ \sigma \mapsto P_\sigma :&\quad (P_\sigma)_x = \sigma(x), \quad x \in M \end{aligned}$$

Diff$M$ acts on the sections of the classifying bundle $\bar{\pi} : F(M)/G \to M$ as $f \cdot \sigma_P = \bar{f} \circ \sigma_P \circ f^{-1}$, where $\bar{f} : F(M)/G \to F(M')/G$ is the map induced from $\tilde{f} : F(M) \to F(M')$. We have

$$\tilde{f}(P) = P' \Leftrightarrow f \cdot \sigma_P = \sigma_{P'}.$$

That is, two G-structures P, P' are equivalent if and only if the corresponding sections $\sigma_P, \sigma_{P'}$ are Diff$M$-equivalent.

DEFINITION 2.2. *A functorial connection* is an assignment $\sigma \mapsto \nabla(\sigma)$, that associates a linear connection $\nabla(\sigma)$ over M to each section σ of the classifying bundle $F(M)/G$, satisfying the following three properties:

(i): $\nabla(\sigma)$is *adapted* to σ; i.e., $\nabla(\sigma)$ is reducible to the subbundle P_σ.

(ii): Naturality: for every diffeomorphism f of M, $\nabla(f \cdot \sigma) = f \cdot \nabla(\sigma)$, where the dot on the right hand side stands for the image of $\nabla(\sigma)$ via f onto $P_{f \cdot \sigma} = \tilde{f}(P_\sigma)$.

(iii): *Continuity*: $\nabla(\sigma)$ depends continuously on σ with respect to the C^∞ topologies of the spaces of sections of the classifying bundle and of the bundle of linear connections. This is equivalent to:

(iii′): *Finiteness*: there exists an integer $r \geq 0$ such that $\nabla(\sigma)(x)$ only depends on $j^r_x \sigma$, for every point $x \in M$.

THEOREM 2.1. *If a functorial connection exists for G-structures, then the first prolongation of the Lie algebra of G must vanish;* i.e., $\mathfrak{g}^{(1)} = 0$.

For the particular case in which $\nabla(\sigma)(x)$ only depends on $j^1_x\sigma$ (i.e., with the additional assumption that the connection only depends on the first contact of the G-structure) the above result is obtained in [39 theorem 1.3].

Recall that the first prolongation of the Lie subalgebra $\mathfrak{g} \subseteq \mathfrak{gl}(n,\mathbb{R}) \cong (\mathbb{R}^n)^* \otimes \mathbb{R}^n$ is defined by the exact sequence (cf. [24]),

$$0 \to \mathfrak{g}^{(1)} \to (\mathbb{R}^n)^* \otimes \mathfrak{g} \xrightarrow{\delta} \bigwedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n,$$

$\delta : \otimes^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n \to \bigwedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ being the antisymmetrization (or Spencer) operator; or equivalently,

$$\mathfrak{g}^{(1)} = \left\{ \begin{array}{l} \text{symmetric bilinear maps } A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \\ \text{such that } A(v,-) \in \mathfrak{g},\ \forall v \in \mathbb{R}^n \end{array} \right\}$$

For example, theorem 2.1 explains why functorial connections do not exist neither in Symplectic Geometry nor in Conformal Geometry. Recall:

$$\mathfrak{sp}\,(n;\mathbb{R})^{(r)} \cong S^{r+2}\left((\mathbb{R}^n)^*\right), \quad \forall r \geq 0.$$

$$\left\{ \begin{array}{ll} \mathfrak{co}\,(n)^{(1)} \cong (\mathbb{R}^n)^*, & \forall n \geq 2, \\ \mathfrak{co}\,(n)^{(r)} = 0,\ \forall r \geq 2, & \forall n \geq 3, \\ \mathfrak{co}\,(2)^{(r)} \cong \mathfrak{gl}\,(1;\mathbb{C})^{(r)} \cong \mathbb{C}, & \forall r \geq 1. \end{array} \right.$$

Remark 2.2. The conformal Cartan connection attached to a conformal structure is not a linear connection.

Remark 2.3. The first prolongation of a general subalgebra $\mathfrak{g} \subseteq \mathfrak{gl}(n,\mathbb{R})$ vanishes. E. Cartan classified the exceptions (cf. [10]).

Theorem 2.2. *Assume the following two conditions hold true:*

1. $\mathfrak{g}^{(1)} = 0$,
2. *there exists a G-invariant subspace W such that,* $\bigwedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n = \delta((\mathbb{R}^n)^* \otimes \mathfrak{g}) \oplus W$.

Then, G-structures admit at least one functorial connection.

Proof. It is not difficult to prove (e.g., see [7, III.6.2]) that in this case there exists a unique linear connection ∇ defined on each G-structure such that the $\delta((\mathbb{R}^n)^* \otimes \mathfrak{g})$-component of its torsion tensor vanishes; or equivalently, Tor_∇ takes values in W. ∎

Remark 2.4. The second condition in the theorem can be re-stated by saying that the injection

$$0 \to (\mathbb{R}^n)^* \otimes \mathfrak{g} \xrightarrow{\delta} \bigwedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n,$$

admits an equivariant retract,

$$0 \to (\mathbb{R}^n)^* \otimes \mathfrak{g} \xleftarrow{\rho} \bigwedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n.$$

Theorem 2.3. *Assume the subalgebra $\mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{R})$ satisfies the following condition:*

$$\left(T \in (\mathbb{R}^n)^* \otimes \mathfrak{g},\ \mathrm{Trace}\,(A \circ i_v \delta\,(T)) = 0,\ \forall A \in \mathfrak{g},\ \forall v \in \mathbb{R}^n\right) \Longrightarrow T = 0.$$

Then $\mathfrak{g}^{(1)} = 0$, and W exists.

Proof. (cf. [39]) First of all we define a subspace $W \subseteq \bigwedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n$ as follows:

$$W = \left\{ T \in \bigwedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n \mid \mathrm{Trace}\ (A \circ i_v T) = 0,\ \forall A \in \mathfrak{g},\ \forall v \in \mathbb{R}^n \right\}.$$

We claim that W is G-invariant. We recall that G acts on $\bigwedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n$ by restricting the natural $Gl(n, \mathbb{R})$-action on $\bigwedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n$ and also that this action is given by

$$(g \cdot T)\,(v, w) = g\left(T\left(g^{-1}(v), g^{-1}(w)\right)\right),\ \forall g \in Gl\,(n, \mathbb{R}),\ \forall v, w \in \mathbb{R}^n.$$

Furthermore, it is not difficult to check that $i_v(g \cdot T) = g \circ i_{g^{-1}(v)} T \circ g^{-1}$ for all $T \in \bigwedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n$, $v \in \mathbb{R}^n$. Hence

$$\begin{aligned} \mathrm{Trace}\ (A \circ i_v\,(g \cdot T)) &= \mathrm{Trace}\ \left(A \circ g \circ i_{g^{-1}(v)} T \circ g^{-1}\right) = \\ &= \mathrm{Trace}\ \left(g \circ \left(g^{-1} \circ A \circ g \circ i_{g^{-1}(v)} T\right) \circ g^{-1}\right) = \\ &= \mathrm{Trace}\ \left(\left(g^{-1} \circ A \circ g\right) \circ i_{g^{-1}(v)} T\right) = 0, \end{aligned}$$

as $A \in \mathfrak{g}$, $g \in G \Rightarrow g^{-1} \circ A \circ g \in \mathfrak{g}$, and $T \in W$. If $T \in \mathfrak{g}^{(1)}$ then $\delta(T) = 0$ and by virtue of the assumption we can conclude that $T = 0$. Hence $\dim \delta((\mathbb{R}^n)^* \otimes \mathfrak{g}) = mn$, where $m = \dim G$. Moreover, if $A_1, \ldots, A_m$ is a basis of $\mathfrak{g}$ and $v_1, \ldots, v_n$ is the standard basis of $\mathbb{R}^n$, then the subspace is defined by the following system of mn linear equations: Trace $(A_i \circ i_{v_j} T) = 0$, $1 \le i \le m$, $1 \le j \le n$. Therefore $\dim W \ge \binom{n}{2} n - mn$. As the assumption implies $\delta((\mathbb{R}^n)^* \otimes \mathfrak{g}) \cap W = \{0\}$ we obtain $\bigwedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n = \delta((\mathbb{R}^n)^* \otimes \mathfrak{g}) \oplus W$, thus finishing the proof. ∎

THEOREM 2.4. *The previous theorem applies to subalgebras such that*

1. $\mathfrak{g}^{(1)} = 0$,
2. $\mathfrak{g}$ *is invariant under transposition;* i.e., $A \in \mathfrak{g} \Rightarrow {}^tA \in \mathfrak{g}$.

In particular the above two conditions hold true for Lie subgroups $G \subseteq O(n)$. *In this way, we obtain a common definition of the complement* W *for a wide class of subalgebras.*

REMARK 2.5. Probably the most part of results still hold true by only using pseudoconnections instead of linear connections, cf. [9]. A pseudoconnection is a $\mathfrak{g}$-valued 1-form ω on a G-structure $P \to M$ whose restriction to each fibre coincides with the Maurer-Cartan form, $\omega(A^*) = A$, $\forall A \in \mathfrak{g}$, but $\omega$is not necessarily AdG -equivariant. Each pseudoconnection gives rise to a pseudotorsion function $T : P \to \bigwedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n$. The reduction of T in

$$[T] : P \to H^{0,2}(\mathfrak{g}) = \bigwedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n / \mathrm{Im}\, \delta$$

is AdG-equivariant and independent of the pseudoconnection chosen. Hence $[T]$ induces a map $M \to H^{0,2}(\mathfrak{g})/G$ canonically associated to P, called the torsion of the G-structure. Nevertheless, linear connections are highly desirable in order to obtain scalar differential invariants. Think of the difficulties in conformal geometry (cf. [4], [6]).

3 – Differential invariants

3.1 – The notion of an invariant

Given a fibred manifold $p : N \to M$, we denote by $p_r : J^r(N) \to M$ the r-jet bundle of local sections of p, and for $r \ge s$, $p_{rs} : J^r(N) \to J^s(N)$

is $p_{rs}(j^r_x\sigma)=j^s_x\sigma$, σ being a local section of p. If $(V; x_1, \dots, x_n, y_1, \dots, y_m)$, $\dim M = n$, $\dim N = m+n$ is a fibred coordinate system for the submersion p, $V \subseteq N$ being an open subset, then we denote by $(x_j; y^i_\alpha)$, $1 \le i \le m$, $1 \le j \le n$, $0 \le |\alpha| \le r$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, with $y^i_0 = y_i$, $1 \le i \le m$, the coordinate system induced on $p^{-1}_{r0}(V)$; i.e., $y^i_\alpha(j^r_x\sigma) = (\partial^{|\alpha|}(y_i \circ \sigma)/\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n})(x)$.

Definition 3.1. (cf. [29]) A (scalar) *rth-order differential invariant* on G-structures is a differentiable function $F : J^r(F(M)/G) \to \mathbb{R}$ such that for every G-structure σ defined on a neigbourhood of $x \in M$ and every diffeomorphism $f : M \to M$, we have $F(j^r_{f(x)}(\bar{f}\circ\sigma\circ f^{-1})) = F(j^r_x\sigma)$.

Remark 3.1. This definition covers "in abstracto" what is usually understood to be an invariant in different settings; e.g., the scalar curvature of a metric, the trace of a tensor field of type $(1,1)$, etc.

3.2 – The infinitesimal definition of an invariant

Each vector field $X \in \mathfrak{X}(M)$ induces a vector field $\bar{X}^r \in \mathfrak{X}(J^r(F(M)/G))$ as follows (cf. [29]). The flow $f_t : M \to M$ of X gives rise to a flow of automorphisms $\tilde{f}_t : F(M) \to F(M)$. Let $\bar{f}_t : F(M)/G \to F(M)/G$ be the flow induced from $\tilde{f}_t$ and let $\bar{f}^r_t : J^r(F(M)/G) \to J^r(F(M)/G)$ be the flow $\bar{f}^r_t(j^r_x\sigma) = j^r_{f_t(x)}(\bar{f}_t\circ\sigma\circ f_{-t})$. By definition, $\bar{X}^r$ is the infinitesimal generator of $\bar{f}^r_t$. The vector field $\bar{X}^r$ is called the natural lifting of X to the r-jet bundle of G-structures by infinitesimal contact transformations. The following properties hold true:

- (i): $\bar{X}^r$ is $\bar{\pi}_r$-projectable onto X.
- (ii): $X \mapsto \bar{X}^r$ is an injection of Lie algebras, $\mathfrak{X}(M) \hookrightarrow \mathfrak{X}(J^r(F(M)/G))$; i.e.,
  - (ii.a): $X \mapsto \bar{X}^r$ is $\mathbb{R}$- linear,
  - (ii.b): $\overline{[X,Y]}^r = [\bar{X}^r, \bar{Y}^r]$, $\forall X, Y \in \mathfrak{X}(M)$.

Definition 3.2. A differentiable function $F : J^r(F(M)/G) \to \mathbb{R}$ is said to be an *infinitesimal differential invariant* if

$$\bar{X}^r(F) = 0,\ \forall X \in \mathfrak{X}(M)\,.$$

It is obvious that if F is a differential invariant, then F also is an infinitesimal differential invariant, as $\bar{X}^r(F) = 0$ if and only if $F \circ \bar{f}^r_t = F$. The converse is "almost" true (see examples below), so we shall mainly work with the infinitesimal definition of an invariant.

3.3 – The basic distribution

The lifting of vector fields gives rise to a differential system on $J^r(F(M)/G)$,

$$\overline{\mathcal{D}}_{j^r_x\sigma}{}^r = \left\{ \bar{X}^r_{j^r_x\sigma} \mid \forall X \in \mathfrak{X}(M) \right\} ,$$

which satisfies the following properties:

1. $\overline{\mathcal{D}}^r$ is involutive.
2. $\overline{\mathcal{D}}^r$ is a generalized distribution (cf. [11]) of locally constant rank on an open dense subset $\mathcal{O}^r \subseteq J^r(F(M)/G)$. (Unfortunately, the singularities of $\overline{\mathcal{D}}^r$ are unavoidable except for $G = \{1\}$).
3. rth-order differential invariants are the *first integrals* of the distribution $\overline{\mathcal{D}}^r$.
4. Hence on the regularity open subset $\mathcal{O}^r$ we have that the maximal number of functionally independent rth-order differential invariants is equal to $\dim J^r(F(M)/G) - \mathrm{rk}.\overline{\mathcal{D}}^r$.

3.4 – Desingularizing $\overline{\mathcal{D}}^r$

Remark 3.2. From now on we assume that G-structures admit a functorial connection.

Definition 3.3. The *desingularizing bundle* is the G-principal bundle

$$p^r : E^r(M) = J^r(F(M)/G) \times_{F(M)/G} F(M) \to J^r(F(M)/G) .$$

The elements of $E^r(M)$ are the pairs $(j^r_x\sigma, u_x)$, such that:

1. σ is a G-structure around x,
2. $u_x \in F(M)$ is a linear frame,
3. $g \in G$,
4. $\sigma(x) = (P_\sigma)_x = u_x \cdot G$.

The action of G is $(j^r_x\sigma, u_x) \cdot g = (j^r_x\sigma, u_x \cdot g)$.

We have an injection of Lie algebras:

$$\mathfrak{X}(M) \to \mathfrak{X}(E^r(M)), \quad X \mapsto X^r = \left(\bar{X}^r, \tilde{X} \right).$$

Similarly, we can define a distribution on $E^r(M)$:

$$\mathcal{D}^r_{(j^r_x\sigma, u_x)} = \left\{ X^r_{(j^r_x\sigma, u_x)} \mid \forall X \in \mathfrak{X}(M) \right\} .$$

PROPOSITION 3.1. *$\mathcal{D}^r$ is an involutive differential system everywhere of constant rank* (i.e., *$\mathcal{D}^r$ is non-singular*).

PROPOSITION 3.2. *rth-order differential invariants can be seen as the first integrals of $\mathcal{D}^r$ which are also invariant under the action of G on $E^r(M)$. Hence invariants are the functions $F : E^r(M) \to \mathbb{R}$ such that:*
1. $X^r(F) = 0$, $\forall X \in \mathfrak{X}(M)$,
2. $F(j^r_x\sigma, u_x \cdot g) = F(j^r_x\sigma, u_x)$.

3.5 – Fibering $E^r(M)$ over $F^{r+1}(M)$

DEFINITION 3.4 (The bundle of rth-order frames). For every $r \geq 1$, we set

$$F^r(M) = \{ j^r_0\varphi \mid \varphi : \mathbb{R}^n \to M,\ \varphi_*(0) \text{ is an isomorphism} \} .$$

LEMMA 3.3. *Let $\exp_\sigma : U_x \subseteq T_x(M) \to M$ be the exponential of the functorial connection attached to the G-structure σ. Then, $j^{r+1}_{0_x}(\exp_\sigma)$ only depends on $j^r_x\sigma$.*

DEFINITION 3.5. We can thus define a projection $\xi^r_M : E^r(M) \to F^{r+1}(M)$ by $\xi^r_M(j^r_x\sigma, u_x) = j^{r+1}_0(\exp_\sigma \circ u_x)$.

THEOREM 3.4. *The bundle $\xi^r_M : E^r(M) \to F^{r+1}(M)$ is trivial: $E^r(M) \cong F^{r+1}(M) \times S^r$. The standard fibre S^r of ξ^r_M is $S^r = (\xi^r_{\mathbb{R}^n})^{-1}(j^{r+1}_0(\mathrm{id}_{\mathbb{R}^n}))$.*

3.6 – From $\mathrm{Diff}(M)$-invariance to G-invariance

We first remark the following basic facts:

1: The group of diffeomorphisms of M act on $E^r(M)$ by

$$f \cdot (j^r_x\sigma, u_x) = \left(\bar{f}^r(j^r_x\sigma), \tilde{f}(u_x) \right) .$$

2: By transporting the action of $\mathrm{Diff}(M)$ to $F^{r+1}(M) \times S^r$ via $E^r(M) \cong F^{r+1}(M) \times S^r$, we have
   2.a: $\mathrm{Diff}(M)$ acts trivially on S^r,
   2.b: $\mathrm{Diff}(M)$ acts naturally on $F^{r+1}(M)$.

3: By transporting the action of G on $E^r(M)$ to $F^{r+1}(M) \times S^r$, we obtain
$$\left(j_0^{r+1}\varphi, j_0^r\tau\right) \cdot g = \left(j_0^{r+1}\left(\varphi \circ g\right), j_0^r\left(\bar{g}^{-1} \circ \tau \circ g\right)\right),$$
where $g : \mathbb{R}^n \to \mathbb{R}^n$ is the linear diffeomorphism induced by $g \in G$, $j_0^{r+1}\varphi \in F^{r+1}(M)$ is a $(r+1)$th frame, and τ is a G-structure defined on a neigbourhood of the origin in $\mathbb{R}^n$.
As a consequence of these properties we can conclude

THEOREM 3.5. *The ring of rth order differential invariants on a manifold of dimension n can be identified to the ring of G-invariant functions on the standard fibre; that is, to the ring $C^\infty(S^r)^G$.*

3.7 – Frame depending invariants

DEFINITION 3.6. An *rth order frame-depending differential invariant* is a differentiable function $F : E^r(M) \to \mathbb{R}$, such that

$$X^r(F) = 0, \ \forall X \in \mathfrak{X}(M).$$

REMARK 3.3. Differential invariants are frame-depending differential invariants which are also G-invariant.

EXAMPLE 3.1. Let ∇ be the linear connection functorially attached to σ. We can define functions:

$$R^i_{hklj_1\cdots j_p} : E^r(M) \to \mathbb{R}, \quad r \geq 2, \ 0 \leq p \leq r-2,$$

$$R^i_{hklj_1\cdots j_p}\left(j_x^r\sigma, u_x\right) = \omega_i\left(\left(\nabla^p R\right)_x\left(X_h, X_k, X_l, X_{j_1}, \ldots, j_p\right)\right)$$
with $R = \operatorname{curv}\nabla$, $u_x = (X_1, \ldots, X_n)$, $\omega_i(X_j) = \delta_{ij}$ (dual coframe).

PROPOSITION 3.6. $R^i_{hklj_1\ldots j_p}$ *is a frame-depending invariant.*

REMARK 3.4. A similar construction works for the torsion tensor, giving rise to functions

$$T^i_{jkl_1\ldots l_p} : E^r(M) \to \mathbb{R}, \ 0 \leq p \leq r-1.$$

Theorem 3.7. *We have*

1. *Two points* $(j^r_x\sigma, u_x)$, $(j^r_{x'}\sigma', u'_{x'})$ *in* $E^r(M)$ *are* $\mathrm{Diff}(M)$*-equivalent if and only if all frame-depending invariants of order* $\leq r$ *coincide on the given points.*
2. *Every frame-depending invariant of order* $\leq r$ *is a differentiable function of* $T^i_{jkl_1\ldots l_p}$, $p \leq r-1$, *and* $R^i_{hklj_1\ldots j_p}$, $p \leq r-2$.
3. *The number of functionally independent functions* $T^i_{jkl_1\ldots l_p}$, $p \leq r-1$, *and* $R^i_{hklj_1\ldots j_p}$, $p \leq r-2$, *is*

$$n + m + \binom{n+r}{r}\left(\frac{n^2 r}{r+1} - n - m\right),$$

with $n = \dim M$, $m = \dim G$.

4 – Some examples

4.1 – Linear parallelisms

Let M be a connected manifold. As we have remarked above, $\{1\}$-structures are linear parallelisms: $\sigma = (X^1, \ldots, X^n) : M \to F(M)$. Let us denote by A_r the ring of rth-order differential invariants on $F(M)$ and by A'_r the ring of rth-order infinitesimal differential invariants. Hence $A_r \subseteq A'_r$. Let ∇ be the flat connection associate with σ; i.e., $\nabla_{X_i}X_j = 0$, $1 \leq i \leq n$, $1 \leq j \leq n$. We can define functions $f^i_{j_1\ldots j_r kl} : J^{r+1}(F(M)) \to \mathbb{R}$ by setting

$$(\nabla^r \mathrm{Tor}_\nabla)\left(X^{j_1}_x, \ldots, X^{j_r}_x, X^k_x, X^l_x\right) = \sum_i f^i_{j_1\ldots j_r kl}\left(j^{r+1}_x\sigma\right) X^i_x.$$

Theorem 4.1 (cf. [19]). *With the above notations we have*

1. *The functions* $f^i_{j_1\ldots j_r kl}$ *are differential invariants.*
2. *The family* $\mathcal{F}_r$ *of functions* $f^i_{j_1\ldots j_m kl}$ *such that* $0 \leq m \leq r-1$, $k < l$, $j_1 \geq \ldots \geq j_m \geq k$, *is functionally independent.*
3. *Every* r*th-order differential invariant can be locally written as a differentiable function of the functions in* $\mathcal{F}_r$.
4. *The number of functions in* $\mathcal{F}_r$ *is*

$$N_{n,r} = \#\mathcal{F}_r = n^2\binom{n+r}{r} + n - n\binom{n+r+1}{r+1}.$$

Furthermore, let $\pi^r : J^r(F(M)) \to \mathbb{R}^{N_{n,r}}$ *be the mapping whose components are the functions of* $\mathcal{F}_r$. *Then,*

(i) *If* M *is non-orientable,* $A_r = A'_r = (\pi^r)^* C^\infty(\mathbb{R}^{N_{n,r}})$.

(ii) *If* M *is orientable and non-reversible,* $A_r = A'_r = (\pi^r)^* C^\infty(\mathbb{R}^{N_{n,r}}) \oplus (\pi^r)^* C^\infty(\mathbb{R}^{N_{n,r}})$.

(iii) *If* M *is reversible,* $A_r = (\pi^r)^* C^\infty(\mathbb{R}^{N_{n,r}})$, $A'_r = A_r \oplus A_r$ *and the injection* $A_r \hookrightarrow A'_r$ *is the diagonal map.*

REMARK 4.1. We recall that the manifold M is said to be reversible if M orientable and M admits an orientation reversing diffeomorphism. For example, $\mathbb{C}P^{2k}$ is non-reversible for every positive $k \in \mathbb{N}$ (cf .[33]). In fact, every connected, oriented manifold M of dimension $\dim M = 2l$, satisfying

1. $\dim H^l(M; \mathbb{R}) = 1$,
2. there exists a closed l-form ω_l such that $[\omega_l]$ is a basis for$H^l(M; \mathbb{R})$, and $\omega_l \wedge \omega_l \neq 0$ everywhere,

   is non-reversible.

4.2 – Projective parallelisms

As we have seen in the example (f) of §2.1, projective parallelisms correspond in a natural way with $\mathbb{R}^*$-structures (cf. [12], [24]) so that the clasifying bundle in this case is $\bar{\pi} : F(M)/\mathbb{R}^* \longrightarrow M$. This is a principal bundle with structure group the full projective group $PGl(n, \mathbb{R}) = Gl(n, \mathbb{R})/\mathbb{R}^+$. We have

THEOREM 4.2 (cf. [38], [39]). *Set* $n = \dim M$. *Then,*

1. *If* $n \geq 3$ *the number of functionally independent invariants on a dense open subset of* $J^r(F(M)/\mathbb{R}^*)$ *is*

$$n + \binom{n+r}{r} \frac{rn^2 - (r+1)(n+1)}{r+1}.$$

2. *If* $n = 2$, *then each differential invariant of first or second order is a constant and for every* $r \geq 3$ *the number of functionally independent invariants on a dense open subset of* $J^r(F(M)/\mathbb{R}^*)$ *is* $(r+1)(r-2)/2$.

4.3 – The metric case

Without doubt $O(n)$-structures and $U(n)$-structures have been the geometries most extensively studied not only in determining scalar differential invariants but also in determining differential forms which depend in a natural way on a Riemannian metric. The literature on this topic is enormous. Among other works and authors, see [2], [3], [13], [16], [20], [21], [23], [25, VI], [26], [27], [28], [34], [38], [42].

Let $O(n^+,n^-)$, $\dim M = n = n^+ + n^-$, be the orthogonal group of the standard quadratic form of signature (n^+,n^-). The classifying bundle of the $O(n^+,n^-)$-structures is $\bar{\pi} : \mathcal{M} = F(M)/O(n^+,n^-) \to M$, which can be canonically identified to the bundle of pseudo-Riemannian metrics on M of signature (n^+,n^-). Functions on $J^r(\mathcal{M})$ which are invariant under diffeomorphisms of M are called metric invariants of order r. They are specially important in General Relativity. The most famous of them is the scalar curvature, which is a second order invariant giving rise to the Hilbert-Einstein Lagrangian density. We have

THEOREM 4.3 (cf. [31]). *Let $i_{n,r}$ be the number of functionally independent metric invariants defined on an open dense subset of the r-jet bundle of metrics of a given signature on an n-dimensional manifold. Then,*

1. *For every $n \geq 1$, $i_{n,0} = i_{n,1} = 0$.*
2. *For every $r \geq 0$, $i_{1,r} = 0$.*
3. $i_{2,2} = 1$ *and for every $r \geq 3$,* $i_{2,r} = \frac{1}{2}(r+1)(r-2)$ (cf. Theorem 4.2–2).
4. *For every $n \geq 3$, $r \geq 2$,*

$$i_{n,r} = n + \frac{(r-1)\,n^2 - (r+1)\,n}{2\,(r+1)} \binom{n+r}{r}.$$

Another problem concerning metric invariants is the following. Let $g = \sum g_{ij} dx_i \otimes dx_j$ be a Riemannian metric on M, ∇ its Levi-Civita connection and R the curvature tensor. Given non-negative integers $\ell_1, \dots, \ell_r$, such that: $\ell_1 + \ldots + \ell_r = 2k$, and covariant indices $1 \leq i_1 < \ldots < i_{k+r} \leq 2k + 3r$, we can apply the isomorphism $g^\sharp : T^*M \to TM$, to the tensor field $(\nabla^{\ell_1} R \otimes \ldots \otimes \nabla^{\ell_r} R)$ of type $(r, 2k+3r)$, thus obtaining a tensor field ${}^{g^\sharp}(\nabla^{\ell_1} R \otimes \ldots \otimes \nabla^{\ell_r} R)^{i_1,\dots,i_{k+r}}$ of type $(k+2r, k+2r)$. If

$j_1, \dots, j_{k+2r}$ is a permutation of $1, \dots, k+2r$, we can finally define an scalar by

$$W_g = c^1_{j_1} \cdots c^{k+2r}_{j_{k+2r}} \left({}^{g^\sharp} \left(\nabla^{\ell_1} R \otimes \ldots \otimes \nabla^{\ell_r} R \right)^{i_1, \dots, i_{k+r}} \right),$$

where c^i_j is the contraction of the i-th contravariant index with the j-th covariant one. The functions constructed in this way are called *Weyl invariants* (cf. [3], [15], [17], [43] and it is not difficult to see that they are invariant under diffeomorphisms (e.g., see [31]). In fact, taking into account that R only depends on the derivatives of order ≤ 2 of the g_{ij}'s and that the local coefficients Γ^k_{ij} of ∇ only depend on the first derivatives of the g_{ij}'s, it follows from the definition that $W_g(x)$ only depends on the derivatives of order $\leq \ell + 2$ of the metric, where $\ell = \max\{\ell_1, \dots, \ell_r\}$; i.e., only depends on $j^{\ell+2}_x(g)$. Therefore, we can define a function $W : J^{\ell+2}(\mathcal{M}) \to \mathbb{R}$ by the formula: $W(j^{\ell+2}_x g) = W_g(x)$, and such a function is invariant under the natural action of diffeomorfisms of M on the jet bundle. If we start with the classical notion of an invariant ([3], [6], [13], [15]), according to which an invariant of order r is a polynomial function $P(g_{ij}, \partial^{|\alpha|} g_{ij}/\partial x^\alpha, [\det(g_{ij})]^{-1})$, $|\alpha| = 1, \dots, r$, then, working in normal coordinates and using classical invariant theory for $O(n)$ (e.g., see [3]), it can be proved that every scalar polynomial invariant is a linear combination of Weyl invariants, but evidently this is no longer true for arbitrary diferential invariants in the sense of Kumpera [29] (cf. see definition (3.1) above). In this case the reasonable hypothesis is to suppose that the Weyl invariants span "differentiably" the ring of all differential invariants; that is, if $W_1, \dots, W_N$ are all distinct Weyl's invariants of order $\leq r$, then for every r-th order metric differential invariant F there exists $G \in C^\infty(\mathbb{R}^N)$ such that $F = G \circ (W_1, \dots, W_N)$. In the last few years several works have been published concerning the number of linerary independent Weyl's invariants (e.g., see [17], [23], [43]). Such works are interesting to determine the minimal basis of Weyl's invariants, but they are no link with the problem that we are dealing with here.

By using the fundamental distribution $\overline{\mathcal{D}}^r$ introduced above we have been able to prove the following result:

Theorem 4.4. *Let* $V^r \subseteq \oplus_{i=0}^{r-2} S^{i+4}((\mathbb{R}^n)^*)$, $r \geq 2$, *the vector subspace of the tensors* $(R^{(0)}, R^{(1)}, \dots, R^{(r-2)})$ *of type* $(0,4)$, $(0,5)$,… ,

$(0, r+2)$, *respectively that satisfy the symmetries* (1.3) – (1.6) *of* [27, Theorem 1.1] *and let* $\mathcal{R}$ *be the Riemann-Christoffel tensor of* g*;* i.e., $\mathcal{R}$ *is the tensor field of type* $(0,4)$ *obtained by lowering the contravariant index of the curvature tensor* R*. As the covariant derivatives of the curvature tensor satisfy the symmetries defining* V^r*, we can define a mapping* $\varphi_r : S^r \to V^r$*,* S^r *being the standard fibre of the fibre bundle* $\xi^r_M : E^r(M) \to F^{r+1}(M)$ *introduced in* Theorem 3.4*, as follows* (cf. [27])*:*

$$\varphi_r\left(j_0^r g\right) = \left(\mathcal{R}_0, \left(\nabla^1\mathcal{R}\right)_0, \dots, \left(\nabla^{r-2}\mathcal{R}\right)_0\right).$$

Then, we have

1. *The mapping* φ_r *is an equivariant diffeomorphism under the natural actions of the orthogonal group.*
2. *If* $P_1, \dots, P_N$ *is a basis of the subring of invariants with respect to the action of* $O(n)$ *on the ring of polynomials on* V^r*;* i.e.,

$$\mathbb{R}\left[P_1, \dots, P_N\right] = \left(S^\bullet\left(V^r\right)^*\right)^G,$$

then, taking into account the identification of Theorem (3.5)*, every metric differential invariant on* M *of order* $\leq r$ *can be written as a differentiable function of* $P_1 \circ \varphi_r, \dots, P_N \circ \varphi_r$*.*

REFERENCES

[1] D. V. Alekseevsky – S. Marchiafava: *Quaternionic-like structures on a manifold: I, II*, Rend. Mat. Acc. Lincei, s. 9, **4** (1993), 43-52.

[2] I. M. Anderson: *Natural Variational Principles on Riemannian Manifolds*, Annals of Math., **120** (1984), 329-370.

[3] M. Atiyah – R. Bott – V. K. Patodi: *On the Heat Equation and the Index Theorem*, Inventiones Math., **19** (1973), 279-330.

[4] T. N. Bailey: *Thomas' D-Calculus, Parabolic Invariant Theory, and Conformal Invariants*, in: Twistor Theory, (S. Huggett, ed.), Lect. Not. in Pure and Applied Math., **169**, Marcel Dekker, Inc., N.Y., 1995, pp. 1-7.

[5] T. N. Bailey – M. G. Eastwood – C. R. Graham: *Invariant Theory for Conformal and CR Geometry*, Annals of Math., **139** (1994), 491-552.

[6] R. J. Baston: *Verma modules and differential conformal invariants*, J. Diff. Geometry, **32** (1990).

[7] D. Bernard: *Sur la Géometrie Différentielle des G-structures*, Ann. Inst. Fourier, Grenoble, **10** (1960), 151-270.

[8] W. Blaschke: *Einführung in die Geometrie der Waben*, Birkhäuser-Verlag, Basel-Stutgart 1955.

[9] R. L. Bryant – P. A. Griffiths: *Characteristic cohomology of differential systems II: conservation laws for a class of parabolic equations. Appendix 1. The equivalence method*, Duke Math. J., **78**, No. **3** (1995), 531-676.

[10] E. Cartan: *Les groupes de transformations, continus, infinis, simples*, Ann. Sci. École Norm. Sup., **26** (1909), 93-161.

[11] D. Cerveau: *Distributions involutives singulères*, Ann. Inst. Fourier, Grenoble, **29**, n° 3, (1979), 261-294.

[12] S. S. Chern: *The Geometry of G-structures*, Bull. AMS, **72** (1966), 167-219.

[13] D. B. A. Epstein: *Natural Tensors on Riemannian Manifolds*, J. Differential Geom., **10** (1975), 631-645.

[14] Ch. Fefferman: *Parabolic Invariant Theory in Complex Analysis*, Adv. Math., **31** (1979), 131-262.

[15] Ch. Fefferman – C. R. Graham: *Conformal Invariants*, Astérisque, (hors série) (1985), 95-116.

[16] A. Ferrández – V. Miquel: *Hermitian Natural Tensors*, Math. Scand., **64** (1989), 233-250.

[17] S. A. Fulling – R. C. King – B. G. Wybourne – C. J. Cummins: *Normal forms for tensor polynomials: I. The Riemann tensor*, Clas. Quantum Grav. **9** (1992), 1151-1197.

[18] P. M. Gadea – J. Muñoz Masqué: *A-differentiability and A-analyticity*, Proc. Amer. Math. Soc., **124** (1996), 1437-1443.

[19] P. L. García Pérez – J. Muñoz Masqué: *Differential invariants on the bundles of linear frames*, J. Geom. Phys., **7**, n° 3 (1990), 395-418.

[20] P. B. Gilkey: *Local invariants of a pseudo-Riemannian manifold*, Math. Scand., **36** (1975), 109-130.

[21] P. B. Gilkey: *Smooth invariants of a Riemannian manifold*, Advances in Math., **28** (1978), 1-10.

[22] V. V. Goldberg: *On a linearizability condition for a three-web on a two-dimensional manifold*, Lect. Notes in Math., **1410** (1989), 223-239.

[23] R. Klinger: *A Basis that Reduces to Zero as many Curvature Components as Possible*, Abh. Math. Sem. Univ. Hamburg, **61** (1991), 243-248.

[24] S. Kobayashi: *Transformation Groups in Differential Geometry*, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[25] S. Kobayashi – K. Nomizu: *Foundations of Differential Geometry*, John Wiley & Sons (Interscience Division), New York, Vol. I, 1963; Vol. II, 1969.

[26] I. Kolář – P. W. Michor – J. Slovak: *Natural Operations in Differential Geometry*, Springer-Verlag, 1993.

[27] O. Kowalski – M. Belger: *Riemannian Metrics with the Prescribed Curvature Tensor and All its Covariant Derivatives at One Point*, Math. Nachr., **169** (1994), 209-225.

[28] D. Krupka – A. Mikolášová: *On the uniqueness of some differential invariants:* d, $[\,,]$, ∇, Czechoslovak Math. J., **34** (1984), 588-597.

[29] A. Kumpera: *Invariants différentiels d'un pseudo groupe de Lie, I, II*, J. Diff. Geom., **10** (1975), 289-345, 347-416.

[30] A. Lichnerowicz: *Global theory of connections and holonomy groups*, Noordhoff International Publishing, Leyden, The Netherlands, 1976.

[31] J. Muñoz Masqué – A. Valdés Morales: *The number of functionally independent invariants of a pseudo-Riemannian metric*, J. Phys. A: Math. Gen., **27** (1994), 7843-7855.

[32] J. Muñoz Masqué – A. Valdés Morales: *Génération des anneaux d'invariants différentiels des métriques riemanniennes*, C.R. Acad. Sci. Paris, **323**, Série I (1996), 643-646.

[33] R. S. Palais: *Extending diffeomorphisms*, Proc. Amer. Math. Soc., **11** (1960), 274-277.

[34] F. Prufer – F. Tricerri – L. Vanhecke: *Curvature invariants, differential operators and local homogeneity*, Trans. Amer. Math. Soc., to appear.

[35] S. Salamon: *Differential geometry of quaternionic manifolds*, Ann. Scient. Ec. Norm. Sup., $4^{\text{ème}}$ série, **19** (1986), 31-55.

[36] V. V. Shurygin: *Manifolds over algebras and their application to the geometry of jet bundles*, Uspekhi Mat. Nauk, **48**:2 (1993), 75-106 (Russian Math. Surveys, **48**:2 (1993), 75-104).

[37] A. Sudbery: *Quaternionic Analysis*, Math. Proc. Cambridge Phil. Soc., **85** (1979), 199-225.

[38] T. Y. Thomas: *The Differential Invariants of Generalized Spaces*, Cambridge University Press, 1934.

[39] A. Valdés: *Differential Invariants of* $\mathbb{R}^*$ *-Structures*, Math. Proc. Cambridge Philos. Soc., **119** (1996), 341-356.

[40] W. C. Waterhouse: *Analyzing some generalized analytic functions*, Expositiones Math., **10** (1992), 183-192.

[41] W. C. Waterhouse: *Differentiable functions on algebras and the equation* $\mathrm{grad}(w) = M\mathrm{grad}(v)$, Proc. Roy. Soc. Edinburgh, **122** (1992), 353-361.

[42] J. E. Wright: *Invariants of Quadratic Differential Forms*, Cambridge University Press Warehouse, Cambridge, 1908.

[43] B. G. Wybourne – J. Meller: *Enumeration of the order*−14 *invariants formed from the Riemann tensor*, J. Phys. A: Math. Gen., **25** (1992), 5999-6003.

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