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Harmonic coordinates, harmonic radius and convergence of Riemannian manifolds

E. HEBEY – M. HERZLICH

Presentazione: *Questo articolo di rassegna presenta lo stato attuale delle ricerche sulla compattezza degli spazi di varietà riemanniane. Si tratta di un importante settore di ricerca, che ha avuto uno sviluppo notevole negli ultimi quindici anni e che ha applicazioni in un ambito assai ampio.*

Nel 1981 M. Gromov stabilì un risultato fondamentale: lo spazio delle varietà riemanniane compatte, con curvatura sezionale limitata, con volume limitato inferiormente e con diametro limitato superiormente, è precompatto nella topologia di Lipschitz. Da allora molti sono stati i lavori che hanno riguardato questioni di convergenza di metriche riemanniane, sotto varie ipotesi di limitatezza di alcuni invarianti metrici fondamentali ed anche rispetto a diverse topologie.

L'articolo, rivolgendosi a un vasto pubblico matematico, è autosufficiente: le prime due sezioni del lavoro sono dedicate alla presentazione del materiale di base e delle nozioni fondamentali; in particolare viene introdotta la nozione cruciale di raggio armonico, legata all'esistenza di sfere geodetiche dove sono definite coordinate armoniche con controllo del tensore metrico.

*I risultati più importanti sono riassunti in un teorema (*Main theorem*) che generalizza alcuni risultati di M. Anderson e di J. Cheeger. Sono anche riportati alcuni teoremi analoghi, validi per domini limitati e per varietà riemanniane puntate complete. L'ultima parte del lavoro è dedicata alla esposizione di alcune significative applicazioni.*

Abstract: *From the appearance of the original works of Cheeger and Gromov on convergence of Riemannian manifolds, the field has considerably evolved. We sum up the question.*

Key Words and Phrases: *Convergence of Riemannian manifolds – Compactness theorems – Harmonic coordinates – Harmonic radius.*
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1 – Introduction and statement of the main theorem

In 1981, M. Gromov stated a striking result about compactness of sets of Riemannian manifolds: given $n \in \mathbb{N}$, $\lambda \geq 0, v > 0, d \geq 0$, the space of compact Riemannian n-manifolds satisfying $|K| \leq \lambda, \text{diam} \leq d, \text{vol} \geq v$ (where K stands for the sectional curvature of the manifold, diam and vol for its diameter and volume) is precompact in the Lipschitz topology. Since 1981, many articles were published in the subject —let us mention the works of M. Anderson [3], [5], M. Anderson and J. Cheeger [4], L. Z. Gao [19], [20], [21], R. Greene and H. Wu [23], J. Jost and H. Karcher [29], A. Kasue [30], A. Katsuda [31], S. Peters [33], [34] and D. Yang [35], [36], [37]— and the result aforementioned has been substantially improved in two different ways: on one hand, one only needs bounds on the Ricci curvature rather than the whole Riemann curvature tensor; on the other hand, we now have precompactness in the $C^{k,\alpha}$-topology, providing much more information than in the rather weak Lipschitz topology, especially useful to those interested in global analysis on manifolds. These results provide us with powerful and precise tools to control the local as well as global geometry of manifolds and they had (and still have) significant applications in Riemannian Geometry or Topology (we refer to section 6 for some of these applications). But we think they could prove very useful in a larger setting, including Nonlinear Analysis and Partial Differential Equations.

The purpose of this paper is then to provide an account of the present state of the field as well as a detailed presentation of the proofs. In order to make the techniques available for a broad mathematical audience, we have tried to make the article as much self-contained as possible: we hope it should be understandable for anyone having only a slight acquaintance with Riemannian metrics and Riemannian Geometry (many good books exist on the subject; let us mention the well-known and excellent [18] and the recent [28]). From its appearance, the field has considerably evolved: whereas the seek for optimal control (or: optimal regularity) on the convergence seems to have come to a climax, the weak (i.e. Lipschitz or Hausdorff) properties of sets of Riemannian metrics are still under intense scrutiny (for a better look at these points, the reader is referred to the recent works of J. Cheeger and T. Colding [12], [13] which will convince him of the wealth of the approach) and are probably far from being completely exhausted. Describing all these developments (and also the structures that occur when some sequences of metrics degenerate) would have taken us too far from our original goal; it's the reason why we have chosen here to report only on the regular side of the theory.

For the sake of simplicity, we decided to focus on a single theorem on compact manifolds (a straightforward generalization of the results of M. Anderson and J. Cheeger [3], [4]) and to stress on some important points (underlying in Anderson and Cheeger's work) which we thought deserved some more attention and further developments.

The paper is organized as follows:

Section 1 is devoted to the presentation of the ground material of this article: definition of the $C^{k,\alpha}$-toplogies and statement of the Main theorem.

Section 2 recalls the basic properties of harmonic coordinates and introduces the crucial notion of $C^{k,\alpha}$- and H_k^p-harmonic radius.

In section 3 comes the heart of the paper. Detailing and extending slightly the work of M. Anderson and J. Cheeger, we derive estimates on the H_k^p-harmonic radius from bounds on the Ricci curvature and injectivity radius. We are then able to state the estimates in a very general form. Though this can be obtained by quite standard bootstrap arguments from Anderson and Anderson-Cheeger's works, they seem to have never appeared in the literature so far.

Section 4 shows how estimates on the harmonic radius implies compactness theorems (following a path known since the pioneer work of J. Cheeger [10]). This provides the Main theorem.

Section 5 lists some analogues of the Main theorem for bounded domains or complete non-compact pointed manifolds, including a local version which is of special interest for nonlinear analysts.

Section 6 eventually presents various applications of compactness or convergence results.

Let us now start with some basic definitions.

Definition 1. Let $n \in \mathbb{N}, \mathrm{k} \in \mathbb{N}, \alpha \in (0,1)$, (M_j, g_j) be a sequence of smooth compact Riemannian n-manifolds, M a smooth compact differentiable n-manifold, and g a $C^{k,\alpha}$ Riemannian metric on M. We say that (M_j, g_j) converges to (M, g) in the $C^{k,\alpha}$-topology if there exists j_o such that the following holds: for any $j \geq j_o$ there exist $C^{k+1,\alpha}$ diffeomorphisms $\Phi_j : M \to M_j$ such that in any chart of the C^∞ complete atlas of M, the components of the metrics $\Phi_j^* g_j$ converge $C^{k,\alpha}_{\mathrm{loc}}$ to the components of g.

Although it is elementary, note that the $C^{k,\alpha}_{\mathrm{loc}}$ convergence in any chart of the C^∞ complete atlas of M is equivalent to the existence of a C^∞ sub-atlas of the C^∞ complete atlas of M, such that in any chart of this sub-atlas, the components of the metrics $\Phi_j^* g_j$ converge $C^{k,\alpha}$ to the components of g. Here, we can look at the components of $\Phi_j^* g_j$ and g as functions defined in some open subset Ω of $\mathbb{R}^n$, and we say that a sequence f_j of $C^{k,\alpha}$ functions defined in Ω converges $C^{k,\alpha}$ to some $C^{k,\alpha}$ function f defined in Ω if $\lim_{j\to\infty} ||f_j - f||_{k,\alpha} = 0$ where

$$||f||_{k,\alpha} = \sum_{0\le|\beta|\le k} \sup_{x\in\Omega} |\partial_\beta f(x)| + \sum_{|\beta|=k} \sup_{x\neq y} \frac{|\partial_\beta f(x) - \partial_\beta f(y)|}{|y-x|^\alpha} .$$

Definition 2. Let $n \in \mathbb{N}, \mathrm{k} \in \mathbb{N}, \alpha \in (0,1)$, and S be some set of smooth compact Riemannian n-manifolds. We say that S is precompact in the $C^{k,\alpha}$-topology if any sequence in S possesses a subsequence which converges in the $C^{k,\alpha}$-topology.

Let us now state the Main theorem of this report. If (M,g) is a Riemannian manifold, $\mathrm{Ric}_{(M,g)}$ denotes its Ricci curvature, $\mathrm{inj}_{(M,g)}$ its injectivity radius, and $\mathrm{vol}_{(M,g)}$ its volume. $D^j \mathrm{Ric}_{(M,g)}$ denotes the j-th covariant derivative of $\mathrm{Ric}_{(M,g)}$. The $C^{0,\alpha}$-part of the theorem has been obtained by Anderson-Cheeger [4], the $C^{1,\alpha}$-part by Anderson [3].

Main theorem. *Let $n \in \mathbb{N}, \lambda \in \mathbb{R}, \mathrm{i} > 0$, and $v > 0$. The space of smooth compact Riemannian n-manifolds (M,g) satisfying*

$$\mathrm{Ric}_{(M,g)} \ge \lambda,\ \mathrm{inj}_{(M,g)} \ge i,\ \mathrm{vol}_{(M,g)} \le v ,$$

is precompact in the $C^{0,\alpha}$-topology for any $\alpha \in (0,1)$. In addition, if instead of the bound $\mathrm{Ric}_{(M,g)} \ge \lambda$ we assume that for some $k \in \mathbb{N}$ and some positive constants $C(j)$,

$$|D^j \mathrm{Ric}_{(M,g)}| \le C(j), \quad \forall\, j = 0, ..., k ,$$

then, it is precompact in the $C^{k+1,\alpha}$-topology for any $\alpha \in (0,1)$.

REMARKS. 1) We stated definition 1 in the class of smooth manifolds, but the real framework for $C^{k,\alpha}$ convergence is the class of $C^{k+1,\alpha}$ manifolds. Namely, let (M_j) be a sequence of $C^{k+1,\alpha}$ compact n-manifolds, M a $C^{k+1,\alpha}$ compact n-manifold, (g_j) a sequence of $C^{k,\alpha}$ Riemannian metrics on M_j, and g a $C^{k,\alpha}$ Riemannian metric on M. We say that (M_j, g_j) converges to (M, g) in the $C^{k,\alpha}$-topology if for large values of j, there exist $C^{k+1,\alpha}$ diffeomorphisms $\Phi_j : M \to M_j$ such that in any chart of the $C^{k+1,\alpha}$ complete atlas of M, the components of the metrics $\Phi_j^* g_j$ converge $C^{k,\alpha}_{\text{loc}}$ to the components of g. Here again, this is equivalent to the existence of a $C^{k+1,\alpha}$ sub-atlas of the $C^{k+1,\alpha}$ complete atlas of M, such that in any chart of this sub-atlas, the components of the metrics $\Phi_j^* g_j$ converge $C^{k,\alpha}$ to the components of g.

2) Under the bounds $\text{Ric}_{(M,g)} \geq \lambda$ and $\text{inj}_{(M,g)} \geq i$ of the Main theorem, an elementary packing argument (CROKE [16]) shows that the bound $\text{vol}_{(M,g)} \leq v$ is equivalent to a diameter bound $\text{diam}_{(M,g)} \leq d$. Independently, under the bound $|K_{(M,g)}| \leq \Lambda$, the bounds $\text{inj}_{(M,g)} \geq i$ and $\text{vol}_{(M,g)} \leq v$ of the Main theorem are equivalent to the bounds $\text{vol}_{(M,g)} \geq v'$ and $\text{diam}_{(M,g)} \leq d$ of the first version of the Gromov convergence theorem mentioned above (CHEEGER-GROMOV-TAYLOR [15]).

3) Let M be a smooth compact Riemannian n-manifold and (g_j) a sequence of smooth Riemannian metrics on M. Suppose that the g_j satisfy the bounds of the Main theorem, for instance $\text{Ric}_{(M,g_j)} \geq \lambda, \text{inj}_{(M,g_j)} \geq i, \text{vol}_{(M,g_j)} \leq v$. Since $M_j = M$ for all j, it is tempting to assert that there exist a $C^{0,\alpha}$ Riemannian metric g on M and a subsequence of (g_j) such that in any chart of the C^∞ complete atlas of M, the components of g_j converge $C^{0,\alpha}$ to the components of g. In other words, with the notations of definition 1, it is tempting to assert that for some subsequence of (g_j) we can take $\Phi_j = \text{Id}$ in the definition of $C^{0,\alpha}$-precompactness. Actually, it is simple to see that this is false in general. Think for instance of (S^n, can), the standard unit sphere of $\mathbb{R}^{n+1}$ with its canonical metric, and set $g_j = f_j^* can$ where the f_j, belonging to the conformal group of (S^n, can), are defined in some given stereographical model by $f_j(y) = jy$. Since the g_j's are isometric to the canonical metric of S^n, they trivially satisfy the bounds of the Main theorem. Independently, if x is the pole of the stereographical projection we consider, we have that for any $y \neq -x$, $\lim_{j\to\infty} f_j(y) = x$, while we have that for any $j, f_j(-x) = -x$.

It is then easy to see that no subsequence of (g_j) converges $C^{0,\alpha}$. On the other hand, the sequence (g_j) converges to can in the $C^{0,\alpha}$-topology since, by construction, for the smooth diffeomorphisms $\Phi_j = f_j^{-1} : S^n \to S^n$, we have that $\Phi_j^* g_j = can$ for all j.

4) The $C^{k,\alpha}$ result on the limit metric is sharp. For instance, PETERS [34] has presented a simple example of a limit metric which is $C^{1,\alpha}$ for any $\alpha < 1$ but not $C^{1,1}$. We refer to [34] for more details. On the other hand, improvements of the convergence are possible in terms of Sobolev spaces. See for instance ANDERSON-CHEEGER [4], PETERS [34], and what is done below.

2 – Harmonic coordinates and harmonic radius

Harmonic coordinates were first used by Einstein, then by Lanczos who observed that they simplify the formula for the Ricci tensor. Namely, in a harmonic coordinate system,

$$(\mathrm{Ric}_{(M,g)})_{ij} = -\, g^{mk}\partial_k\partial_m g_{ij} + \\ + \text{“terms involving at most one derivative of the metric”}.$$

From the works of DETURCK-KAZDAN [17], it is now well known that we obtain optimal regularity by using harmonic coordinates, while the basic intuition that one obtains optimal regularity by using geodesic normal coordinates is false in general. In this very short section, we recall the definition of harmonic coordinates, we give the main properties they satisfy, and we briefly introduce the crucial notion of harmonic radius. For more details on what is done here, one should look at DETURCK-KAZDAN [17] and JOST-KARCHER [29].

First, we recall the definition of harmonic coordinates.

DEFINITION 3. A coordinate chart $(x^1, ..., x^n)$ on a Riemannian manifold (M, g) is called harmonic if $\Delta x^k = 0$ for all $k = 1, ..., n$. Since $\Delta x^k = g^{ij}\Gamma_{ij}^k$, where the Γ's are the Christoffel symbols of the connection associated to g, we get that a coordinate chart $(x^1, ..., x^n)$ is harmonic if and only if for any $k = 1, ..., n$,

$$g^{ij}\,\Gamma_{ij}^k = 0\,.$$

Then, we have the following result of DeTurck-Kazdan [17, lemma 1.2, theorem 2.1], which, roughly speaking, states that we get optimal regularity in harmonic coordinates. As already mentioned, this is not true for geodesic normal coordinates.

Theorem 4. *Let the metric on a Riemannian manifold* (M, g) *be of class* $C^{k,\alpha}(k \geq 1)$ *in a local coordinate chart about some point* x. *Then, there is a neighborhood of* x *in which harmonic coordinates exist, these new coordinates being* $C^{k+1,\alpha}$ *functions of the original coordinates. Moreover:*

1) *we can choose this coordinate system such that* $g_{ij}(x) = \delta_{ij}$ *for any* i, j *in* $1, ..., n$, *where the* $g_{ij}(x)$ *are the components of* g *at* x,

2) *all harmonic coordinate charts defined near* x *have this* $C^{k+1,\alpha}$ *regularity,*

3) *the metric* g *is of class* $C^{k,\alpha}$ *in any harmonic coordinate chart near* x.

Proof. It is based on the formula

$$\Delta u = -g^{ij}\partial_i\partial_j u - \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}\, g^{ij}\right)\partial_j u$$

($|g|$ stands for the determinant of the matrix g_{ij} in the coordinate chart considered) and on the fact that since g belongs to $C^{k,\alpha}$, there is always a solution u in $C^{k+1,\alpha}$ of $\Delta u = 0$ with $u(x)$ and $\partial_i u(x)$ prescribed. If the $y^j, j = 1, ..., n$, are the solutions of $\Delta y^j = 0$ with $y^j(x) = 0$ and $\partial_i y^j(x) = \delta_i^j$, the functions y^j are the desired harmonic coordinates. Part 2) of the theorem is then a consequence of standard elliptic regularity theorems, while to prove part 3) one just has to note that the expression of any tensor in these coordinates involves at most the first derivatives of these coordinates. Finally, to prove that we can choose harmonic coordinates such that $g_{ij}(x) = \delta_{ij}$, just note that composing with linear transformations do not affect the fact that coordinates are harmonic. □

One of the main differences between geodesic normal coordinates and harmonic coordinates is that under the assumption that g is of class $C^{k,\alpha}$ in a local chart around some point x, the only thing one can say is that g is of class $C^{k-2,\alpha}$ in geodesic normal coordinates at x. In general, one can do no better and changing to geodesic normal coordinates involves a loss of two derivatives. The example 2.3 of DeTurck-Kazdan [17] illustrates this fact. Independently, many other results are available with the use of harmonic coordinates. For instance, it is possible to prove that if in harmonic coordinates $\mathrm{Ric}_{(M,g)} \in C^{k,\alpha}$, then in these coordinates g is $C^{k+2,\alpha}$. We refer to DeTurck-Kazdan [17] for more details.

Let us now define the concept of harmonic radius.

Definition 5. Let (M, g) be a smooth Riemannian n-manifold without boundary and let x in M. Given $Q > 1, k \in \mathbb{N}$, and $\alpha \in (0,1)$, we define the $C^{k,\alpha}$-harmonic radius at x as the largest number $r_H = r_H(Q,k,\alpha)(x)$ such that on the geodesic ball $B_x(r_H)$ of center x and radius r_H, there is a harmonic coordinate chart such that the metric tensor is $C^{k,\alpha}$ controlled in these coordinates. Namely, if $g_{ij}, i,j = 1,...,n$, are the components of g in these coordinates, then

1) $Q^{-1}\,\delta_{ij} \leq g_{ij} \leq Q\,\delta_{ij}$ as bilinear forms,

2) $\sum\limits_{1\leq|\beta|\leq k} r_H^{|\beta|} \sup_x |\partial^\beta g_{ij}(x)| + \sum\limits_{|\beta|=k} r_H^{k+\alpha} \sup_{y\neq z} \frac{|\partial^\beta g_{ij}(y)-\partial^\beta g_{ij}(z)|}{d_g(y,z)^\alpha} \leq Q-1$

where d_g is the distance associated to g. The harmonic radius $r_H(Q,k,\alpha)$ (M) of (M,g) is now defined by $r_H(Q,k,\alpha)(M) = \inf_{x\in M} r_H(Q,k,\alpha)(x)$.

According to theorem 4, the harmonic radius is positive for any fixed smooth compact Riemannian manifold. Note that if (M,g) and (N,h) are isometric, then, for any $Q > 1, k \in \mathbb{N}$, and $\alpha \in (0,1), r_H(Q,k,\alpha)(M) = r_H(Q,k,\alpha)(N)$.

3 – Estimates on the harmonic radius

The purpose of this section is to obtain estimates on the harmonic radius in terms of bounds on the Ricci curvature and the injectivity radius. Roughly speaking, we prove that bounds on the Ricci curvature and the injectivity radius give lower bounds on the harmonic radius. As one can see, the result is essentially local. In particular, this provides us with a very local form of the Main theorem. One just has to use the Arzela-Ascoli theorem. We will insist on that in section 5. Note that the $C^{0,\alpha}$-part of the theorem has been obtained by Anderson-Cheeger [4], the $C^{1,\alpha}$-part by Anderson [3].

Theorem 6. *Let $\alpha \in (0,1), Q > 1, \delta > 0$. Let (M,g) be an arbitrary smooth Riemannian n-manifold without boundary, and Ω an open subset of M. Set*

$$\Omega(\delta) = \{x \in M \text{ s.t. } d_g(x,\Omega) < \delta\}$$

where d_g is the distance associated to g. Suppose that for some $\lambda \in \mathbb{R}$ and $i > 0$, we have that for all $x \in \Omega(\delta)$,

$$\mathrm{Ric}_{(M,g)}(x) \geq \lambda \quad \textit{and} \quad \mathrm{inj}_{(M,g)}(x) \geq i\,,$$

where $\mathrm{inj}_{(M,g)}(x)$ is the injectivity radius at x. Then, there exists a positive constant $C = C(n,Q,\alpha,\delta,i,\lambda)$, depending only on $n,Q,\alpha,\delta,i,$ and λ, such that for any $x \in \Omega$,

$$r_H(Q,0,\alpha)(x) \geq C\,.$$

In addition, if instead of the bound $\mathrm{Ric}_{(M,g)}(x) \geq \lambda$ we assume that for some $k \in \mathbb{N}$ and some positive constants $C(j)$,

$$|D^j \,\mathrm{Ric}_{(M,g)}(x)| \leq C(j) \quad \textit{for all } j = 0,..,k \textit{ and all } x \in \Omega(\delta)\,,$$

then, there exists a positive constant $C = C(n,Q,k,\alpha,\delta,i,C(j),0 \leq j \leq k)$, depending only on $n,Q,k,\alpha,\delta,i,C(j),0 \leq j \leq k$, such that for any $x \in \Omega$,

$$r_H(Q,k+1,\alpha)(x) \geq C\,.$$

Corollary. *Let $\alpha \in (0,1), Q > 1$. Let (M,g) be a smooth complete Riemannian n-manifold. Suppose that for some $\lambda \in \mathbb{R}$ and $i > 0$,*

$$\mathrm{Ric}_{(M,g)} \geq \lambda \quad \textit{and} \quad \mathrm{inj}_{(M,g)} \geq i\,.$$

Then, there exists a positive constant $C = C(n,Q,\alpha,i,\lambda)$, depending only on $n,Q,\alpha,i,$ and λ, such that

$$r_H(Q,0,\alpha)(M) \geq C\,.$$

In addition, if instead of the bound $\mathrm{Ric}_{(M,g)} \geq \lambda$ we assume that for some $k \in \mathbb{N}$ and some positive constants $C(j)$,

$$|D^j \,\mathrm{Ric}_{(M,g)}| \leq C(j) \quad \textit{for all } j = 0,...,k\,,$$

then, there exists a positive constant $C = C(n, Q, k, \alpha, i, C(j), 0 \le j \le k)$, *depending only on* n, Q, k, α, i, *and the* $C(j)$, *such that*

$$r_H(Q, k+1, \alpha)(M) \ge C\,.$$

REMARKS. 1) Let (M, g) be an arbitrary smooth Riemannian n-manifold without boundary and let Ω be an open subset of M. Let us say that $\Omega' \subset M$ is a δ-neighborhood of Ω if for any $x \in M \setminus \Omega'$, $d_g(x, \Omega) \ge \delta$. Roughly speaking, theorem 6 says that if the Ricci curvature and the injectivity radius are controlled in a δ-neighborhood of some open subset Ω of M, then there exists a uniform lower bound C for the harmonic radius at any point of Ω, C depending only on δ and the constants which control the Ricci curvature and the injectivity radius.

2) It is now classical that analogous estimates are available if one works with geodesic normal coordinates instead of harmonic coordinates. In general, these estimates are much rougher. We do not enter into too many details but, for instance, it is possible to prove the following "$C^{0,\alpha}$-analogue" of theorem 6. For more details, we refer to HEBEY-VAUGON [27]. Let $Q > 1$ and let (M, g) be a smooth Riemannian n-manifold without boundary. Suppose that for some $x \in M$ there exist positive constants C_1 and C_2 such that

$$|\,\mathrm{Rm}_{(M,g)}\,| \le C_1 \;\text{ and }\; |D\ \mathrm{Rm}_{(M,g)}\,| \le C_2 \text{ on } B_x(\mathrm{inj}_{(M,g)}(x))\,,$$

where $\mathrm{Rm}_{(M,g)}$ is the Riemann curvature of g. Then, there exist positive constants $K = K(n, C_1, C_2)$ and $\delta = \delta(n, Q, C_1, C_2)$ such that the components (g_{ij}) of g in geodesic normal coordinates at x satisfy for any $i, j, k = 1, ..., n$, and any $y \in \mathbb{R}^n$ such that $|y| < \inf(\delta, \mathrm{inj}_{(M,g)}(x))$,

(1) $(1/Q)\,\delta_{ij} \le g_{ij} \le Q\,\delta_{ij}$ (as bilinear forms),

(2) $|g_{ij}(y) - \delta_{ij}| \le K|y|^2$ and $|\partial_k g_{ij}(y)| \le K|y|$,

where $|y|$ is the euclidean distance from 0 to y and where $g_{ij}(y) = g_{ij}(\exp_x(y))$. As one can see, when changing from harmonic coordinates to geodesic normal coordinates, the bound $|\,\mathrm{Ric}_{(M,g)}\,| \le C$ "has to" be replaced by the much demanding bounds $|\,\mathrm{Rm}_{(M,g)}\,| \le C_1$ and $|D\ \mathrm{Rm}_{(M,g)}\,| \le C_2$. On the other hand, such results are sometimes useful, for instance in Nonlinear Analysis...

Let us now start with the proof of theorem 6. We closely follow the lines of ANDERSON [3] and ANDERSON-CHEEGER [4]. In particular, we work with the Sobolev spaces $H^p_{k+1}, p > n$, in place of $C^{k,\alpha}$. Note that since $H^p_{k+1} \subset C^{k,\alpha}$, with $\alpha = 1 - n/p$, this will give stronger results. First, we define the notion of H^p_k-harmonic radius.

DEFINITION 7. Let (M, g) be a smooth Riemannian n-manifold without boundary and let $x \in M$. Given $Q > 1, k \in \mathbb{N}^*$, and $p > n$, we define the H^p_k-harmonic radius at x as the largest number $r_H = r_H(Q, k, p)(x)$ such that, on the geodesic ball $B_x(r_H)$ of center x and radius r_H, there is a harmonic coordinate chart such that the metric tensor is H^p_k-controlled in these coordinates. Namely, if g_{ij} $(i, j$ in $1, ..., n)$ are the components of g in these coordinates, then

$$1)\quad Q^{-1}\,\delta_{ij} \le g_{ij} \le Q\,\delta_{ij} \text{ as bilinear forms,}$$

$$2)\ \sum_{1\le|\beta|\le k} r_H^{|\beta|-n/p} ||\partial_\beta g_{ij}||_{L^p} \le Q - 1\,.$$

The harmonic radius $r_H(Q, k, p)(M)$ of (M, g) is now defined as

$$r_H(Q, k, p)(M) = \inf_{x\in M} r_H(Q, k, p)(x)\,.$$

Since conditions 1) and 2) are invariant under rescalings of the metric, the H^p_k-harmonic radius scales as the distance function (indeed, if g satisfies 1) and 2) in some harmonic coordinates $(x^1, ..., x^n)$, then, it is easy to see that $\lambda^2 g$, for $\lambda > 0$, satisfies 1) and 2) in the harmonic coordinates $(y^1, ..., y^n)$ defined by $y^i = \lambda x^i, i = 1, ..., n$, with λr_H in place of r_H). Note that this is also true for the $C^{k,\alpha}$-harmonic radius.

Another basic property of the H^p_k-harmonic radius is that it is increasing and upper semicontinuous with respect to Q. Namely, we have the following:

LEMMA 8. *Let (M, g) be a smooth Riemannian n-manifold without boundary, $x \in M, k \in \mathbb{N}^*$, and $p > n$. Then, for any $1 < Q \le Q' < \infty$,*

$$r_H(Q, k, p)(x) \le r_H(Q', k, p)(x)\,,$$

and for any $Q > 1$,

$$\lim_{\varepsilon \to 0^+} r_H(Q+\varepsilon, k, p)(x) = r_H(Q, k, p)(x)\,.$$

PROOF. By definition, $r_H(Q, k, p)(x)$ is clearly increasing with respect to Q. Hence, to prove the lemma, we just have to prove that for any $Q > 1$,

$$\overline{\lim_{\varepsilon \to 0^+}}\, r_H(Q+\varepsilon, k, p)(x) \le r_H(Q, k, p)(x)\,.$$

Fix $r < \overline{\lim}\, r_H(Q+\varepsilon, k, p)(x)$. For a decreasing sequence of $\varepsilon > 0$ converging to 0, we have a harmonic coordinate chart Φ_ε on $B_x(r)$ which satisfies the bounds 1) and 2) of definition 7 with $Q+\varepsilon$ in place of Q (and r in place of r_H). Now, by standard elliptic theory, we get that a subsequence of (Φ_ε) converges H^p_{k+1} to a limiting chart $\Phi : B_x(r) \to \mathbb{R}^n$ (for more details, we refer to the first part of the proof of lemma 10, where such a result is proved in a more general setting where the background metric is also changing with ε). Since the bounds 1) and 2) of definition 7 are clearly preserved under H^p_{k+1}-convergence, we get that $r_H(Q, k, p)(x) \ge r$. Since $r < \overline{\lim}\, r_H(Q+\varepsilon, k, p)(x)$ was chosen arbitrarily, this ends the proof of the lemma. □

LEMMA 9. *Let (M, g) be a smooth Riemannian n-manifold without boundary, $Q > 1, k \in \mathbb{N}^*$, and $p > n$. Then, $x \longmapsto r_H(Q, k, p)(x)$ is 1-lipschitzian on M.*

PROOF. By definition, it is obvious that for any $x, y \in M$,

$$r_H(Q, k, p)(y) \ge r_H(Q, k, p)(x) - d_g(x, y)\,,$$

where d_g is the distance associated to g.

Then, by symmetry, we get that $x \longmapsto r_H(Q, k, p)(x)$ is 1-lipschitzian on M. □

For convenience, given $(M,g), x \in M, Q > 1, k \in \mathbb{N}^*$, and $p > n$, we set $r_H(g,Q) = r_H(Q,k,p)(x)$ and we now prove the following:

LEMMA 10. *Let M be a smooth differentiable n-manifold without boundary, $x \in M$, (g_m) a sequence of smooth Riemannian metrics on M, $Q > 1, k \in \mathbb{N}^*$, and $p > n$. Suppose that (g_m) converges H_k^p to some H_k^p Riemannian metric g on M. Then,*

$$r_H(g,Q) \geq \overline{\lim_{m\to\infty}}\, r_H(g_m,Q)$$

and for any $0 < \varepsilon < Q-1$,

$$r_H(g,Q-\varepsilon) \leq \underline{\lim}_{m\to\infty}\, r_H(g_m,Q)\,.$$

REMARK. Although we defined the H_k^p-harmonic radius for smooth metrics, the definition easily extends to H_k^p metrics (note that $k \geq 1$). Indeed, one may speak of harmonic functions on M, which are at least in H_{k+1}^p. Of course, lemma 8 and lemma 9 still hold for H_k^p metrics. Independently, we say that (g_m) converges H_k^p to g if in any chart of the C^∞ complete atlas of M, the components of g_m converge $H_{k,\text{loc}}^p$ to the components of g, or, equivalently, if there exists a C^∞ sub-atlas of the C^∞ complete atlas of M such that in any chart of this sub-atlas, the components of g_m converge H_k^p to the components of g.

PROOF. First, we prove that

$$r_H(g,Q) \geq \overline{\lim}\, r_H(g_m,Q)\,.$$

Let $\Phi_m : B_x(r_m) \to \mathbb{R}^n$, where $B_x(r_m)$ is the geodesic ball for g_m of center x and radius $r_m = r_H(g_m,Q)$, be harmonic coordinate charts satisfying the bounds 1) and 2) of definition 7. We may suppose that $\overline{\lim}\, r_m > 0$. Since the metrics g_m converge H_k^p to a limit metric g, by standard elliptic theory, we get that for any $r < \overline{\lim}\, r_m$ a subsequence of (Φ_m) converges H_{k+1}^p to a limiting chart $\Phi : B_x(r) \to \mathbb{R}^n$, where $B_x(r)$ is the geodesic ball for g of center x and radius r. For instance, given a local coordinate chart $(x^1,...,x^n)$ on $B_x(r)$, we have for any $\theta = 1,...,n$,

$$g_m^{ij}\frac{\partial^2\Phi_m^\theta}{\partial x^i\partial x^j} = g_m^{ij}(\Gamma_m)_{ij}^s\frac{\partial\Phi_m^\theta}{\partial x^s}\,,$$

where the g_{ij}^m are the components of g_m in the chart $(x^1, ..., x^n)$, (g_m^{ij}) is the inverse matrix of (g_{ij}^m), and $(\Gamma_m)_{ij}^s$ are the Christoffel symbols of g_m.

Now, since (g_m) converges H_k^p to a limit metric g, and since Φ_m satisfies 1) of definition 7, we get that for any θ, (Φ_m^θ) is C^1 bounded. Then, by standard elliptic theory, we get that for any θ, (Φ_m^θ) is a bounded sequence in H_{k+1}^p. Hence, for any θ and after passing to a subsequence, (Φ_m^θ) converges in $C^{k,\alpha'}$, $\alpha' < 1 - n/p$. Now, if we write that, for any pair (m, q),

$$g^{ij}\frac{\partial^2(\Phi_m^\theta - \Phi_q^\theta)}{\partial x^i \partial x^j} = (g^{ij} - g_m^{ij})\frac{\partial^2 \Phi_m^\theta}{\partial x^i \partial x^j} \ - \ (g^{ij} - g_q^{ij})\frac{\partial^2 \Phi_q^\theta}{\partial x^i \partial x^j} + \\ + \ (g_m^{ij}(\Gamma_m)_{ij}^s - g_q^{ij}(\Gamma_q)_{ij}^s)\frac{\partial \Phi_m^\theta}{\partial x^s} + g_q^{ij}(\Gamma_q)_{ij}^s\left(\frac{\partial \Phi_m^\theta}{\partial x^s} - \frac{\partial \Phi_q^\theta}{\partial x^s}\right),$$

we get, again by standard elliptic theory, that for any θ, (Φ_m^θ) is a Cauchy sequence in H_{k+1}^p. Hence, (Φ_m) converges H_{k+1}^p to a limiting map $\Phi : B_x(r) \to \mathbb{R}^n$. In addition, since Φ_m satisfies 1) of definition 7, Φ is also a chart. Now, since the bounds 1) and 2) of definition 7 are clearly preserved under H_k^p-convergence, we get that $r_H(g, Q) \geq r$ for any $r < \overline{\lim}\, r_m$. Therefore,

$$r_H(g, Q) \geq \overline{\lim}\, r_H(g_m, Q)\,.$$

Let us now prove the more significant inequality

$$r_H(g, Q - \varepsilon) \leq \underline{\lim}\, r_H(g_m, Q), \quad \forall \varepsilon \in (0, Q - 1)\,.$$

Fix $r < r_H(g, Q)$ and let $(x^1, ..., x^n)$ be harmonic coordinates for g on $B = B_x(r)$, the geodesic ball for g of center x and radius r. Let Δ_m be the Laplace operator of g_m. In the chart $(x^1, ..., x^n)$ we have with the notations defined above

$$\Delta_m = -g_m^{ij}\frac{\partial^2}{\partial x^i \partial x^j} \ + \ g_m^{ij}(\Gamma_m)_{ij}^s\frac{\partial}{\partial x^s} = -\frac{1}{\sqrt{|g_m|}}\frac{\partial}{\partial x^i}\left(g_m^{ij}\sqrt{|g_m|}\frac{\partial}{\partial x^j}\right)$$

where $|g_m|$ is the determinant of (g_{ij}^m) in these coordinates. Let $y_m^1, ..., y_m^n$ be the solutions of

$$\Delta_m y_m^\theta = 0 \text{ in } B, \ \ y_m^\theta = x^\theta \text{ on } \partial B, \ \ \forall \theta = 1, ..., n\,,$$

and set $\omega_m^\theta = x^\theta - y_m^\theta$. We have

$$\Delta_m \omega_m^\theta = \Delta_m x^\theta \text{ in } B, \quad \omega_m^\theta = 0 \text{ on } \partial B\,.$$

Now, by standard elliptic theory, we get the following.

SUBLEMMA. *For any compact subset $B' \subset B$, $\lim_{m\to\infty} ||\omega_m^\theta||_{H_{k+1}^p(B')} = 0$.*

PROOF OF THE SUBLEMMA Since (g_m) converges $H_k^p \subset C^{0,\alpha}$ to g, by [22, theorem 8.16] we obtain the existence of a positive constant C, independent of m, such that

$$||\omega_m^\theta||_{C^0(B)} \leq C\,||\Delta_m x^\theta||_{L^p(B)}\,.$$

On the other hand, it is easy to see that $\lim_{m\to\infty} ||\Delta_m x^\theta||_{L^p(B)} = 0$. Hence,

$$\lim_{m\to\infty} ||\omega_m^\theta||_{C^0(B)} = 0\,.$$

Now, by [22, theorem 8.33] (see also the remark page 212 of [22]), we get that

$$\lim_{m\to\infty} ||\omega_m^\theta||_{C^{1,\alpha}(B)} = 0\,, \text{ where } \alpha = 1 - n/p.$$

Then, writing the equation $\Delta_m \omega_m^\theta = \Delta_m x^\theta$ under the form

$$g_m^{ij}\frac{\partial^2 \omega_m^\theta}{\partial x^i \partial x^j} = g_m^{ij}(\Gamma_m)_{ij}^s \frac{\partial \omega_m^\theta}{\partial x^s} - \Delta_m x^\theta\,,$$

we obtain by [22 theorem 9.13] that

$$\lim_{m\to\infty} ||\omega_m^\theta||_{H_2^p(B)} = 0\,.$$

Finally, by induction and noting that for any multi-index γ,

$$g_m^{ij}\frac{\partial^2}{\partial x^i \partial x^j}\partial_\gamma \omega_m^\theta = \partial_\gamma\left(g_m^{ij}\frac{\partial^2 \omega_m^\theta}{\partial x^i \partial x^j}\right) + \text{ terms involving derivatives of } \omega_m^\theta \text{ of order } \leq |\gamma|+1$$

and

$$\partial_\gamma \left(g_m^{ij} \frac{\partial^2 \omega_m^\theta}{\partial x^i \partial x^j} \right) = \partial_\gamma \left(g_m^{ij} (\Gamma_m)_{ij}^s \frac{\partial \omega_m^\theta}{\partial x^s} \right) \; - \; \partial_\gamma \Delta_m x^\theta \, ,$$

we get by [22, theorem 9.11] what we announced, namely that for any compact subset $B' \subset B$,

$$\lim_{m \to \infty} ||\omega_m^\theta||_{H_{k+1}^p(B')} = 0 \, .$$

This ends the proof of the sub-lemma. □

Proof of lemma 10 (continued). From the sublemma, it is easy to see that for any compact subset $B' \subset B$ there exists m_0 such that for $m \geq m_0, (y_m^1, ..., y_m^n)$ is a harmonic coordinate chart on B' for g_m, and since the bounds 1) and 2) of definition 7 are continuous in the (strong) H_k^p-topology, we get that the charts $(y_m^1, ..., y_m^n)$ and g_m satisfy 1) and 2) of definition 7 on B', with constants Q_m satisfying $\lim\limits_{m \to \infty} Q_m = Q$. As an immediate consequence, we have that for any $\varepsilon > 0$,

$$r \leq \underline{\lim}\; r_H(g_m, Q_m) \leq \underline{\lim}\; r_H(g_m, Q + \varepsilon) \, .$$

Since $r \leq r_H(g, Q)$ was arbitrary, this ends the proof of the lemma. □

Remarks. 1) If $k \geq 2$, we do not need to use [22, theorem 8.33]. Actually, since if $k \geq 2$, $g_m^{ij}(\Gamma_m)_{ij}^s$ belongs to C^0, we directly obtain by [22, theorem 9.13] the existence of a positive constant C, independent of m, such that

$$||\omega_m^\theta||_{H_2^p(B)} \leq C \left(||\omega_m^\theta||_{L^p(B)} + ||\Delta_m x^\theta||_{L^p(B)} \right) \, .$$

2) According to the first part of the proof of lemma 10, we have that if a sequence (g_m) of metrics converges $H_k^p, k \geq 1$, and if Φ_m are harmonic charts for g_m satisfying 1) of definition 7, then, after passing to a subsequence, (Φ_m) converges H_{k+1}^p to a limiting chart. Actually, with the same ideas, it is easy to see that the result still holds with $C^{k,\alpha}$ convergence instead of H_k^p-convergence. Namely, if a sequence (g_m) of metrics converges $C^{k,\alpha}, k \geq 0$, and if Φ_m are harmonic charts for g_m satisfying 1) of definition 7, then, after passing to a subsequence, (Φ_m) converges $C^{k+1,\alpha}$ to a limiting chart.

Let us now state a modified version of theorem 6, based on the H^p_k-harmonic radius instead of the $C^{k,\alpha}$ harmonic radius.

Theorem 11. *Let (M,g) be a smooth Riemannian n-manifold without boundary, $Q>1, p>n, \delta>0$, and Ω an open subset of M. Set*

$$\Omega(\delta)=\{x\in M \text{ s.t. } d_g(x,\Omega)<\delta\}$$

where d_g is the distance associated to g. Suppose that for some $\lambda\in\mathbb{R}$ and $i>0$, we have that for all $x\in\Omega(\delta)$,

$$\mathrm{Ric}_{(M,g)}(x)\geq\lambda \quad \textit{and} \quad \mathrm{inj}_{(M,g)}(x)\geq i\,,$$

where $\mathrm{inj}_{(M,g)}(x)$ is the injectivity radius at x. Then, there exists a positive constant $C=C(n,Q,p,\delta,i,\lambda)$, depending only on n,Q,p,δ,i, and λ, such that for any $x\in\Omega$, the H^p_1-harmonic radius $r_H(Q,1,p)(x)$ satisfies

$$r_H(Q,1,p)(x)\geq C\,.$$

In addition, if instead of the bound $\mathrm{Ric}_{(M,g)}(x)\geq\lambda$ we assume that for some $k\in\mathbb{N}$ and some positive constants $C(j)$,

$$|D^j\,\mathrm{Ric}_{(M,g)}(x)|\leq C(j),\quad \forall j=0,..,k,\ \forall x\in\Omega(\delta)\,,$$

then, there exists a positive constant $C=C(n,Q,k,p,\delta,i,C(j),0\leq j\leq k)$, depending only on n,Q,k,p,δ,i, and the $C(j)$, such that for any $x\in\Omega$, the H^p_{k+2}-harmonic radius $r_H(Q,k+2,p)(x)$ satisfies

$$r_H(Q,k+2,p)(x)\geq C\,.$$

The proof of theorem 11 is by contradiction. The general idea is to construct a sequence (M_m,x_m,g_m) of Riemannian manifolds, $x_m\in M_m, r_H(x_m)=1$, which converges in the H^p_k-topology to a limiting manifold $(M,x,g), x=\lim x_m$, then, to prove that (M,g) is necessarily isometric to $(\mathbb{R}^n,\delta)$, where δ is the euclidean metric, finally, to get the contradiction from lemma 9 and lemma 10, since one should have $r_H(x)=1$, while, obviously, $\mathbb{R}^n$ has infinite harmonic radius. We present the proof of the theorem for the H_k^p-harmonic radius when $k \geq 2$. We refer to Anderson-Cheeger [4] for the H_1^p-part of the theorem.

Proof. As already mentioned, the proof of theorem 11 is by contradiction. Hence, we assume that for some $n \in \mathbb{N}, \mathrm{Q} > 1, \mathrm{p} > \mathrm{n}, \delta > 0, \mathrm{i} > 0, \mathrm{k} \in \mathbb{N}$, and $C(j) \in \mathbb{R}_+^*, j = 0, ..., k$, there exists a sequence (M_m, g_m) of Riemannian n-manifolds without boundary, there exists a sequence (Ω_m) of open subsets of M_m, and there exists a sequence (x_m) of points of Ω_m, such that

$$\forall x \in \Omega_m(\delta),\ \mathrm{inj}_{(M_m,g_m)}(x) \geq i,$$
$$\forall x \in \Omega_m(\delta),\ \forall j = 0, \ldots, k,\ |D^j \operatorname{Ric}_{(M_m,g_m)}(x)| \leq C(j),$$
$$\text{and} \quad \lim_{m\to\infty} r_H(g_m, x_m) = 0$$

where $r_H(g_m, x_m)$ is the H_{k+2}^p-harmonic radius $r_H(Q, k+2, p)(x_m)$ of (M_m, g_m) at x_m. Now, for a fixed m, we would like to choose x_m so that $r_H(g_m, x_m)$ is minimal at x_m. But, a priori, no such point needs to exist. To remedy this, we consider instead the sequence (B_m, g_m) where $B_m = B_{x_m}(\inf(\delta, i))$ is the geodesic ball for g_m with center x_m and radius $\inf(\delta, i)$. For $x \in B_m$, let $\mathrm{inj}_{(B_m,g_m)}(x)$ be the injectivity radius of (B_m, g_m) at x. We have

$$\mathrm{inj}_{(B_m,g_m)}(x) = d_{g_m}(x, \partial B_m)\,,$$

where d_{g_m} is the distance associated to g_m. Hence,

$$\lim_{x\to\partial B_m} \mathrm{inj}_{(B_m,g_m)}(x) = 0, \quad \text{while} \quad \mathrm{inj}_{(B_m,g_m)}(x_m) = \inf(\delta, i)\,.$$

As a consequence, if we continue to define r_H with respect to M_m, we get that, for any m, there exists $y_m \in B_m$ such that

$$x \longmapsto \frac{r_H(g_m, x)}{\mathrm{inj}_{(B_m,g_m)}(x)}$$

attains its minimal value at y_m, i.e.:

$$\forall x \in B_m, \quad \frac{r_H(g_m, y_m)}{\mathrm{inj}_{(B_m,g_m)}(y_m)} \leq \frac{r_H(g_m, x)}{\mathrm{inj}_{(B_m,g_m)}(x)}\,.$$

In particular, since $\mathrm{inj}_{(B_m,g_m)}(y_m) \leq \inf(\delta,i)$ and $\mathrm{inj}_{(B_m,g_m)}(x_m) = \inf\,(\delta,i)$,

$$\frac{1}{\inf(\delta,i)}\, r_H(g_m,y_m) \leq \frac{r_H(g_m,y_m)}{\mathrm{inj}_{(B_m,g_m)}(y_m)} \leq \frac{1}{\inf(\delta,i)}\, r_H(g_m,x_m)\,,$$

and we get that

$$\lim_{m\to\infty} r_H(g_m,y_m) \;=\; \lim_{m\to\infty} \frac{r_H(g_m,y_m)}{\mathrm{inj}_{(B_m,g_m)}(y_m)} \;=\; 0\,.$$

From now on, set $h_m = r_H(g_m,y_m)^{-2} g_m$. Since the harmonic radius scales as the distance function under rescalings of the metric (see above), we get that

$$r_H(h_m,y_m) = 1\,,$$

while

$$\begin{aligned}
&\lim_{m\to\infty} ||\,\mathrm{Ric}_{(B_m,h_m)}\,||_{C^k} = 0,\\
&\lim_{m\to\infty} \mathrm{inj}_{(B_m,h_m)}(y_m) \quad = +\infty,\\
&\lim_{m\to\infty} d_{h_m}(y_m,\partial B_m) \quad = +\infty\,,
\end{aligned}$$

and

$$\begin{aligned}
\forall y\in B_m,,\quad \forall\, m,\quad r_H(h_m,y) = \frac{r_H(g_m,y)}{r_H(g_m,y_m)} &\geq \frac{d_{g_m}(y,\partial B_m)}{d_{g_m}(y_m,\partial B_m)}\\
&\geq \frac{d_{h_m}(y,\partial B_m)}{d_{h_m}(y_m,\partial B_m)}\,.
\end{aligned}$$

Set

$$\delta_m = \frac{r_H(g_m,y_m)}{\mathrm{inj}_{(B_m,h_m)}(y_m)} = \frac{1}{d_{h_m}(y_m,\partial B_m)}\,.$$

Then,

$$\lim_{m\to\infty} \delta_m = 0\,,$$

and for all $y \in B_{y_m}(\frac{1}{2\delta_m})$, where $B_{y_m}(\frac{1}{2\delta_m})$ is the geodesic ball for h_m with center y_m and radius $\frac{1}{2\delta_m}$, we have

$$r_H(h_m,y) \geq \frac{1}{2}\,.$$

In particular, given $R < \infty$, $r_H(h_m, y) \geq 1/2$ on $B_{y_m}(R)$ provided m is sufficiently large. As a consequence, given $R < \infty$ and (z_m) a sequence of points in $B_{y_m}(R)$, there exist harmonic coordinate charts $U_m : \Omega_m \to B_0(\frac{1}{2\sqrt{Q}})$, centered at z_m, such that

$$Q^{-1}\delta_{ij} \leq ((U_m^{-1})^* h_m)_{ij} \leq Q\delta_{ij} \quad \text{as bilinear forms}$$

and

$$||((U_m^{-1})^* h_m)_{ij}||_{H^p_{k+2}} \leq C(Q)$$

where $C(Q)$ depends only on Q, and $B_0(\frac{1}{2\sqrt{Q}})$ is the euclidean ball of $\mathbb{R}^n$ with center 0 and radius $1/2\sqrt{Q}$. For convenience, we set $\widetilde{h}_m = (U_m^{-1})^* h_m$. Our first claim is that (B_m, y_m, h_m) converges H^p_{k+2}, uniformly on compact subsets, to a limiting complete manifold (M, y, h). First, since the $\widetilde{h}_m$ are H^p_{k+2}-bounded, after passing to a subsequence we can assume that they converge in $C^{k+1,\alpha}$. Now, as already mentioned in section 2 (see also[17, lemma 4.1]), the equation for the Ricci curvature of $\widetilde{h}_m$ is

$$\widetilde{h}^{st}_m \frac{\partial^2 \widetilde{h}^m_{ij}}{\partial x^s \partial x^t} = -2\left(\mathrm{Ric}_{(B,\widetilde{h}_m)}\right)_{ij} - A(\widetilde{h}_m)$$

where $A(\widetilde{h}_m)$ is a quadratic term in the first derivatives of $\widetilde{h}_m$.

But, $||\,\mathrm{Ric}_{(B_m,h_m)}\,||_{C^k}$ goes to 0. Hence, $(\mathrm{Ric}_{(B,\widetilde{h}_m)})_{ij}$ converges C^k to 0, and, by standard elliptic theory, we easily get that $(\widetilde{h}_m)$ converges in H^p_{k+2}. Now, with the same kind of arguments than those used to prove proposition 12 of section 4 (which are completely independent of the present proof of theorem 11), we obtain the existence of a H^p_{k+3} n-manifold M, the existence of y in M, and the existence of a H^p_{k+2} Riemannian metric h on M, such that for any compact domain $D \subset M$, with $y \in D$, and after passing to a subsequence, there exist compact domains $D_m \subset B_m, y_m \in D_m$, and there exist H^p_{k+3} diffeomorphisms $\Phi_m : D \to D_m$, such that $\lim\limits_{m\to\infty} \Phi_m^{-1}(y_m) = y$, and such that $(\Phi_m^* h_m)$ converges H^p_{k+2} to h in D. In addition, since $\lim\limits_{m\to\infty} d_{h_m}(y_m, \partial B_m) = +\infty$, (M, h) is necessarily a complete Riemannian manifold. This proves our first claim.

At this point, given a compact domain $D \subset M$, with y in D, we set $\hat{h}_m = \Phi_m^* h_m$. Now, let $x \in D$, and let $U_m : B_x(r) \to \mathbb{R}^n, r > 0$, be harmonic coordinate charts for $\hat{h}_m$ satisfying 1) and 2) of definition 7. According to what we have said above (see the remark following the proof of lemma 10) and since $(\hat{h}_m)$ converges H^p_{k+2} to h, (U_m) converges H^p_{k+3} to a limiting chart $U : B_x(r) \to \mathbb{R}^n$. Hence, coming back to the equation for the Ricci curvature in harmonic coordinates, we get that $H = (U^{-1})^* h$ satisfies

$$H^{st} \frac{\partial^2 H_{ij}}{\partial x^s \partial x^t} + A(H) = 0 \,.$$

Therefore, by standard elliptic theory, $(U^{-1})^* h$ is smooth, and since the left hand size of this equation is in fact the expression of the Ricci curvature of $(U^{-1})^* h$, it has null Ricci curvature: (M, h) is a smooth Ricci-flat complete Riemannian manifold.

Now, our second claim is that (M, h) is isometric to $(\mathbb{R}^n, \delta)$, where δ is the euclidean metric of $\mathbb{R}^n$. First, let $v \in T_y M$ be a unit tangent vector for h, and let γ be the geodesic of (M, h) with $\gamma(0) = y, \gamma'(0) = v$. Given a compact domain $D \subset M$, with $y \in D$, set $\hat{h}_m = \Phi_m^* h_m$ as above. Since $\hat{h}_m$ converges to h, there exists λ_m such that $\lambda_m \to 1$ and such that $\lambda_m v$ is a unit-length vector for $\hat{h}_m$. Let γ_m be the geodesic for $\hat{h}_m$ with $\gamma_m(0) = y$ and $(\gamma_m)'(0) = \lambda_m v$. Since $\hat{h}_m$ converges to h, and since $\lim_{m \to \infty} \mathrm{inj}_{(B_m, h_m)}(y_m) = +\infty$, γ_m converges to γ and γ is length minimizing. As a consequence, for any $v \in T_y M$, we have a line in the Ricci-flat manifold (M, h), in the direction v. Now, the Cheeger-Gromoll theorem [14], see also [8, chapter 6], implies that $(M, h) = (\mathbb{R}^n, \delta)$.

Finally, we get the contradiction as follows. Since (h_m) converges to h in the H^p_{k+2} topology, by lemma 9 and lemma 10 we should have that

$$r_H(Q', k+2, p)(y) \le \lim_{m \to \infty} r_H(Q, k+2, p)(y_m) \,,$$

for some $Q' < Q$. But, by construction $r_H(Q, k+2, p)(y_m) = 1$, while, obviously, $\mathbb{R}^n$ has infinite harmonic radius for any Q' and this ends the proof of the theorem. ∎

Remark. As one can see, the previous proof (as presented above) does not work when we deal with the H^p_1-harmonic radius. For instance, $\mathrm{Ric}_{(M_m, h_m)}$ does not converge anymore to 0, and we just have that $\mathrm{Ric}_{(M_m,h_m)} \geq \lambda\, r_H(h_m, y_m)^2 \to 0$, where λ is as in the statement of the theorem. To remedy this, the proof presented by ANDERSON-CHEEGER in [4] relies on the study of the distance function and does not use anymore the equation for the Ricci curvature in harmonic coordinates. Anyway, the structure of the proof is unchanged. First, one has to prove that (B_m, y_m, h_m) converges in the H_1^p-topology to a complete manifold (M, y, h), then, to prove that $(M, h) = (\mathbb{R}^n, \delta)$. Hence, here again, the contradiction comes from lemma 9 and lemma 10.

Let us now prove how theorem 11 implies theorem 6. The basic idea is just that H_{k+1}^p is continuously embedded in $C^{k,\alpha}$, $\alpha = 1 - n/p$, so that if the metric tensor is H_{k+1}^p-controlled in a harmonic coordinate chart, then it is also $C^{k,\alpha}$ controlled. Roughly speaking, one just has to show that the constant of the embedding of H_{k+1}^p in $C^{k,\alpha}$ can be chosen independently of the domain where the embedding is considered.

PROOF OF THEOREM 6. Let $\delta > 0$, (M, g) an arbitrary smooth Riemannian n-manifold without boundary, and Ω an open subset of M. Set

$$\Omega(\delta) = \{x \in M \text{ s.t. } d_g(x, \Omega) < \delta\}\,,$$

and suppose that for some $i > 0, k \in \mathbb{N}, \lambda \in \mathbb{R}$, and $C(j) > 0, j = 0, ..., k$, we have for all $x \in \Omega(\delta)$,

(i) $\mathrm{inj}_{(M,g)} \geq i$,

and

(iia) $\mathrm{Ric}_{(M,g)} \geq \lambda$,

respectively

(iib) $|D^j \,\mathrm{Ric}_{(M,g)}| \leq C(j), \quad \forall j = 0, ..., k$.

Set $K = 0$ in case condition (iia) is satisfied, $K = k+1$ in case condition (iib) is satisfied, and let $Q > 1$, $p > n$. According to theorem 11, there exists a positive constant c, depending only on n, Q, K, p, δ, i, and either λ or $C(j), j = 0, ..., k$, such that for any $x \in \Omega$ there exists a harmonic coordinate chart $\Phi : B_x(c) \to \mathbb{R}^n$ such that if $g_{ij}, i, j = 1, ..., n$, are the components of g in these coordinates, then

$$Q^{-1}\,\delta_{ij} \leq g_{ij} \leq Q\,\delta_{ij} \quad \text{as bilinear forms}\;, \tag{1}$$

$$\sum_{1 \leq |\beta| \leq K+1} c^{|\beta| - n/p} ||\partial_\beta g_{ij}||_{L^p} \leq Q - 1\,. \tag{2}$$

Now, by condition (1), if $B_\eta = B_0(c/\sqrt{Q})$ denotes the euclidean ball of $\mathbb{R}^n$ with center 0 and radius $\eta = c/\sqrt{Q}$, we have that $B_\eta \subset \Phi(B_x(c))$. Hence, if we look at the g_{ij}'s as functions on B_η, and if the L^p norm is now taken with respect to the euclidean metric, we get that

$$\sum_{1\le|\beta|\le K+1} \eta^{|\beta|-n/p} ||\partial_\beta g_{ij}||_{L^p} \le Q^{n/2(K+1)}(Q-1)\,. \tag{3}$$

Let S, given by the Sobolev embedding theorem (see for instance Adams [2]), be such that

$$\begin{aligned}\sum_{1\le|\beta|\le K} \eta^{|\beta|} \sup_x |\partial_\beta g_{ij}(x)| + \eta^{K+\alpha} \sum_{|\beta|=K} \sup_{y\ne z} \frac{|\partial_\beta g_{ij}(y) - \partial_\beta g_{ij}(z)|}{|z-y|^\alpha} \\ \le S \sum_{1\le|\beta|\le K+1} \eta^{|\beta|-n/p} ||\partial_\beta g_{ij}||_{L^p}\quad ,\end{aligned} \tag{4}$$

where $\alpha = 1 - n/p$. *A priori*, since S comes from the embedding of $H^p_{K+1}(B_\eta)$ in $C^{k,\alpha}(B_\eta)$, it depends on η and K. We claim that S can be chosen such that it does not depend on η (the point here is that η depends on Q, and that we do not want S to depend on Q). Actually, our claim just comes from the fact that if $h(x) = f(\frac{\eta}{r}x), r > 0$ and $|x| \le r$, then

$$r^{|\beta|}|\partial_\beta h(x)| = \eta^{|\beta|}|\partial_\beta f\Big(\frac{\eta}{r}x\Big)|$$

and

$$r^{|\beta|-n/p}||\partial_\beta h||_{L^p(B_r)} = \eta^{|\beta|-n/p}||\partial_\beta f||_{L^p(B_\eta)}\,.$$

Hence, S can be chosen as the constant coming from the embedding of $H^p_{K+1}(B_1)$ in $C^{K,\alpha}(B_1)$, where B_1 is the unit ball of $\mathbb{R}^n$. In particular, S can be chosen such that it depends only on K, and this proves our claim.

Now, since by (1), $B_x(c/Q) \subset \Phi^{-1}(B_\eta)$, we get by (3) and (4) that

$$\begin{aligned}\sum_{1\le|\beta|\le K} r^{|\beta|} \sup_x |\partial_\beta g_{ij}(x)| + \sum_{|\beta|=K} r^{K+\alpha} \sup_{z\ne y} \frac{|\partial_\beta g_{ij}(z) - \partial_\beta g_{ij}(y)|}{d_g(z,y)^\alpha} \\ \le S\, Q^{(\alpha+n/(K+1))/2}(Q-1)\end{aligned} \tag{5}$$

on $B_x(r)$ where $r = c/Q$. Independently, given $Q' > 1$ and $\alpha' \in (0,1)$, and since S depends only on K, it is easy to see that we can find $Q > 1$, Q close to 1, and that we can find $p > n$, such that

$$\alpha' = 1 - n/p, \quad Q < Q', \quad \text{and} \quad SQ^{(\alpha'+n/(K+1))/2}(Q-1) \leq Q' - 1 \,.$$

Hence, according to what we have said above and (5), we get that for any $x \in \Omega$, the $C^{K,\alpha'}$-harmonic radius $r_H(Q', K, \alpha')(x)$ at x is greater than c/Q, namely than a constant which depends only on $n, Q', K, \alpha', \delta, i$, and either λ or $C(j), j = 0, ..., k$. This ends the proof of theorem 6. □

4 – Proof of the Main theorem

Before we proceed to prove the Main theorem stated in section 1, let us establish the following result. The Main theorem can then be seen as an easy corollary of it. The proof we present closely follows the lines of Kasue [30] and is completely independent of theorem 11.

Proposition 12. *Let (M_m, g_m) be a sequence of smooth complete Riemannian n-manifolds, (x_m) a sequence of points in M_m, $\lambda \in \mathbb{R}, Q > 1, k \in \mathbb{N}$, p > n, and $\alpha \in (0,1)$. Suppose*

1) *for any m, $\mathrm{Ric}_{(M_m,g_m)} \geq \lambda$,*
2) *there exists $r > 0$ such that for any sequence (y_m) of points in M_m there is a harmonic chart $H_m : \Omega_m \to B_0(r)$, where Ω_m is some open neighbourhood of y_m in M_m and $B_0(r)$ is the euclidean ball of $\mathbb{R}^n$ with center 0 and radius r, such that*
3) *for any m, $Q^{-1}\,\delta_{ij} \leq ((H_m^{-1})^* g_m)_{ij} \leq Q\,\delta_{ij}$ as bilinear forms*

and

4a) *a subsequence of $((H_m^{-1})^* g_m)$ converges in $C^{k,\alpha}(B_0(r))$*

respectively

4b) *a subsequence of $((H_m^{-1})^* g_m)$ converges in $H^p_{k+1}(B_0(r))$.*

Then, there exists a complete Riemannian n-manifold (M, g), M of class $C^{k+1,\alpha}$ (respectively H^p_{k+2}) and g of class $C^{k,\alpha}$ (respectively H^p_{k+1}), and there exists $x \in M$, such that the following holds: for any compact domain $D \subset M$, with $x \in D$, there exist, up to passing to a subsequence, compact *domains* $D_m \subset M_m$*, with points* $x_m \in D_m$*, and* $C^{k+1,\alpha}$ *(respectively* H^p_{k+2}*) diffeomorphisms* $\Phi_m : D \to D_m$*, satisfying:*

5) $\lim_{m\to\infty} \Phi_m^{-1}(x_m) = x$,
6) $(\Phi_m^* g_m)$ *converges* $C^{k,\alpha}$ *(respectively* H^p_{k+1}*) to* g *in any chart of the induced* $C^{k+1,\alpha}$ *(respectively* H^p_{k+2}*) complete atlas of* D*.*

Proof. First, let (M, x, g) be one of the pointed Riemannian manifolds of the sequence (M_m, x_m, g_m). In other words, fix m and let $(M, x, g) = (M_m, x_m, g_m)$.

We now take $\delta \in (0, r/\sqrt{Q})$ and let $\mathcal{N}$ be a $(\delta/4)$-net in M (recall that an ε-net is a maximal set of points x_i of M such that all the balls $B_{x_i}(\varepsilon)$ are pairwise disjoint). The existence of $\mathcal{N}$ is given by Zorn's lemma, and, as an easy consequence of the maximality of $\mathcal{N}$, M is necessarily covered by the balls $B_{x_i}(\delta/2)$. Suppose also $x_1 = x$.

Now, let D be a bounded domain in M, containing the base point x and of diameter less than d, and let

$$D_\delta = \{y \in M \text{ s.t. } d_g(y, D) < \delta\}$$

as usual, where d_g is the distance associated to g. By the Bishop-Gromov comparison theorem (see for instance [18, theorem 4.19]) we get that

$$\begin{aligned}\operatorname{vol} B_x(\frac{d}{2} + 2\delta) &\geq \sum_{x_i \in \mathcal{N} \cap D_\delta} \operatorname{vol} B_{x_i}(\delta/4)\\ &\geq \#(\mathcal{N} \cap D_\delta)\ \operatorname{vol} B_x(\frac{d}{2} + 2\delta)\frac{V^\lambda(\delta/4)}{V^\lambda(d/2 + 2\delta)}\end{aligned}$$

where $V^\lambda(r)$ stands for the volume of any ball of radius r in the simply connected space of (possibly negative) constant curvature $\lambda/(n-1)$. Thus, the number of points in $\mathcal{N} \cap D_\delta$ is finite and uniformly bounded from above by a constant μ depending only on n, λ, d, and δ.

Now, we produce an embedding of D into an euclidean space, using an analog of the Whitney construction. For any x_i in $\mathcal{N}$, let H_i denote the harmonic coordinate chart onto $B_0(r)$ given by the hypotheses of proposition 12, and let φ be a cut-off function defined on $\mathbb{R}^n$ by

$$\varphi = 1 \text{ in } B_0(r/2), \quad \varphi = 0 \text{ outside } B_0(r).$$

Set $\varphi_i = \varphi \circ H_i$, and let $\Psi_D : M \to \mathbb{R}^N$, where $N = n\mu + \mu$, be defined by

$$\Psi_D(y) = (\varphi_1 H_1(y), ..., \varphi_\mu H_\mu(y), \varphi_1(y), ..., \varphi_\mu(y)).$$

Clearly, Ψ_D is smooth, the image of Ψ_D is contained in a fixed ball of $\mathbb{R}^N$ whose radius depends only on n, μ, and r, and, finally, Ψ_D, when restricted to D, turns out to be a smooth embedding. Moreover, for any $i = 1, ..., \mu$, $\Psi_D(H_i^{-1}(B_0(r/2))$ is a graph over $B_0(r/2)$. Namely, for every z in $B_0(r/2)$,

$$\Psi_D(H_i^{-1}(z)) = (\varphi_1 H_1(H_i^{-1}(z)), \dots, \varphi_{i-1} H_{i-1}(H_i^{-1}(z)), \quad z \quad ,$$
$$\varphi_{i+1} H_{i+1}(H_i^{-1}(z)), \dots, \varphi_\mu H_\mu(H_i^{-1}(z)), \varphi_1(H_i^{-1}(z)), \dots, \varphi_\mu(H_i^{-1}(z))).$$

We now return to the sequence (M_m, x_m, g_m) of pointed Riemannian manifolds of proposition 12. According to what we have just said, for any $R > 0$ and each m, we can build an embedding $\Psi_R^m : B_{x_m}(R) \to \mathbb{R}^N$, $N = N(R)$ (considering only a subsequence if necessary, we shall assume that μ is independent of m). Furthermore, and this is important, for m fixed, any two embeddings Ψ_R^m and $\Psi_{R'}^m$, $R < R'$, are compatible on $B_{x_m}(R)$ in the sense that one is obtained from the other by canonical embeddings of $\mathbb{R}^{N(R)}$ in $\mathbb{R}^{N(R')}$.

Now, in any "graphing chart" $B_0(r/2)$, a subsequence of $((H_i^m)^{-1})^* g_m$ converges $C^{k,\alpha}$ (respectively H_{k+1}^p, depending on which condition (4a) or (4b) is satisfied). Hence, according to what we have said in section 3, see for instance remark 2 following the proof of lemma 10, a subsequence of $H_j^m \circ (H_i^m)^{-1}$ converges either $C^{k+1,\alpha}$ or H_{k+2}^p to transition functions H_{ji}. Since we can repeat the argument for any i, and since there is only a finite number of these for a fixed R, this implies a subsequence of the images of the embeddings Ψ_R^m converges $C^{k+1,\alpha}$ or H_{k+2}^p, as submanifolds of $\mathbb{R}^{N(R)}$, to a submanifold M_R of class $C^{k+1,\alpha}$ or H_{k+2}^p embedded in $\mathbb{R}^{N(R)}$. Remember our manifolds are obtained by a collection of charts over which they can be seen as graphs of functions, and convergence should here be understood as convergence of these functions as well as the transition functions between these charts. Locally, we define x as

$$x = \lim_{m\to\infty} \Psi_R^m \circ (H_1^m)^{-1}(0) = \lim_{m\to\infty} \Psi_R^m(x_m).$$

Obviously, x does not depend on R.

Set $M_R^m = \Psi_R^m(B_{x_m}(R))$ and $\tilde{g}_m = ((\Psi_R^m)^{-1})^* g_m$. If $\Pi_m : M_R^m \to M_R$ denotes the projection along the normals onto M_R, then, for m large enough and after passing to a subsequence, Π_m is well defined, Π_m induces a $C^{k+1,\alpha}$ (respectively H_{k+2}^p) diffeomorphism from M_R^m onto M_R, and $(\Pi_m^{-1})^* \tilde{g}_m$ converges $C^{k,\alpha}$ (respectively H_{k+1}^p) to a $C^{k,\alpha}$ (respectively H_{k+1}^p) Riemannian metric g on M_R. Now, let R_j be an increasing sequence of numbers going to infinity. From a diagonal sequence argument, we get a $C^{k+1,\alpha}$ (respectively H_{k+2}^p) limiting manifold M which is the increasing union of the M_{R_j}'s. In the same way, M is endowed with a limiting $C^{k,\alpha}$ (respectively H_{k+1}^p) Riemannian metric g, and since M has an exhaustion by closed and bounded domains which are compact, (M, g) is complete from Hopf-Rinow's theorem. Clearly, this ends the proof of the proposition. □

Let us now prove the Main theorem.

Proof of the Main theorem. Let (M_m, g_m) be a sequence of smooth compact Riemannian n-manifolds such that

(i) for any m, $\mathrm{inj}_{(M_m,g_m)} \geq i, \mathrm{vol}_{(M_m,g_m)} \leq v$,

and

(iia) for any m, $\mathrm{Ric}_{(M_m,g_m)} \geq \lambda$,

respectively

(iib) for any m, $|D^j \mathrm{Ric}_{(M_m,g_m)}| \leq C(j)$ for all $j = 0, ..., k$,

where $i > 0, v > 0, \lambda \in \mathbb{R}, k \in \mathbb{N}$, and $C(j) > 0, j = 0, ..., k$ are given constants independent of m. First note that in both cases (iia) and (iib), condition (1) of proposition 12 is obviously satisfied. Now, conditions (2) and (3) of proposition 12 come from theorem 6, namely from the existence of harmonic coordinate charts (with control on the metric tensor) on balls of radii uniformly bounded from below. Furthermore, depending on which condition (iia) or (iib) is satisfied, we get by the Arzela-Ascoli compactness criterion that for any $\alpha \in (0, 1)$, a subsequence of the metrics converges either $C^{0,\alpha}$ or $C^{k+1,\alpha}$ "in any of these charts". In other words, condition (4a) of proposition 12 is satisfied. Finally, using remark 2) following the statement of the Main theorem, the diameter of any of the (M_m, g_m) is bounded from above by a constant d independent of m. Hence, applying proposition 12 with $D = B_x(R)$, $R > d$, we get the desired result. Namely, $D_m = M_m$ for m large enough, and, up to passing to a subsequence, there exist diffeomorphisms $\Phi_m : M \to M_m$ such that $(\Phi_m^* g_m)$ converges either $C^{0,\alpha}$ or $C^{k+1,\alpha}$ to g in M. In particular, M possesses a smooth (sub)structure coming, for instance, from one of the diffeomorphisms with M_m. This ends the proof of the Main theorem. □

Remark. The first proofs of Gromov compactness theorem (including Gromov's original one) involved techniques from the Hausdorff and Lipschitz topologies on sets of Riemannian (or, more generally, metric) spaces. They relied on the following fact: in case of "bounded geometry" (e.g controlled curvature and injectivity radius), the class of quasi-isometry of any such manifold is well described by the geometry of an ε-net $\mathcal{N}_\varepsilon$ in M, provided ε is small enough. Then, the proof of Gromov's theorem only requires an understanding of the behaviour of the nets $\mathcal{N}_\varepsilon$ when passing to the limit (for details, see for instance Cheeger [11], Greene-Wu [23], Gromov-Lafontaine-Pansu [24], Katsuda [31], Peters [34], and the forthcoming book of Bridson and Haefliger [9]). Our proof (a mixture of those who appeared in the literature) uses an analog of this fact for the $C^{k,\alpha}$-topology.

5 – Other convergence results

As already mentioned, the Main theorem admits various versions, e.g for bounded domains as well as for complete noncompact but pointed Riemannian manifolds. First, as a basic application of theorem 11, we have the following.

Theorem 13. *Let (M_m, g_m) a sequence of smooth Riemannian n-manifolds without boundary, (x_m) a sequence of points in M_m, and $\delta > 0$. Suppose that for some $\lambda \in \mathbb{R}$ and $i > 0$,*

$$\mathrm{Ric}_{(M_m,g_m)}(x) \geq \lambda \quad \textit{and} \quad \mathrm{inj}_{(M_m,g_m)}(x) \geq i$$

for all $x \in B_{x_m}(\delta)$, the geodesic ball for g_m of center x_m and radius δ. Then, for any $Q > 1$ and any $p > n$, there exists a positive constant $r = r(n, Q, p, \delta, i, \lambda)$, depending only on n, Q, p, δ, i, and λ, and there exist smooth diffeomorphisms $\Phi_m : B_0(r) \to M_m$ such that $\Phi_m(0) = x_m$ and

1) $Q^{-1}\,\delta \leq \Phi_m^* g_m \leq Q\,\delta$ in $B_0(r)$ and as bilinear forms,

2) $(\Phi_m^* g_m)$ *is* H_1^p*-bounded in* $B_0(r)$,

3) *a subsequence of* $(\Phi_m^* g_m)$ *converges* $C^{0,\alpha}$ *in* $B_0(r)$,

where $B_0(r)$ *is the euclidean ball of* $\mathbb{R}^n$ *with center* 0 *and radius* r, δ *is the euclidean metric of* $\mathbb{R}^n$, *and* $\alpha = 1 - n/p$. *In addition, if instead of the bound* $\mathrm{Ric}_{(M_m,g_m)}(x) \geq \lambda$ *we assume that for some* $k \in \mathbb{N}$ *and some positive constants* $C(j), j = 0, ..., k$,

$$|D^j \,\mathrm{Ric}_{(M_m,g_m)}(x)| \leq C(j), \quad \forall j = 0, ..., k, \ , \quad \forall x \in B_{x_m}(\delta)\,,$$

then, for any $Q > 1$ *and any* $p > n$ *there exists a positive constant* r, *depending only on* n, Q, k, p, δ, i *and the* $C(j), j = 0, ..., k$, *and there exist smooth diffeomorphisms* $\Phi_m : B_0(r) \to M_m$, *such that* $\Phi_m(0) = x_m$ *and*

1′) $Q^{-1}\,\delta \leq \Phi_m^* g_m \leq Q\,\delta$ *in* $B_0(r)$ *and as bilinear forms,*

2′) $(\Phi_m^* g_m)$ *is* H_{k+2}^p*-bounded in* $B_0(r)$

3′) *a subsequence of* $(\Phi_m^* g_m)$ *converges* $C^{k+1,\alpha}$ *in* $B_0(r)$.

More generally, when we are concerned with bounded domains or complete noncompact but pointed Riemannian manifolds, the Main theorem admits the following versions.

Theorem 14. *Let* (M_m, g_m) *be a sequence of smooth complete Riemannian n-manifolds,* (Ω_m) *a sequence of smooth domains in* M_m, *and* $\varepsilon > 0$. *Suppose that for some* $\lambda \in \mathbb{R}, i > 0$, *and* $v_1, v_2 > 0$,

$$\mathrm{Ric}_{(M_m,g_m)}(x) \geq \lambda, \quad \mathrm{inj}_{(M_m,g_m)}(x) \geq i, \quad v_1 \leq \mathrm{vol}_{(\Omega_m,g_m)} \leq v_2\,,$$

for all $x \in \Omega_m$. *Then, there exist smooth domains* $N_m \subset \Omega_m$, *satisfying*

$$\forall x \in \partial N_m, \quad \frac{\varepsilon}{2} \leq d_{g_m}(x, \partial\Omega_m) \leq \varepsilon\,,$$

such that the sequence of open manifolds (N_m, g_m) *is precompact in the* $C^{0,\alpha}$*-topology for any* $\alpha \in (0,1)$. *In addition, the same conclusion holds with precompactness in the* $C^{k+1,\alpha}$*-topology if instead of the bound* $\mathrm{Ric}_{(M_m,g_m)}(x) \geq \lambda$ *we assume that for some* $k \in \mathbb{N}$ *and some positive constants* $C(j), j = 0, ..., k$,

$$|D^j \,\mathrm{Ric}_{(M_m,g_m)}(x)| \leq C(j), \quad \forall j = 0, ..., k, \quad \forall x \in \Omega_m\,.$$

THEOREM 15. *Let (M_m, g_m) be a sequence of smooth complete non compact Riemannian n-manifolds, (x_m) a sequence of points in M_m and $\alpha \in (0,1)$. Suppose that for some $\lambda \in \mathbb{R}$, and $i > 0$,*

$$\mathrm{Ric}_{(M_m,g_m)} \geq \lambda \quad \textit{and} \quad \mathrm{inj}_{(M_m,g_m)} \geq i.$$

Then, there exists a complete Riemannian n-manifold (M,g), M of class $C^{1,\alpha}$ and g of class $C^{0,\alpha}$, and there exists $x \in M$, such that the following holds: for any compact domain $D \subset M$, $x \in D$, there exist, up to passing to a subsequence, compact domains $D_m \subset M_m, x_m \in D_m$, and $C^{1,\alpha}$ diffeomorphisms $\Phi_m : D \to D_m$, such that

1) $\lim_{m\to\infty} \Phi_m^{-1}(x_m) = x$,
2) $(\Phi_m^* g_m)$ *converges* $C^{0,\alpha}$ *to* g *in the induced* $C^{1,\alpha}$ *complete atlas of* D.

In addition, if instead of the bound $\mathrm{Ric}_{(M_m,g_m)} \geq \lambda$ we assume that for some $k \in \mathbb{N}$ and some positive constants $C(j), j = 0, ..., k$,

$$|D^j \,\mathrm{Ric}_{(M_m,g_m)}| \leq C(j), \quad \forall j = 0, ..., k,$$

then, there exists a complete Riemannian n-manifold (M,g), M of class $C^{k+2,\alpha}$ and g of class $C^{k+1,\alpha}$, and there exists $x \in M$, such that the following holds: for any compact domain $D \subset M, x \in D$, there exist, up to passing to a subsequence, compact domains $D_m \subset M_m, x_m \in D_m$, and $C^{k+2,\alpha}$ diffeomorphisms $\Phi_m : D \to D_m$, such that

1) $\lim_{m\to\infty} \Phi_m^{-1}(x_m) = x$,
2) $(\Phi_m^* g_m)$ *converges* $C^{k+1,\alpha}$ *to* g *in the induced* $C^{k+2,\alpha}$ *complete atlas of* D.

For sake of completeness, we mention that convergence of Riemannian manifolds with integral bounds on the curvature is studied in ANDERSON [3], [5], GAO [19], [20], [21], and YANG [37], [36], [35]. Anyway, note that with only minor modifications of the proof of theorem 11, we get the following.

THEOREM 16. *Let (M,g) be a smooth Riemannian n-manifold without boundary, $Q > 1$, $p > n$, $\delta > 0, k \in \mathbb{N}$, and Ω an open subset of M. Set*

$$\Omega(\delta) = \{x \in M \text{s.t. } d_g(x,y) < \delta\},$$

and suppose that for some $\lambda \in \mathbb{R}, i > 0$, *and* $C(j), j = 0, ..., k$, *we have that*

1) $\mathrm{Ric}_{(M,g)}(x) \geq \lambda$ *and* $\mathrm{inj}_{(M,g)}(x) \geq i$ *for all* $x \in \Omega(\delta)$

2) $\int_{\Omega(\delta)} |D^j \mathrm{Ric}_{(M,g)}|^p \, d\mathrm{vol}(g) \leq C(j)$ *for all* $j = 0, \ldots, k$.

Then, there exists a positive constant $C = C(n, Q, k, p, \delta, i, \lambda, C(j), 0 \leq j \leq k)$, *depending only on* $n, Q, k, p, \delta, i, \lambda$, *and the* $C(j)$*'s, such that for any* $x \in \Omega$, *the* H^p_{k+2}*-harmonic radius* $r_H(Q, k+2, p)(x)$ *satisfies*

$$r_H(Q, k+2, p)(x) \geq C\,.$$

In particular, the $C^{k+1,\alpha}$ *harmonic radius at any point of* $\Omega, \alpha = 1 - n/p$, *is bounded below by a positive constant depending only on* $n, Q, k, p, \delta, i, \lambda$, *and the* $C(j)$*'s.*

Then, according to the proof of proposition 12, we get the following.

THEOREM 17. *Let* $n \in \mathbb{N}$, $p > n$, $i > 0, v > 0, \lambda > 0, k \in \mathbb{N}$, *and* $C(j) > 0, j = 0, ..., k$. *The space of smooth compact Riemannian* n*-manifolds* (M, g) *such that*

$$1)\ \ \mathrm{Ric}_{(M,g)} \geq \lambda, \quad \mathrm{inj}_{(M,g)} \geq i, \quad \mathrm{vol}_{(M,g)} \leq v,$$

and

$$2)\ \ \int_M |D^j \mathrm{Ric}_{(M,g)}|^p \, d\,\mathrm{vol}(g) \leq C(j), \quad \forall j = 0, \ldots, k,$$

is precompact in the $C^{k+1,\alpha}$*-topology for* $\alpha = 1 - n/p$.

Finally, to end this section, we mention that compactness of conformal metrics with integral bounds on curvature is studied in GURSKY [25].

6 – Examples of applications

We present in this section some applications of the convergence theorems stated above. We have no pretention to exhaustivity, and we just chose the few examples listed below with the intention to show that convergence theorems can be helpful in many branches of Mathematics, from pure Riemannian Geometry to Nonlinear Analysis. The choices also owe a lot to the different interests of the authors. Note that the general scheme of application of convergence theorems is the following: if one wants to prove some result where geometric quantities are bounded, then, by contradiction, build a sequence converging by the results stated above to a limit manifold, and... try to get a contradiction!

6.1 – Finiteness theorems (Cheeger [11])

THEOREM. *Let $n \in \mathbb{N}, \Lambda > 0, \mathrm{d} > 0$, and $v > 0$. Then, there are only finitely many diffeomorphism types of n-manifolds satisfying*

$$|K_{(M,g)}| \leq \Lambda, \quad \mathrm{vol}_{(M,g)} \geq v, \quad \textit{and } \mathrm{diam}_{(M,g)} \leq d.$$

The proof is by contradiction. If there is an infinite number of diffeomorphism types, build a sequence by picking a manifold in each class. By the Main theorem (see also remark 2 of section I), such a sequence subconverges. Hence, we get diffeomorphisms between these manifolds, and this is the contradiction we were looking for. Note that the same result holds if one replaces the bounds

$$|K_{(M,g)}| \leq \Lambda, \quad \mathrm{vol}_{(M,g)} \geq v, \quad \mathrm{diam}_{(M,g)} \leq d$$

by the bounds (ANDERSON-CHEEGER [4])

$$\mathrm{Ric}_{(M,g)} \geq \lambda, \quad \mathrm{inj}_{(M,g)} \geq i, \quad \mathrm{vol}_{(M,g)} \leq v.$$

6.2 – Pinching just below 1/4 (Berger [7], Abresch-Meyer [1])

This has been the first historical example of application of Gromov compactness theorem in Riemannian Geometry.

Theorem. *For any $n \in \mathbb{N}$, there exists $\varepsilon(n) > 0$ such that any compact simply connected Riemannian n-manifold (M, g) with*

$$1/4 - \varepsilon(n) \le K_{(M,g)} \le 1$$

is either homeomorphic to the standard sphere, or diffeomorphic to a compact rank one symmetric space.

Assume there is a sequence of manifolds whose pinching constants converge to 1/4. By a well-known result of Klingenberg in even dimensions or a recent result due to Abresch and Meyer in odd dimensions, the injectivity radius of these manifolds is bounded below by π. Hence, by Croke [16] the volume of these manifolds is uniformly bounded from below, while, according to Myers theorem, the diameter of these manifolds is uniformly bounded from above. The Main theorem then yields a limiting 1/4-pinched manifold, and a well-known rigidity result ends the proof of the theorem.

6.3 – Differentiable sphere theorems (Pacelli Bessa [32])

Theorem. *Given an integer $n \geq 2$ and $i > 0$, there exists a positive $\varepsilon = \varepsilon(n, i)$, such that if a compact Riemannian n-manifold (M, g) satisfies*

$$\mathrm{Ric}_{(M,g)} \geq (n-1), \quad \mathrm{inj}_{(M,g)} \geq i, \quad \mathrm{diam}_{(M,g)} \geq \pi - \varepsilon$$

then M is diffeomorphic to the standard unit sphere S^n in $\mathbb{R}^{n+1}$ and the metric g is $\varepsilon' = \varepsilon'(\varepsilon)$-close in the $C^{0,\alpha}$-topology to the canonical metric of curvature $+1$ on S^n.

Let (M_m, g_m) be a sequence of compact Riemannian n-manifolds such that

$$\mathrm{Ric}_{(M_m,g_m)} \geq n-1, \quad \mathrm{inj}_{(M_m,g_m)} \geq i, \quad \mathrm{diam}_{(M_m,g_m)} \geq \pi - \varepsilon_m$$

where $\varepsilon_m \to 0$ as m goes to ∞. By the Main theorem, (M_m, g_m) subconverges to a limiting manifold (M, g). The point is to prove that (M, g) is isometric to the standard sphere S^n.

6.4 – Existence and control of optimal Sobolev inequalities on complete manifolds (Hebey [26])

THEOREM. *Let (M, g) be a smooth complete Riemannian n-manifold, $n \geq 3$, with Ricci curvature bounded below by some arbitrary $\lambda \in \mathbb{R}$ and positive injectivity radius bounded below by some arbitrary $i > 0$. Then, for any $\varepsilon > 0$ and any $q \in [1, n)$, there exists a positive constant $C = C(\varepsilon, n, q, \lambda, i)$, depending only on $\varepsilon, n, q, \lambda$, and i, such that for any $u \in H_1^q(M)$,*

$$\left(\int_M |u|^p \, d\,\mathrm{vol}(g)\right)^{q/p} \leq (K(n,q)+\varepsilon) \int_M |Du|^q \, d\mathrm{vol}(g) + C \int_M |u|^q \, d\,\mathrm{vol}(g)$$

where $1/p = 1/q - 1/n$ and $K(n, q)$, an explicit constant depending only on n and q, is the smallest constant having this property.

The point is that we just need to establish local versions of these inequalities. Theorem 13 yields such inequalities.

6.5 – Estimates on the Yamabe quotient of a compact Einstein manifold (Aviles-Escobar [6])

Let (M, g) be a smooth compact Riemannian n-manifold, $n \geq 3$. The Yamabe quotient $Q(M, [g])$ of the conformal class of (M, g) is then defined by

$$Q(M, [g]) = \inf_{u \in C^\infty(M)} \frac{\int_M (|Du|^2 + (n-2)/4(n-1)u^2) \, d\,\mathrm{vol}(g)}{||u||^2_{2n/(n-2)}}.$$

THEOREM. *Given an integer $n \geq 3$, there exists a positive constant $\varepsilon(n)$, depending only on n, such that for any smooth compact Einstein n-manifold (M, g), not conformally diffeomorphic to the standard n-dimensional sphere S^n in $\mathbb{R}^{n+1}$,*

$$Q(M, [g]) \ \leq \ Q(S^n, can) - \varepsilon(n)\,.$$

Suppose that the result is false. Then, there exists a sequence of Einstein manifolds (M_m, g_m), which are not conformally diffeomorphic to the standard sphere S^n, such that $Q(M_m) \to Q(S^n)$ as m goes to ∞. The point is to prove that (M_m, g_m) subconverges to a limiting manifold (M, g), then, to prove that (M, g) is isometric to the standard sphere S^n. The contradiction comes from the well-known fact that on the sphere, the standard metric (up to diffeomorphism) is an isolated Einstein metric.

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REFERENCES

[1] U. Abresch – W. Meyer: *Pinching below* 1/4*, injectivity radius and conjugate radius*, J. Diff. Geom., **40** (1994), 643-691.

[2] R. A. Adams: *Sobolev spaces*, Academic Press, 1978.

[3] M. T. Anderson: *Convergence and rigidity of manifolds under Ricci curvature bounds*, Invent. Math., **102** (1990), 429-445.

[4] M. T. Anderson – J. Cheeger: C^α *compactness for manifolds with Ricci curvature and injectivity radius bounded below*, J. Diff. Geom., **35** (1992), 265-281.

[5] M. T. Anderson: *Ricci curvature bounds and Einstein metrics on compact manifolds*, J. Amer. Math. Soc., **2** (1989), 455-490.

[6] P. Avilès – J. F. Escobar: *On the Sobolev quotient of an Einstein manifold*, Indiana Univ. Math. J., **41** (1992), no. 2, 435-438.

[7] M. Berger: *Sur les variétés riemanniennes pincées juste au-dessous de* 1/4, Ann. Inst. Fourier, **33** (1983), 135-150.

[8] A. L. Besse: *Einstein manifolds*, Ergeb. Math. Grenzgeb., vol. 10, Springer, Berlin, 1987.

[9] M. Bridson – A. Haefliger: book in preparation.

[10] J. Cheeger: *Comparison and finiteness theorems for Riemannian manifolds*, Ph.D. thesis, Princeton University, 1967.

[11] J. Cheeger: *Finiteness theorems for Riemannian manifolds*, Amer. J. Math., **92** (1970), 61-74.

[12] J. Cheeger – T. H. Colding: *Lower bounds on Ricci curvature and the almost rigidity of warped products*, Annals of Math., **144** (1996), 189-237.

[13] J. Cheeger – T. H. Colding: *Almost rigidity of warped products and the structure of spaces with Ricci curvature bounded below*, C. R. Acad. Sci. Paris, **320** (1996), 353-357.

[14] J. Cheeger – D. Gromoll: *The splitting theorem for manifolds of non-negative Ricci curvature*, J. Diff. Geom, **6** (1971), 119-128.

[15] J. Cheeger – M. Gromov – M. Taylor: *Finite propagation speed, kernel estimates for functions of the Laplace operator and the geometry of complete Riemannian manifolds*, J. Diff. Geom., **17** (1982), 15-53.

[16] C. B. Croke: *Some isoperimetric inequalities and eigenvalues estimates*, Ann. Scient. Ec. Norm. Sup. Paris, **14** (1980), 249-260.

[17] D. De Turck – J. Kazdan: *Some regularity theorems in Riemannian Geometry*, Ann. Scient. Ec. Norm. Sup. Paris, **14** (1981), 249-260.

[18] S. Gallot – D. Hulin – J. Lafontaine: *Riemannian Geometry*, 2nd ed., Springer, 1993.

[19] L. Z. Gao: *Convergence of Riemannian manifolds: Ricci and* $L^{\frac{n}{2}}$ *pinching*, J. Diff. Geom., **32** (1990), 349-381.

[20] L. Z. Gao: *Einstein metrics*, J. Diff. Geom., **32** (1990), 155-183.

[21] L. Z. Gao: $L^{\frac{n}{2}}$*-curvature pinching*, J. Diff. Geom., **32** (1990), 713-774.

[22] D. Gilbarg – N. S. Trudinger: *Elliptic Partial Differential Equations of Second Order*, Grundlehr. Math. Wiss., vol. 224, Springer, Berlin, 1977.

[23] R. Greene – H. Wu: *Lipschitz convergence of Riemannian manifolds*, Pacific J. Math., **131** (1988), 119-141.

[24] M. Gromov – J. Lafontaine – P. Pansu: *Structures métriques pour les variétés riemanniennes*, Cedic-Fernand Nathan, Paris, 1981.

[25] M. J. Gursky: *Compactness of conformal metrics with integral bounds on curvature*, Duke Math. J., **72** (1993), 339-367.

[26] E. Hebey: *Optimal Sobolev inequalities on complete Riemannian manifolds with Ricci curvature bounded below and positive injectivity radius*, Amer. J. Math., **118** (1996), 291-300.

[27] E. Hebey – M. Vaugon: *The best constant problem in the Sobolev embedding theorem for complete Riemannian manifolds*, Duke Math. J., **79** (1995), 235-279.

[28] J. Jost: *Riemannian Geometry and Geometric Analysis*, Springer, 1995.

[29] J. Jost – H. Karcher: *Geometrischen Methoden für gewinnung von a priori Schranken für harmonische Abbildungen*, Compositio Math., **40** (1982), 27-77.

[30] A. Kasue: *A convergence theorem for Riemannian manifolds and some applications*, Nagoya Math. J., **114** (1989), 21-51.

[31] A. Katsuda: *Gromov's convergence theorem and its application*, Nagoya Math. J., **100** (1985), 11-48.

[32] G. Pacelli Bessa: *Differentiable sphere theorems for the Ricci curvature*, Math. Z., **214** (1993), 245-259.

[33] S. Peters: *Cheeger's finiteness theorem for diffeomorphism classes of Riemannian manifolds*, J. reine angew. Math., **349** (1984), 77-82.

[34] S. Peters: *Convergence of Riemannian manifolds*, Compositio Math., **62** (1987), 3-16.

[35] D. Yang: *Convergence of Riemannian manifolds with integral bounds on curvature, I*, Ann. Scient. Ec. Norm. Sup. Paris, **25** (1992), 77–105.

[36] D. Yang: *Convergence of Riemannian manifolds with integral bounds on curvature, II*, Ann. Scient. Ec. Norm. Sup. Paris, **25** (1992), 179-199.

[37] D. Yang: L^p *pinching and compactness theorems for compact Riemannian manifolds*, Forum Math., **4** (1992), 323-333.

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Indirizzo degli Autori:

E. Hebey – 9, rue Villehardouin – 75003 Paris, France and Département de Mathématiques – Université de Cergy-Pontoise – Site de Saint Martin – 95302 Cergy-Pontoise Cedex, France
Email: hebey@u-cergy.fr

M. Herzlich – Centre de Mathématiques de l'Ecole Polytechnique – CNRS URA 169 – 91128 Palaiseau Cedex, France and Département de Mathématiques – Université de Cergy-Pontoise – Site de Saint Martin – 95302 Cergy-Pontoise Cedex, France
Email: herzlich@math.polytechnique.fr