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Compact cosymplectic manifolds with transversally positive definite Ricci tensor

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RIASSUNTO: *Si dimostra che il gruppo fondamentale di una varietà cosimplettica M, il cui tensore di Ricci sia trasversalmente definito positivo, è isomorfo a $\mathbb{Z}$. Se inoltre M ha curvatura φ-sezionale positiva e costante, allora M è isometrica quasi di contatto al prodotto di uno spazio proiettivo complesso con S^1.*

ABSTRACT: *In this paper we study compact cosymplectic manifolds with transversally positive definite Ricci tensor, that is, compact cosymplectic manifolds such that its Ricci tensor is positive definite on vector fields orthogonal to the Reeb vector field of the cosymplectic structure. We prove that the fundamental group of a cosymplectic manifold M with transversally positive definite Ricci tensor is isomorphic to $\mathbb{Z}$ and, in particular, if M is of positive constant φ-sectional curvature we show that there exists a certain cosymplectic structure on the product of a complex projective space of positive constant holomorphic sectional curvature with the circle S^1 such that M is almost contact isometric to such a product.*

1 – Introduction

The curvature properties of a compact orientable Riemannian manifold affect its topological structure (see, for instance, [4], [9], [14] and [15]):

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1. A compact Riemannian manifold with positive definite Ricci tensor has finite fundamental group.
2. A complete connected Riemannian even-dimensional manifold of positive constant sectional curvature is isometric to a sphere or to a real projective space.

For compact Kähler manifolds we also have the two following results (see [7], [8] and [11]):

1. A compact Kähler manifold with positive definite Ricci tensor is simply connected.
2. A compact Kähler manifold with positive constant holomorphic sectional curvature is holomorphically isometric to a complex projective space of positive constant holomorphic sectional curvature.

The odd-dimensional counterpart of Kähler manifolds are cosymplectic manifolds. Let us recall that an almost contact metric structure (φ, ξ, η, g) on a manifold M is cosymplectic if it is integrable and the 1-form η and the fundamental 2-form of the structure are closed (see [1] and subsection 2.2). The canonical example of cosymplectic manifold is given by the product of a Kähler manifold with $\mathbb{R}$ or with the circle S^1 (see [2], [5] and subsection 2.2). In fact, a complete simply connected cosymplectic manifold is the product of a complete simply connected Kähler manifold with $\mathbb{R}$ and, a compact cosymplectic manifold has similar topological properties to that of the product of a compact Kähler manifold with S^1 (see [2] and [6]). However, we remark the following facts:

a. A compact simply connected manifold cannot admit a cosymplectic structure and, moreover, the Ricci tensor of a cosymplectic manifold cannot be positive definite (see subsection 2.2). In particular, the Ricci tensor S of a cosymplectic manifold $(M, \varphi, \xi, \eta, g)$ of positive constant φ-sectional curvature is transversally positive definite, that is, for all $x \in M$, S_x is positive definite on the orthogonal subspace to ξ_x (see remark 2.1).
b. There exist examples of compact cosymplectic manifolds which are not topologically equivalent to a global product of a compact Kähler manifold with S^1 (see [6]).

In this paper, we study compact cosymplectic manifolds with transversally positive definite Ricci tensor and prove the following results (see theorems 3.2 and 3.3):

1′. The fundamental group of a compact cosymplectic manifold with transversally positive definite Ricci tensor is isomorphic to $\mathbb{Z}$.
2′. If $(M, \varphi, \xi, \eta, g)$ is a $(2m+1)$-dimensional compact cosymplectic manifold with positive constant φ-sectional curvature k, then M is almost contact isometric to the product manifold $P_m(\mathbb{C}^{m+1})(k) \times S^1$ endowed with a certain cosymplectic structure (φ, ξ, η, g), $P_m(\mathbb{C}^{m+1})(k)$ being the m-dimensional complex projective space of positive constant holomorphic sectional curvature k. The cosymplectic structure (φ, ξ, η, g) is obtained by deformating the natural cosymplectic structure of $P_m(\mathbb{C}^{m+1})(k) \times S^1$. This deformation depends on a real constant $c, c \neq 0$, and on a matrix A of $\mathfrak{su}(m+1)$ (the Lie algebra of the special unitary group $SU(m+1)$). In particular, if $c = 1$ and $A = 0$ then (φ, ξ, η, g) is the natural cosymplectic structure of $P_m(\mathbb{C}^{m+1})(k) \times S^1$.

The results 1′ and 2′ are the version for cosymplectic manifolds of the properties 1 and 2 of Kähler manifolds above mentioned. In order to prove 1′ and 2′ we obtain the relation between the fundamental group of a compact cosymplectic manifold $(M, \varphi, \xi, \eta, g)$ and the fundamental group of the leaves of the foliation $\mathfrak{F}^\perp$, provided that the first Betti number of M is equal to 1. As a consequence, we deduce that the foliation $\mathfrak{F}^\perp$ is a smooth bundle over S^1 (see theorem 3.1). We notice that there exist examples of compact cosymplectic manifolds with first Betti number equal to 1 which are not a global product of a Kähler manifold with S^1 (see remark 3.1).

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2 – Curvature in Kähler and cosymplectic manifolds

2.1 – Kähler manifolds with positive constant holomorphic sectional curvature

All the manifolds considered in this paper are assumed to be connected and of class C^∞.

Let V be an **almost hermitian manifold** with metric h and **almost complex structure** J. Denote by $\mathfrak{X}(V)$ the Lie algebra of vector fields on V and by T_xV the tangent space to V at a point x of V. The **Kähler 2-form** Ω is defined by $\Omega(X,Y) = h(X, JY)$ for $X, Y \in \mathfrak{X}(V)$. An almost hermitian manifold (V, J, h) is said to be **Kähler** if $[J, J] = 0$ and $d\Omega = 0$ or equivalently if J is paralell.

If π is a plane in T_xV, $x \in V$, which is invariant by the almost complex structure J and u is a unit vector in π, then $\{u, J_x(u)\}$ is an orthonormal basis for π and, hence, the sectional curvature $H_x(\pi) = H_x(u)$ is defined by $H_x(u) = R_x(u, J_x(u), u, J_x(u))$, where R is the Riemannian curvature tensor of V. The sectional curvature $H_x(u)$ is called the **holomorphic sectional curvature** by u. If $H_x(u)$ is a constant for all unit vector u in T_xV and for all point $x \in V$, then V is called a **space of constant holomorphic sectional curvature**.

A map F between the almost hermitian manifolds (V, J, h) and (V', J', h') is said to be a **holomorphic isometry** if F is an isometry which verifies $F_* \circ J = J' \circ F_*$. If $V = V'$ then F is called a holomorphic isometry of V.

It is well known that for any positive number k, the complex projective space $P_m(\mathbb{C}^{m+1})$ carries a complete Kähler metric of constant holomorphic sectional curvature k (see [12]). With respect to an inhomogeneous coordinate system $z^1, \dots, z^m$ it is given by

$$h = \frac{4}{k}\,\frac{(1 + \sum z^\alpha \overline{z}^\alpha)(\sum dz^\alpha d\overline{z}^\alpha) - (\sum \overline{z}^\alpha dz^\alpha)(\sum z^\alpha d\overline{z}^\alpha)}{(1 + \sum z^\alpha \overline{z}^\alpha)^2}\,. \tag{2.1}$$

We denote by $P_m(\mathbb{C}^{m+1})(k)$ the Kähler manifold with this structure.

In fact, if (V, J, h) is a $2m$-dimensional complete simply connected Kähler manifold of positive constant holomorphic sectional curvature k then (V, J, h) is holomorphically isometric to $P_m(\mathbb{C}^{m+1})(k)$ (see [7] and [8]).

Now, let $SU(m + 1)$ be the special unitary group and C a matrix of $SU(m + 1)$. If k is a positive real number, then the matrix C induces, in a natural form, a holomorphic isometry of $P_m(\mathbb{C}^{m+1})(k)$ (see, for instance, [12]).

The center of $SU(m + 1)$ is the discrete normal subgroup V_{m+1} given by,

$$V_{m+1} = \{\alpha I_{m+1} / \alpha \in \mathbb{C}, \quad \alpha^{m+1} = 1\}\,,$$

where I_{m+1} is the identity $(m + 1) \times (m + 1)$ matrix.

It is clear that if $C \in SU(m + 1)$ and $C' \in V_{m+1}$, the matrices C and CC' induce the same holomorphic isometry of $P_m(\mathbb{C}^{m+1})(k)$. In fact, if $I(P_m(\mathbb{C}^{m+1})(k))$ is the isometry group of $P_m(\mathbb{C}^{m+1})$ with the metric defined by (2.1) then (see [15], p. 385),

$$I(P_m(\mathbb{C}^{m+1})(k)) \simeq \frac{SU(m+1)}{V_{m+1}} \cup \alpha \frac{SU(m+1)}{V_{m+1}} \tag{2.2}$$

where α is the isometry of $P_m(\mathbb{C}^{m+1})(k)$ induced by the complex conjugation in $\mathbb{C}^{m+1}$, that is, if $(z_0, z_1, \ldots, z_m) \in \mathbb{C}^{m+1} - \{0\}$ we have

$$\alpha[(z_0, z_1, \ldots, z_m)] = [(\overline{z}_0, \overline{z}_1, \ldots, \overline{z}_m)] \, .$$

Let J be the complex structure of $P_m(\mathbb{C}^{m+1})(k)$. Then, the isometry α verifies $\alpha_* \circ J = -J \circ \alpha_*$. Therefore, from (2.2), we deduce that

$$A(P_m(\mathbb{C}^{m+1})(k)) \simeq \frac{SU(m+1)}{V_{m+1}} \, , \tag{2.3}$$

$A(P_m(\mathbb{C}^{m+1})(k))$ being the holomorphic isometry group of $P_m(\mathbb{C}^{m+1})(k)$.

2.2 – Cosymplectic manifolds of constant φ-sectional curvature

Let $(M, \varphi, \xi, \eta, g)$ be an **almost contact metric manifold**. Then, we have

$$\varphi^2 = -I + \eta \otimes \xi \, , \quad \eta(\xi) = 1 \, , \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \, , \tag{2.4}$$

for all $X, Y \in \mathfrak{X}(M)$, I being the identity transformation. The vector field ξ is called the **Reeb vector field** of the almost contact metric structure (φ, ξ, η, g).

From (2.4), we deduce that

$$\varphi\xi = 0, \quad \eta(X) = g(X, \xi) \, , \tag{2.5}$$

for all $X \in \mathfrak{X}(M)$.

On an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ we shall denote by $\mathfrak{F}$ the foliation defined by the Reeb vector field ξ and by $\mathfrak{F}^\perp$ the distribution determined by the normal bundle of $\mathfrak{F}$. Using (2.5), it is clear that $\mathfrak{F}^\perp$ is the distribution given by $\eta = 0$.

The **fundamental** 2-**form** Φ of M is defined by $\Phi(X,Y)=g(X,\varphi Y)$, for $X,Y \in \mathfrak{X}(M)$.

A mapping F between the almost contact metric manifolds (M, φ, ξ, η, g) and (M', φ', ξ', η', g') is said to be an **almost contact isometry** if F is an isometry which verifies $F_* \circ \varphi = \varphi' \circ F_*$ and $F^*\eta' = \eta$. The above conditions imply that $F_*\xi = \xi'$.

Let (M,φ,ξ,η,g) be an almost contact metric manifold and x a point of M. A plane section π in the tangent space T_xM is called a φ-**section** if there exists a unit vector u in T_xM orthogonal to ξ_x such that $\{u,\varphi_x u\}$ is an orthonormal basis of π. The sectional curvature $K_x u = R_x(u,\varphi_x u,u,\varphi_x u)$ is called a φ-**sectional curvature**. If $K_x u$ is a constant for all unit vector u in T_xM orthogonal to ξ_x and for all point $x \in M$, then M is called a **space of constant** φ-**sectional curvature**.

Denote by $T\mathfrak{F}^\perp$ the vector subbundle of the tangent bundle of M, TM, which consists of the tangent vectors to the distribution $\mathfrak{F}^\perp$ and, by $T_x\mathfrak{F}^\perp$ the fiber of $T\mathfrak{F}^\perp$ over x, for a point x of M, i.e., if $\tau_M : TM \longrightarrow M$ is the canonical projection then

$$T\mathfrak{F}^\perp = \{v \in TM/\eta_{\tau_M(v)}(v) = 0\}\ , \quad T_x\mathfrak{F}^\perp = \{v \in T_xM/\eta_x(v) = 0\}\ .$$

Let S be the Ricci curvature tensor of M; S is said to be **transversally positive definite** if S_x is positive definite on the subspace $T_x\mathfrak{F}^\perp$ for all $x \in M$.

An almost contact metric manifold (M,φ,ξ,η,g) is said to be **integrable** if $[\varphi,\varphi] = 0$ and **cosymplectic** if it is integrable and $d\eta = 0, d\Phi = 0$ (see [1]). On a cosymplectic manifold (M,φ,ξ,η,g) the Reeb vector field ξ and the 1-form η are parallel (see, for instance, [1]). Thus, if S is the Ricci curvature tensor of M then $S(\xi,\xi) = 0$, which implies that S cannot be positive definite.

We also have that the leaves of the foliation $\mathfrak{F}^\perp$ with the induced almost hermitian structure are Kähler manifolds (see [3]). Moreover, since $\mathfrak{F}^\perp$ is a totally geodesic foliation, we deduce that M is of constant φ-sectional curvature k if and only if the leaves of $\mathfrak{F}^\perp$ are of constant holomorphic sectional curvature k.

Now, let (V,J,h) be a Kähler manifold and M the product manifold $V \times \mathbb{R}$. Consider on M the almost contact metric structure (φ,ξ,η,g)

defined by

$$\begin{aligned}\varphi = J \circ (pr_1)_* \,, \quad \xi = \frac{\partial}{\partial t} \,, \quad \eta = (pr_2)^*(dt) \,, \\ g = (pr_1)^*(h) + (pr_2)^*(dt \otimes dt) \,,\end{aligned} \tag{2.6}$$

where $pr_1 : M \longrightarrow V$ and $pr_2 : M \longrightarrow \mathbb{R}$ are the canonical projections of $V \times \mathbb{R}$ onto the first and second factor respectively and t is the usual coordinate on $\mathbb{R}$. Then $(M, \varphi, \xi, \eta, g)$ is a cosymplectic manifold (see, for instance, [2] and [5]).

From (2.6), we also obtain that if V is of constant holomorphic sectional curvature k then $(M, \varphi, \xi, \eta, g)$ is a cosymplectic manifold of constant φ-sectional curvature k. In particular, for all positive real number k, the product manifold $P_m(C^{m+1})(k) \times \mathbb{R}$, with the almost contact metric structure (φ, ξ, η, g) given by (2.6), is a $(2m+1)$-dimensional complete simply connected cosymplectic manifold of constant φ-sectional curvature k.

In fact, if M is a complete simply connected cosymplectic manifold then M is almost contact isometric to the product of a complete simply connected Kähler manifold V with $\mathbb{R}$ (see [2]). Moreover, if M is of constant φ-sectional curvature k, using (2.6), we have that V is of constant holomorphic sectional curvature k.

The natural example of compact cosymplectic manifold is given by the product of a compact Kähler manifold (V, J, h) with the circle S^1. The cosymplectic structure (φ, ξ, η, g) on the product manifold $M = V \times S^1$ is defined by

$$\begin{aligned}\varphi = J \circ (pr_1)_* \,, \quad \xi = E \,, \quad \eta = (pr_2)^*(\theta) \,, \\ g = (pr_1)^*(h) + (pr_2)^*(\theta \otimes \theta) \,,\end{aligned} \tag{2.7}$$

where $pr_1 : M \longrightarrow V$ and $pr_2 : M \longrightarrow S^1$ are the canonical projections of $V \times S^1$ onto the first and second factor respectively, θ is the length element of S^1 and E is its dual vector field (see [2]).

If the Kähler manifold (V, J, h) is of constant holomorphic sectional curvature k then, from (2.7), we deduce that $(M, \varphi, \xi, \eta, g)$ is a cosymplectic manifold of constant φ-sectional curvature k. In particular, for all positive real number k, the product manifold $P_m(\mathbb{C}^{m+1})(k) \times S^1$ with the almost contact metric structure (φ, ξ, η, g) given by (2.7) is a compact cosymplectic manifold of constant φ-sectional curvature k.

Next, we shall obtain another examples of cosymplectic structures on the product of a compact Kähler manifold with the circle S^1.

Let (V, J, h) be a compact Kähler manifold and A^* a Killing vector field on V, that is, A^* satisfies

$$\mathfrak{L}_{A^*} h = 0 \,,$$

$\mathfrak{L}$ being the Lie derivate operator on V.

Define the almost contact metric structure (φ, ξ, η, g) on the product manifold $M = V \times S^1$ by

$$\begin{aligned} &\varphi = J \circ (pr_1)_* + c(pr_2)^*(\theta) \otimes JA^*, \qquad \xi = -A^* + \frac{E}{c}, \qquad \eta = c(pr_2)^*\theta, \\ &g = (pr_1)^*(h) + c[(pr_1)^*(\alpha^*) \otimes (pr_2)^*\theta + (pr_2)^*\theta \otimes (pr_1)^*(\alpha^*)] + \\ &\quad + c^2(1 + h(A^*, A^*))[(pr_2)^*(\theta) \otimes (pr_2)^*(\theta)] \,, \end{aligned} \tag{2.8}$$

where $pr_1 : M \to V$ and $pr_2 : M \to S^1$ are the canonical projections of $M = V \times S^1$ onto the first and second factor respectively, θ is the lenght element of S^1, E is its dual vector field, c is a real number, $c \neq 0$, and α^* is the 1-form on V defined by

$$\alpha^*(X) = h(X, A^*)$$

for all $X \in \mathfrak{X}(V)$.

We remark that if $c = 1$ and $A^* = 0$ then the almost contact metric structure (φ, ξ, η, g) is given as in (2.7). Moreover, we have

PROPOSITION 2.1. *Let (V, J, h) be a compact Kähler manifold, A^* a Killing vector field on V and (φ, ξ, η, g) the almost contact metric structure on the product manifold $M = V \times S^1$ given by* (2.8)*. Then:*

1. *$(M, \varphi, \xi, \eta, g)$ is a compact cosymplectic manifold.*
2. *The cosymplectic manifold $(M, \varphi, \xi, \eta, g)$ is of constant φ-sectional curvature k if and only if the Kähler manifold (V, J, h) is of constant holomorphic sectional curvature k.*

Proof. Since V is a compact Kähler manifold and A^* a Killing vector field on V then we deduce that A^* is an infinitesimal holomorphic transformation of V (see [9]), that is,

$$\mathfrak{L}_{A^*}J = 0\,. \tag{2.9}$$

A direct computation, using (2.8) and (2.9) shows that

$$\begin{aligned}
[\varphi,\varphi]((X,0),(Y,0)) &= ([J,J](X,Y),0) = 0,\\
[\varphi,\varphi]((X,0),(0,E)) &= -[\varphi,\varphi]((0,E),(X,0)) =\\
&= (c\,([J,J](X,A^*) + (\mathfrak{L}_{A^*}J)JX),0) = 0,
\end{aligned}$$

for all $X, Y \in \mathfrak{X}(V)$.

Thus, the almost contact metric manifold (M,φ,ξ,η,g) is integrable.

Now, if Φ is the fundamental 2-form of the structure (φ,ξ,η,g) and Ω is the Kähler 2-form of (V,J,h), from (2.8), we obtain that

$$\Phi = (pr_1)^*(\Omega) + 2c(pr_2)^*(\theta) \wedge (pr_1)^*(\alpha^* \circ J)\,. \tag{2.10}$$

Moreover, since J is parallel we have that

$$2d(\alpha^* \circ J)(X,Y) = (\mathfrak{L}_{A^*}h)(X,JY) + h(X,(\mathfrak{L}_{A^*}J)(Y)) = 0 \tag{2.11}$$

for all $X, Y \in \mathfrak{X}(V)$.

Therefore, using (2.10) and (2.11), we conclude that the 2-form Φ is closed. This proves 1.

Let x_0 be a point of V and $t_0 \in \mathbb{R}$.

From (2.8), we deduce that the leaf of the foliation $\mathfrak{F}^\perp$ over $(x_0,[t_0]) \in M = V \times S^1$ is $V \times \{[t_0]\}$. Furthermore, the induced Kähler structure on $V \times \{[t_0]\}$ by the cosymplectic structure (φ,ξ,η,g) is just (J,h) (see (2.8)). This proves 2. ∎

Next, we shall suppose that the compact Kähler manifold V is $P_m(\mathbb{C}^{m+1})(k)$, with $k \in \mathbb{R}, k > 0$.

The Lie algebra $\mathfrak{su}(m+1)$ of the special unitary group $SU(m+1)$ consists of all complex skew-hermitian matrices with null trace:

$$\mathfrak{su}(m+1) = \{A \in \mathrm{gl}(m+1,\mathbb{C}) \;/\; {}^T\bar{A} = -A, \mathrm{trace} A = 0\}\,.$$

Thus, if A is a matrix of $\mathfrak{su}(m+1)$ and A^* is the infinitesimal generator of the action of $SU(m+1)$ on $P_m(\mathbb{C}^{m+1})(k)$ corresponding to A then, from (2.3), we obtain that A^* is a Killing vector field on $P_m(\mathbb{C}^{m+1})(k)$. In fact, A^* is also an infinitesimal holomorphic transformation of $P_m(\mathbb{C}^{m+1})(k)$.

Suppose that (φ, ξ, η, g) is the almost contact metric structure given by (2.8) on the product manifold $M = P_m(\mathbb{C}^{m+1})(k) \times S^1$, taking as Killing vector field on $V = P_m(\mathbb{C}^{m+1})(k)$ the vector field A^*.

Then, using proposition 2.1, we have that $(P_m(\mathbb{C}^{m+1})(k) \times S^1, \varphi, \xi, \eta, g)$ is a compact cosymplectic manifold with positive constant φ-sectional curvature k. We shall denote by $(P_m(\mathbb{C}^{m+1})(k) \times S^1)(c, A)$ ($A \in \mathfrak{su}(m+1)$ and $c \in \mathbb{R}, c \neq 0$) the cosymplectic manifold with this structure.

Now, let M be a compact cosymplectic manifold.

We remark that if $H_1(M, \mathbb{Z})$ is the first integral homology group of M and $b_1(M)$ is the first Betti number then, since $b_1(M) \geq 1$ (see [2] and [6]), we have that the rank of $H_1(M, \mathbb{Z})$ is greater or equal than 1. Because of $H_1(M, \mathbb{Z})$ is isomorphic to the quotient group $\frac{\pi_1(M)}{[\pi_1(M), \pi_1(M)]}$, $\pi_1(M)$ being the fundamental group of M and $[\pi_1(M), \pi_1(M)]$ the commutator subgroup of $\pi_1(M)$, we deduce that the fundamental group $\pi_1(M)$ is infinite. Therefore, we conclude that a compact simply connected manifold cannot admit a cosymplectic structure.

Finally, using a result of Ludden [13] (see also theorem 3.5 of [10]), we shall obtain an explicit expression for the Ricci curvature tensor on a cosymplectic manifold of constant φ-sectional curvature.

Proposition 2.2. *If $(M, \varphi, \xi, \eta, g)$ is a $(2m+1)$-dimensional cosymplectic manifold of constant φ-sectional curvature k and S is the Ricci curvature tensor of M then,*

$$S(X, Y) = \frac{k(m+1)}{2}(g(X, Y) - \eta(X)\eta(Y)) ,$$

for all $X, Y \in \mathfrak{X}(M)$.

Remark 2.1. Let $(M, \varphi, \xi, \eta, g)$ be a cosymplectic manifold of positive constant φ-sectional curvature and S the Ricci curvature tensor of M. From proposition 2.2, we deduce that S is transversally positive definite.

3 – Compact cosymplectic manifolds with transversally positive definite Ricci tensor

In this section, we shall study the fundamental group of a compact cosymplectic manifold with transversally positive definite Ricci tensor. As a consequence, we shall obtain that if M is a compact cosymplectic manifold of positive constant φ-sectional curvature k then there exists a constant $c, c \neq 0$, and a matrix A of $\mathfrak{su}(m+1)$ such that M is almost contact isometric to the cosymplectic manifold $(P_m(\mathbb{C}^{m+1})(k)\times S^1)(c,A)$. First, we prove a result which will be useful in the sequel.

Theorem 3.1. *Let (M,φ,ξ,η,g) be a compact cosymplectic manifold such that $b_1(M)=1$, $b_1(M)$ being the first Betti number of M.*

1. *If L is a leaf of the foliation $\mathfrak{F}^\perp$ given by $\eta=0$ and $\pi_1(M)$ (respectively $\pi_1(L)$) is the fundamental group of M (respectively L) then the inclusion of the leaf L induces a monomorphism $i:\pi_1(L)\longrightarrow\pi_1(M)$ onto a normal subgroup and the quotient group $\frac{\pi_1(M)}{\pi_1(L)}$ is isomorphic to $\mathbb{Z}$.*
2. *There exists a fibration $\tau: M\longrightarrow S^1$ of M onto the circle S^1 such that the leaves of $\mathfrak{F}^\perp$ are the fibres of τ.*
3. *The leaves of $\mathfrak{F}^\perp$ are compact.*

Proof. Denote by $\tilde{M}$ the universal covering space of M with covering map $\pi:\tilde{M}\longrightarrow M$ and let $(\tilde{\varphi},\tilde{\xi},\tilde{\eta},\tilde{g})$ be the induced cosymplectic structure on $\tilde{M}$, i.e.,

$$\pi_*\circ\tilde{\varphi}=\varphi\circ\pi_*\ ,\quad \pi_*(\tilde{\xi})=\xi\ ,\quad \tilde{\eta}=\pi^*\eta\ ,\quad \tilde{g}=\pi^*g\ . \tag{3.1}$$

We have that $(\tilde{M},\tilde{\varphi},\tilde{\xi},\tilde{\eta},\tilde{g})$ is almost contact isometric to the product of a simply connected complete Kähler manifold (V,J,h) with $\mathbb{R}$ and thus, from (2.6), we obtain that the leaves of the foliation $\widetilde{\mathfrak{F}^\perp}$ are of the form $V\times\{t_0\}$, with $t_0\in\mathbb{R}$, $\widetilde{\mathfrak{F}^\perp}$ being the lift of the foliation $\mathfrak{F}^\perp$ to $\tilde{M}\simeq V\times\mathbb{R}$. Moreover, if $(\tilde{y},t_0)$ is a point of $V\times\mathbb{R}$, and L is the leaf of the foliation $\mathfrak{F}^\perp$ over $y=\pi(\tilde{y},t_0)$ then $V\times\{t_0\}$ is the universal covering space of L and $\pi_{|V\times\{t_0\}}:V\times\{t_0\}\longrightarrow L$ is the covering map. In fact, the mapping $\pi_{|V\times\{t_0\}}:V\times\{t_0\}\longrightarrow L$ is a local holomorphic isometry between the Kähler manifolds $V\times\{t_0\}$ and L.

Now, realize $\pi_1(M)$ as the group of covering transformations on $\tilde{M} = V \times \mathbb{R}$. Let $\tilde{C} : V \times \mathbb{R} \longrightarrow V \times \mathbb{R}$ be a covering transformation. If we consider on $V \times \mathbb{R}$ the cosymplectic structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ then, using (3.1), we deduce that the mapping $\tilde{C}$ is an almost contact isometry. Consequently, from (2.6), we obtain

$$\tilde{C}_* \circ \tilde{\varphi} = \tilde{\varphi} \circ \tilde{C}_* \ , \quad \tilde{C}_*(\frac{\partial}{\partial t}) = \frac{\partial}{\partial t} \ , \quad \tilde{C}^*(dt) = dt, \quad \tilde{C}^*(h) = h \ .$$

This implies that there exists a real number t_1 and a holomorphic isometry $C : (V, J, h) \longrightarrow (V, J, h)$ such that

$$\tilde{C}(\tilde{y}, t) = (C(\tilde{y}), t + t_1) \ , \tag{3.2}$$

for all $(\tilde{y}, t) \in V \times \mathbb{R}$.

Denote by L the leaf of the foliation $\mathfrak{F}^{\perp}$ over x, where x is a point of M such that $x = \pi(\tilde{x}, t_0)$, with $(\tilde{x}, t_0) \in V \times \mathbb{R}$. Realize $\pi_1(L)$ as the group of covering transformations on $V \times \{t_0\}$.

Let $\overline{C} : V \times \{t_0\} \longrightarrow V \times \{t_0\}$ be a covering transformation. Suppose that $\overline{C}(\tilde{y}, t_0) = (C(\tilde{y}), t_0)$, for all $(\tilde{y}, t_0) \in V \times \{t_0\}$. It is clear that $(i\overline{C})(\tilde{x}, t_0) = (C(\tilde{x}), t_0)$, where $i : \pi_1(L) \longrightarrow \pi_1(M)$ is the canonical homomorphism. Thus, from (3.2), we obtain that

$$(i\overline{C})(\tilde{y}, t) = (C(\tilde{y}), t) \ , \tag{3.3}$$

for all $(\tilde{y}, t) \in V \times \mathbb{R}$. Therefore i is a monomorphism. The subgroup $i(\pi_1(L))$ will also be denoted by $\pi_1(L)$.

Using (3.2), we can define a homomorphism $\rho : \pi_1(M) \longrightarrow \mathrm{Diff}(\mathbb{R})$ of $\pi_1(M)$ onto the group of diffeomorphisms of $\mathbb{R}$. The fundamental group $\pi_1(L)$ is the normal subgroup $\mathrm{Ker}\rho$ and the image of ρ, $\mathrm{Im}\,\rho$, is a group of translations and, hence, is abelian.

Since M is compact, $\pi_1(M)$ is finitely generated. Therefore, $\mathrm{Im}\,\rho \simeq \frac{\pi_1(M)}{\pi_1(L)}$ is also finitely generated and abelian, so it is isomorphic to $\mathbb{Z}^r$, with $r \geq 0$.

If $r = 0$, then $\pi_1(L) = \pi_1(M)$. Consequently, using (3.3) and the fact that $M \simeq \frac{V \times \mathbb{R}}{\pi_1(M)}$ and $L \simeq \frac{V \times \{t_0\}}{\pi_1(L)}$, we deduce that $M \simeq L \times \mathbb{R}$ which, in view of the compactness of M, is a contradiction.

If $r \geq 2$, then the group $\frac{\pi_1(M)}{\pi_1(L)} \simeq \mathbb{Z}^r$ is abelian, which implies that $[\pi_1(M), \pi_1(M)] \subseteq \pi_1(L)$, $[\pi_1(M), \pi_1(M)]$ being the commutator subgroup of $\pi_1(M)$. Thus, an epimorphism $\beta : \frac{\pi_1(M)}{[\pi_1(M),\pi_1(M)]} \longrightarrow \frac{\pi_1(M)}{\pi_1(L)} \simeq \mathbb{Z}^r$ is naturally induced. But, it is not possible, since the quotient group $\frac{\pi_1(M)}{[\pi_1(M),\pi_1(M)]}$ is isomorphic to the first integral homology group $H_1(M, \mathbb{Z})$ and the rank of $H_1(M, \mathbb{Z})$ is equal to the first Betti number of M.

From the above considerations, we conclude that the group $\operatorname{Im}\rho \simeq \frac{\pi_1(M)}{\pi_1(L)}$ is isomorphic to $\mathbb{Z}$.

Let $\gamma : \mathbb{Z} \longrightarrow \operatorname{Im}\rho \simeq \frac{\pi_1(M)}{\pi_1(L)}$ be an isomorphism. Suppose that $\gamma(1)$ is the translation defined by

$$\gamma(1) : \mathbb{R} \longrightarrow \mathbb{R} \quad t \longrightarrow \gamma(1)(t) = t + c\,,$$

with $c \in \mathbb{R}$, $c \neq 0$. Denote the diffeomorphism $\gamma(1)$ by T_c. Then, $\operatorname{Im}\rho = \{T_{cn}/n \in \mathbb{Z}\}$.

Thus, we obtain a fibration $\tau : M \simeq \frac{V\times\mathbb{R}}{\pi_1(M)} \longrightarrow S^1 \simeq \frac{\mathbb{R}}{\mathbb{Z}}$ such that the following diagram is commutative:

$$\begin{array}{ccc} V \times \mathbb{R} & \xrightarrow{\ \pi\ } & M \\ \Big\downarrow{\scriptstyle pr_2} & & \Big\downarrow{\scriptstyle \tau} \\ \mathbb{R} & \xrightarrow{\ p\ } & S^1 \end{array}$$

where p is the mapping of $\mathbb{R}$ onto S^1 given by $p(t) = [\frac{t}{c}]$, for all $t \in \mathbb{R}$.

We notice that $\tau^*(c\theta) = \eta$ and $\tau_*(\xi) = \frac{E}{c}$, θ being the length element of S^1 and E its dual vector field. Moreover, it is clear that the leaves of the foliation $\mathfrak{F}^\perp$ are just the fibres of the fibration $\tau : M \longrightarrow S^1$. This implies that the leaves of $\mathfrak{F}^\perp$ are closed submanifolds of M and, since M is compact, they are also compact. $\square$

Remark 3.1. In [6], we have constructed an example of compact cosymplectic manifold M which is not a global product of a Kähler manifold with the circle S^1. The manifold M is flat and its first Betti number is 1.

Next, we shall show that the fundamental group of a compact cosymplectic manifold with transversally positive definite Ricci tensor is isomorphic to $\mathbb{Z}$.

THEOREM 3.2. *Let M be a compact cosymplectic manifold with transversally positive definite Ricci tensor and $\pi_1(M)$ the fundamental group of M. Then $\pi_1(M)$ is isomorphic to $\mathbb{Z}$. In particular, the first integral homology group of M is also isomorphic to $\mathbb{Z}$.*

PROOF. Denote by S the Ricci curvature tensor of M and by (φ, ξ, η, g) the cosymplectic structure of M.

Let β be a harmonic 1-form on M such that $\beta(\xi) = 0$, and B the metrically equivalent vector field to the 1-form β, i.e., B is the vector field on M which verifies $g(X, B) = \beta(X)$, for all $X \in \mathfrak{X}(M)$.

We remark that

$$\eta(B) = g(\xi, B) = \beta(\xi) = 0\,. \tag{3.4}$$

Using a well-known result (see [9], pag. 86) we obtain that $\int_M (S(B, B) + \|\nabla\beta\|^2) * 1 = 0$, where $*$ denotes the Hodge star isomorphism and ∇ the Riemannian connection of g. Thus, from (3.4), we deduce that $B = 0$ which implies $\beta = 0$.

Therefore, if $\Omega^1_{H\xi}(M)$ is the subspace of harmonic 1-forms γ such that $\gamma(\xi) = 0$, we have that $\Omega^1_{H\xi}(M) = \{0\}$.

On the other hand, in [6] it is proved that the dimension of the subspace $\Omega^1_{H\xi}(M)$ is $b_1(M) - 1$, where $b_1(M)$ is the first Betti number of M (see theorem 5 of [6]). Consequently, we obtain that $b_1(M) = 1$.

Now, let L be a leaf of the foliation $\mathfrak{F}^\perp$ on M given by $\eta = 0$. Since the Reeb vector field ξ is parallel then, using theorem 3.1 and the fact that $\mathfrak{F}^\perp$ is a totally geodesic foliation, we deduce that L is a compact Kähler manifold with positive definite Ricci tensor. Thus, from theorem A of [11], we conclude that L is simply connected. In view of theorem 3.1, this fact shows that $\pi_1(M)$ is isomorphic to $\mathbb{Z}$.

Finally, since the first integral homology group of M, $H_1(M, \mathbb{Z})$, is isomorphic to the quotient group $\frac{\pi_1(M)}{[\pi_1(M), \pi_1(M)]}$, we have that $H_1(M, \mathbb{Z})$ is also isomorphic to $\mathbb{Z}$. □

Next, we shall prove, for a compact cosymplectic manifold with positive constant φ-sectional curvature, the announced result at the beginning of this section.

THEOREM 3.3. *Let $(M, \varphi, \xi, \eta, g)$ be a $(2m + 1)$-dimensional compact cosymplectic manifold with positive constant φ-sectional curvature*

k. Then there exist a constant $c, c \neq 0$, and a matrix A of $\mathfrak{su}(m+1)$ such that M is almost contact isometric to the cosymplectic manifold $(P_m(\mathbb{C}^{m+1})(k) \times S^1)(c, A)$.

Proof. Let $\tilde{M}$ be the universal covering space of M and $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ the induced cosymplectic structure on $\tilde{M}$. Then, we obtain that $(\tilde{M}, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is almost contact isometric to the product manifold $P_m(\mathbb{C}^{m+1})(k) \times \mathbb{R}$.

Moreover, from theorem 3.2, we deduce that the first Betti number of M is equal to 1. Thus, if L is a leaf of the foliation $\mathfrak{F}^\perp$ given by $\eta = 0$ then, from theorem 3.1, we have that L is a compact Kähler manifold with positive constant holomorphic sectional curvature k. In particular, the Ricci curvature tensor of L is positive definite (see, for instance, [12] pag. 168) which implies that L is simply connected (see theorem A of [11]).

Now, realize the fundamental group of M, $\pi_1(M)$, as the group of covering transformations on $\tilde{M} = P_m(\mathbb{C}^{m+1})(k) \times \mathbb{R}$. Then, there exist a real number $c \neq 0$ and an isomorphism γ of $\mathbb{Z}$ onto $\pi_1(M)$

$$\mathbb{Z} \longrightarrow \pi_1(M) , \quad \gamma(n) = \tilde{C}_n ,$$

such that the covering transformation $\tilde{C}_n : P_m(\mathbb{C}^{m+1})(k) \times \mathbb{R} \longrightarrow P_m(\mathbb{C}^{m+1})(k) \times \mathbb{R}$ is given by

$$\tilde{C}_n(\tilde{x}, t) = (C_n(\tilde{x}), t + cn) ,$$

for all $(\tilde{x}, t) \in P_m(\mathbb{C}^{m+1})(k) \times \mathbb{R}$, being $C_n : P_m(\mathbb{C}^{m+1})(k) \longrightarrow P_m(\mathbb{C}^{m+1})(k)$ a holomorphic isometry of $P_m(\mathbb{C}^{m+1})(k)$ (see the proof of theorem 3.1). Therefore, M is almost contact isometric to $\frac{P_m(\mathbb{C}^{m+1})(k) \times \mathbb{R}}{\{\tilde{C}_n\}_{n \in \mathbb{Z}}}$.

It is clear that $C_n = (C_1)^n$, for all $n \in \mathbb{Z}$, and, if C_1 is the identity transformation of $P_m(\mathbb{C}^{m+1})(k)$, then M is almost contact isometric to the product manifold $(P_m(\mathbb{C}^{m+1})(k) \times S^1)(c, 0)$. If that is not the case, we proceed as follows.

Using (2.3), we deduce that there exists a matrix C of the special unitary group $SU(m+1)$ which induces the holomorphic isometry C_1 of $P_m(\mathbb{C}^{m+1})(k)$. Since the group $SU(m+1)$ is compact it admits a complete bi-invariant metric (in fact, if B is the Killing form of $SU(m+1)$ then, using that $SU(m+1)$ is semisimple, we obtain that $-B$ is a bi-invariant metric on $SU(m+1)$). Thus, the geodesics of $SU(m+1)$ starting at the identity $(m+1)\times(m+1)$ matrix are just the one-parameter subgroups of $SU(m+1)$. Consequently, there exists a one-parameter subgroup $\alpha : \mathbb{R} \longrightarrow SU(m+1)$ of $SU(m+1)$ such that $\alpha(c) = C$.

Let A be the left invariant vector field on $SU(m+1)$ associated to the one-parameter subgroup α. A can be viewed as a matrix of $\mathfrak{su}(m+1)$ (the Lie algebra of $SU(m+1)$, see subsection 2.2). We shall denote by A^* the infinitesimal generator of the action of $SU(m+1)$ on $P_m(\mathbb{C}^{m+1})(k)$ corresponding to A and by α^* the 1-form on $P_m(\mathbb{C}^{m+1})(k)$ given by

$$\alpha^*(X) = h(X, A^*)$$

for all $X \in \mathfrak{X}(P_m(\mathbb{C}^{m+1})(k))$, where (J, h) is the Kähler structure of $P_m(\mathbb{C}^{m+1})(k)$. We consider on the product manifold $P_m(\mathbb{C}^{m+1})(k) \times \mathbb{R}$ the almost contact metric structure $(\tilde{\varphi}', \tilde{\xi}', \tilde{\eta}', \tilde{g}')$ defined by

$$\begin{aligned} \tilde{\varphi}' &= J \circ (pr_1)_* + (pr_2)^*(dt) \otimes JA^*, \quad \tilde{\xi}' = -A^* + \frac{\partial}{\partial t}, \quad \tilde{\eta}' = (pr_2)^*(dt)\,, \\ \tilde{g}' &= (pr_1)^*(h) + (pr_2)^*(dt) \otimes (pr_1)^*\alpha^* + (pr_1)^*\alpha^* \otimes (pr_2)^*(dt) + \\ &\quad + (1 + h(A^*, A^*))[(pr_2)^*(dt) \otimes (pr_2)^*(dt)]\,, \end{aligned} \tag{3.5}$$

where $pr_1 : P_m(\mathbb{C}^{m+1})(k) \times \mathbb{R} \to P_m(\mathbb{C}^{m+1})(k)$ and $pr_2 : P_m(\mathbb{C}^{m+1})(k) \times \mathbb{R} \to \mathbb{R}$ are the canonical projections of $P_m(\mathbb{C}^{m+1})(k) \times \mathbb{R}$ onto the first and second factor respectively and t is the coordinate on $\mathbb{R}$.

Proceeding as in the proof of proposition 2.1, we have that $(P_m(\mathbb{C}^{m+1})(k) \times \mathbb{R}, \tilde{\varphi}', \tilde{\xi}', \tilde{\eta}', \tilde{g}')$ is a cosymplectic manifold of positive constant $\tilde{\varphi}'$-sectional curvature k.

Now, we define the diffeomorphism $\tilde{F}$ of $P_m(\mathbb{C}^{m+1})(k) \times \mathbb{R}$ onto $P_m(\mathbb{C}^{m+1})(k) \times \mathbb{R}$ by

$$\tilde{F}(\tilde{x}, t) = (\alpha(t)\tilde{x}, t) \tag{3.6}$$

for all $(\tilde{x}, t) \in P_m(\mathbb{C}^{m+1})(k) \times \mathbb{R}$. Then, from (2.3), (2.6), (3.5) and (3.6), we deduce that $\tilde{F}$ is an almost contact isometry of the cosymplectic manifold $(P_m(\mathbb{C}^{m+1})(k) \times \mathbb{R}, \tilde{\varphi}', \tilde{\xi}', \tilde{\eta}', \tilde{g}')$ onto the manifold $P_m(\mathbb{C}^{m+1})(k) \times \mathbb{R}$ with the usual cosymplectic structure. Moreover, the diffeomorphism $\tilde{F}$ induces a diffeomorphism F between the cosymplectic manifolds

$(P_m(\mathbb{C}^{m+1})(k) \times S^1)(c, A)$ and M in such a way that the following diagram

$$\begin{array}{ccc} P_m(\mathbb{C}^{m+1})(k) \times \mathbb{R} & \xrightarrow{\tilde{F}} & P_m(\mathbb{C}^{m+1})(k) \times \mathbb{R} \\ \pi' \Big\downarrow & & \Big\downarrow \pi \\ (P_m(\mathbb{C}^{m+1})(k) \times S^1)(c, A) \simeq & & \\ \simeq \dfrac{P_m(\mathbb{C}^{m+1})(k) \times \mathbb{R}}{\{1_{P_m(\mathbb{C}^{m+1})(k)} \times T_{cn}\}_{n\in\mathbb{Z}}} & \xrightarrow[F]{} & M \simeq \dfrac{P_m(\mathbb{C}^{m+1})(k) \times \mathbb{R}}{\{\tilde{C}_n\}_{n\in\mathbb{Z}}} \end{array}$$

is commutative, being π and π' the canonical projections and T_{cn} the translation of $\mathbb{R}$ defined by $T_{cn}(t) = t + cn$ for all $t \in \mathbb{R}$. It is clear that F is also an almost contact isometry. This ends the proof of the theorem. □

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