Rendiconti di Matematica, Serie VII
Volume 17, Roma (1997), 625-658

Singular holomorphic foliations with attractive leaves

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In memoriam PETER THULLEN † June 24, 1996

Riassunto: *L'articolo contiene una rassegna sulle fogliazioni olomorfe singolari coerenti su varietà complesse connesse. Queste fogliazioni sono studiate usando tecniche della teoria dei sistemi dinamici; nozioni come quelle di insieme limite, bacino di attrazione, attrattore sono definite geometricamente. Sotto le ipotesi che vi siano foglie ovunque, che esistano solo "poche" foglie compatte e che lo spazio delle foglie non compatte sia di Hausdorff fortemente o debolmente, viene mostrato che: il dominio di attrazione di ogni foglia compatta L è una sottovarietà analitica di dimensione maggiore di quella delle foglie; essa consiste, se L è quasi attrattiva, nell'unione di L con l'intero spazio delle foglie non compatte; nel caso invece in cui la fogliazione definisca una relazione di equivalenza aperta, L è sempre attrattiva. Per ogni foglia non compatta L il numero degli attrattori quasi globali è limitato superiormente dal numero delle componenti connesse dell'insieme limite di L; questo è a sua volta limitato superiormente dal numero degli estremi di L. Questi risultati sono interpretati per $\mathbb{C}$-azioni olomorfe. Le fogliazioni olomorfe associate sono per lo più singolari; su $\mathbb{P}^n$ per esempio esse sono sempre singolari. La teoria generale è illustrata da $\mathbb{C}$-azioni diagonali su $\mathbb{P}^n$ di tipo generale.*

Key Words and Phrases: *Singular holomorphic foliations – Few compact leaves – Almost global attractors – Limit cycles – Ends of leaves – Holomorphic $\mathbb{C}$-actions – Thullen's singularity theorem.*

A.M.S. Classification: 58F12 – 58F18 – 58F21

This paper was written while the authors were supported by the Swiss National Science Foundation under grant 21-36185.95.

Il contenuto di questo lavoro è stato oggetto di una conferenza tenuta dal primo Autore, H. Holmann, al Convegno "Recenti sviluppi in Geometria Differenziale", Università "La Sapienza", Roma, 11-14 Giugno 1996.

Abstract: *The article starts with a survey on coherent singular holomorphic foliations on connected complex manifolds. These foliations are studied using techniques of dynamical systems, notions like limit sets, basins of attraction, attractors being defined geometrically. Under the assumptions that there are leaves everywhere, that there exist only "few" compact leaves and that the space of non-compact leaves is hausdorff respectively weakly hausdorff, it is shown: The domain of attractivity of a compact leaf L is an analytic subvariety of dimension bigger than the dimension of the leaves; it consists of the union of L with the whole space of non-compact leaves, if L is almost attractive; in case the foliation defines an open equivalence relation, L is always attractive. For every non-compact leaf L the number of almost global attractors is bounded from above by the number of connected components of the limit set of L, which again is bounded from above by the number of ends of L. These results are interpreted for holomorphic* $\mathbb{C}$*-actions. The associated holomorphic foliations are mostly singular, on* $\mathbb{P}^n$ *for example they are always singular. The general theory is illustrated by diagonal* $\mathbb{C}$*-actions on* $\mathbb{P}^n$ *of general type.*

– Introduction

A holomorphic $\mathbb{C}$-action on a complex manifold X without fixed points defines a regular holomorphic foliation on X, the leaves being the orbits of the action. In order to study the dynamics of such an action it is useful (compare [8] and [10]) to develop a theory of limit sets, attractors etc. for holomorphic foliations and then apply it to the special case of group actions.

"Compact complex manifolds very rarely foliate without singularities. Foliations with singularities, however, exist in great abundance" (see [2], remark (c) on page 287). Especially on $\mathbb{P}^n$ there are no nontrivial regular holomorphic foliations. Therefore in this article we shall study the dynamics of singular holomorphic foliations. A theory of such foliations is already developped in [2], [4], [18] and [19].

In section 1 we recall the basic definitions of the theory of singular holomorphic foliations following [19]. Via the notion of integrals we get a connection to the type of foliations with singularities given in [6].

In section 2 we define integral varieties and leaves for singular holomorphic foliations, following [19]. The notion of a leaf in this article is a generalization of that in [6]. In general there exist points where no leaf passes through (for a simple example see 2.8). We shall consider only foliations with leaves everywhere. For these we get a leaf space $X/\mathcal{F}$ and a corresponding equivalence relation $R_{\mathcal{F}}$, the leaf $L(x)$ passing through x being the equivalence class of $x \in X$.

In the sections 3, 4 and 5, we follow [19] and [10]. But here we admit foliations with singularities.

In section 3 we start with the basic definitions of the theory of ends. Then we define the limit set $\lim L$ of a leaf L and study the connection between the number $l(L)$ of connected components of $\lim L$ and the number $e(L)$ of ends of L. Under suitable conditions we get $l(L) \leq e(L)$ (compare the propositions 3.12 and 3.13).

In section 4 we define the basin of attraction $\mathcal{A}(M)$ of an arbitrary non-void, closed and $\mathcal{F}$-saturated subset M of X as follows:

$\mathcal{A}(M) := M \ \cup\ \{x \in X;\ \lim L(x) \neq \emptyset$ and a connected component of $\lim L(x)$ is contained in $M\}$.

We always make the following **assumptions**:

(a) Every leaf L of $\mathcal{F}$ is a locally analytic subset of X of the same pure dimension $p < \dim X$.

(b) The compact leaves form a non-void locally finite family $(\Gamma_j)_{j \in J}$.

(c) On $X' := X \setminus \bigcup_{j \in J} \Gamma_j$, the restriction $R' \subset X' \times X'$ of $R_{\mathcal{F}}$ is an open analytic equivalence relation.

In the following main results of our article the numbers $c(\mathcal{F})$ and $c'(\mathcal{F})$ of compact resp. compact almost attractive leaves of $\mathcal{F}$ and different notions of attractivity (compare section 4) play an important role.

THEOREM.

(1) *For every compact leaf* Γ *of* $\mathcal{F}$ *the basin of attraction* $\mathcal{A}(\Gamma)$ *is an analytic subset of* $X' \cup \Gamma$ *of dimension* $> p$ *everywhere.*

(2) *If a compact leaf* Γ *of* $\mathcal{F}$ *is almost attractive then it is attractive and almost globally attractive, even* $\mathcal{A}(\Gamma) = X' \cup \Gamma$.

(3) *For every non compact leaf* L *of* $\mathcal{F}$ *we have* $c'(\mathcal{F}) \leq l(L) \leq e(L)$.

(4) *If* codim$\mathcal{F} = 1$, *then* $\mathcal{A}(\Gamma) = X' \cup \Gamma$ *for all compact leaves of* $\mathcal{F}$, i.e. $c'(\mathcal{F}) = c(\mathcal{F})$.

(5) *If* X *is compact, then there exists at least one almost attractive compact leaf,* i.e. $c'(\mathcal{F}) \geq 1$ (compare (2)).

THEOREM. *If* $R_{\mathcal{F}}$ *is an open equivalence relation, then*

(1) *For every non-compact leaf* L *of* $\mathcal{F}$ *we get* $\lim L = \bigcup_{j \in J} \Gamma_j$.

(2) *For every compact leaf* Γ *of* $\mathcal{F}$ *we have* $\mathcal{A}(\Gamma) = X' \cup \Gamma$.

(3) $c'(\mathcal{F}) = c(\mathcal{F}) = l(L) \leq e(L)$ *for all noncompact leaves* L *of* $\mathcal{F}$.

We obtain these results by studying equivalence relations under analogous assumptions in the first part of section 4. The main tool for the proofs is the extension theorem of Thullen-Remmert-Stein (compare [22]).

In section 5 we study holomorphic $\mathbb{C}$-actions $\Phi : \mathbb{C} \times X \to X$ on complex manifolds X together with the corresponding singular holomorphic foliation $\mathcal{F}_\Phi$.

We work under assumptions (compare 5.7) which are partially weaker then those in section 4. However, they imply the stronger ones. Since the non-compact leaves of $\mathcal{F}_\Phi$ have 1 or 2 ends, we can strengthen the results of section 4 (compare proposition 5.9); in particular, if X is compact and Φ operates effectively, then all non-compact leaves have only one end and there exists exactly one compact leaf and this is a global attractor. In this case one can even weaken the hausdorff condition (compare the assumption 5.12 and proposition 5.14).

In section 6 we study holomorphic $\mathbb{C}$-actions on $\mathbb{P}^n$ which are diagonal of general type (compare definition 6.1). We show that they define singular foliations with leaves everywhere and give an explicit description of their leaves, limit sets and attractors. We get examples satisfying the assumptions of section 5.

1 – Holomorphic foliations with singularities

In this section we give a survey of the theory of holomorphic foliations with singularities. For details and more general aspects we refer to [19].

Let X be an n-dimensional paracompact connected complex manifold. We denote the sheaf of holomorphic vector fields by $\Theta = \Theta_X$ and the sheaf of holomorphic 1-forms by $\Omega = \Omega_X$. A **regular foliation $\mathcal{F}$ on X** is a holomorphic foliation in the usual sense, all leaves being complex manifolds of a fixed dimension called **dim $\mathcal{F}$**. Such a regular foliation defines holomorphic subsheaves $\Theta_\mathcal{F}$ of Θ and $\Omega_\mathcal{F}$ of Ω in a canonical way.

Proposition 1.1. *For a regular foliation $\mathcal{F}$, the sheaf $\Omega_\mathcal{F}$ has the following properties:*

(1) *$\Omega/\Omega_\mathcal{F}$ is locally free.*

(2) *$\Omega_\mathcal{F}$ is involutive,* i.e. $d\Omega_\mathcal{F} \wedge \Lambda^q \Omega_\mathcal{F} = 0$ *for* $q :=\ \mathrm{rank}\, \Omega_\mathcal{F}$.

The Frobenius theorem ([16, p. 117 f]) tells us:

THEOREM 1.2. *If Ω' is a holomorphic subsheaf of Ω with properties* (1) *and* (2), *then there is a regular holomorphic foliation $\mathcal{F}$ on X such that $\Omega' = \Omega_{\mathcal{F}}$.*

An analogous result holds for $\Theta_{\mathcal{F}}$.

We are going to explain now what we understand by a holomorphic foliation with singularities.

For this purpose we consider analytic subsets A of X with $\dim A < \dim X$ and regular holomorphic foliations $\mathcal{F}_A$ on $X\backslash A$. Two such regular foliations $\mathcal{F}_A$ and $\mathcal{F}_B$ are called **equivalent** iff the restrictions of $\mathcal{F}_A$ and $\mathcal{F}_B$ to $X\backslash(A\cup B)$ coincide. For each such regular foliation $\mathcal{F}_A$ there exists a unique equivalent one, called $\mathcal{F}_S$, where the analytic set S is minimal.

DEFINITION 1.3. The equivalence class of a regular holomorphic foliation $\mathcal{F}_A$ (as described above) is called a (singular) **holomorphic foliation on** X, which we denote by $\mathcal{F}$. The uniquely associated analytic set S is called the **singular locus** of $\mathcal{F}$, which we denote by $S(\mathcal{F})$. By the **regular locus** of $\mathcal{F}$ we understand $X^r := X\backslash S(\mathcal{F})$ while $\mathcal{F}^r := \mathcal{F}_S$ is called the **maximal regular foliation** in $\mathcal{F}$. The numbers $\dim \mathcal{F}$ and $\operatorname{codim} \mathcal{F}$ are defined canonically.

DEFINITION 1.4. Via the sheaf $\Omega_{\mathcal{F}^r}$ on X^r one defines on X a holomorphic subsheaf $\Omega_{\mathcal{F}}$ of Ω by setting for open subsets U of X

$$\Omega_{\mathcal{F}}(U) := \{\omega \in \Omega(U);\ \omega|_{U\cap X^r} \in \Omega_{\mathcal{F}^r}(U \cap X^r)\}\,.$$

The subsheaf $\Theta_{\mathcal{F}}$ of Θ is defined in an analogous way.

The subsheaf $\mathcal{O}_{\mathcal{F}}$ of the holomorphic structure sheaf $\mathcal{O}$ of X is defined by setting for open subsets U of X:

$$\mathcal{O}_{\mathcal{F}}(U) := \{f \in \mathcal{O}(U);\ df \in \Omega_{\mathcal{F}}(U)\}\,.$$

PROPOSITION AND DEFINITION 1.5. *The holomorphic foliation $\mathcal{F}$ is called* **coherent** *iff it satisfies the following equivalent properties* (*compare* [19]) :

(1) $\Omega_{\mathcal{F}}$ *is coherent.*

(2) *There exists a coherent subsheaf of* Ω *on* X *extending* $\Omega_{\mathcal{F}^r}$.
(3) $\Theta_{\mathcal{F}}$ *is coherent.*
(4) *There exists a coherent subsheaf of* Θ *on* X *extending* $\Theta_{\mathcal{F}^r}$.
(5) $\operatorname{codim} S(\mathcal{F}) \geq 2$.

REMARK 1.6. Coherent holomorphic foliations are exactly those in the sense of BAUM-BOTT (compare [2]). The singular locus $S(\mathcal{F})$ of a coherent holomorphic foliation $\mathcal{F}$ on X can be described as

$$S(\mathcal{F}) = \{x \in X;\ \Omega_x/(\Omega_{\mathcal{F}})_x \text{ is not free}\}\,.$$

If $\Omega_{\mathcal{F}} = \mathcal{O} \cdot \omega$ is generated by a single $1-$form ω, then $S(\mathcal{F}) = \{x \in X;\ \omega(x) = 0\}$.

In the following all foliations are assumed to be coherent.

Usually one defines regular foliations locally by holomorphic submersions. We generalize this concept:

DEFINITION 1.7. Let $U \subset X$ be an open subset and $f\colon U \to Z$ an open holomorphic mapping into a reduced complex space. f is called an **integral** of the holomorphic foliation $\mathcal{F}$ on X iff there exists a nowhere dense analytic subset A of U such that
(1) $A \supset U \cap S(\mathcal{F})$, $\quad f(U\backslash A) \subset Z\backslash \mathrm{Sing} Z$.
(2) $\tilde{f} := f|_{U\backslash A} : U\backslash A \to Z\backslash \mathrm{Sing} Z$ is a holomorphic submersion defining $\mathcal{F}^r$ on $U\backslash A$, i.e. the connected components of the fibres of $\tilde{f}$ are the leaves of $\mathcal{F}^r|_{U\backslash A}$.

$\mathcal{F}$ is called **integrable in a point** $x \in X$ iff there exists an open neighborhood U of x and an integral $f : U \to Z$ of $\mathcal{F}$. $\mathcal{F}$ is called **integrable** iff $\mathcal{F}$ is integrable in every point $x \in X$.

It is no restriction to demand in definition 1.7 that f is surjective and Z a normal complex space.

In general, $\mathcal{F}$ is not integrable in points $x \in S(\mathcal{F})$ (see example 2.8).

In [19] the definition of an integral is more general. An integral in the sense of our definition 1.7 is called an open integral there.

In [6] special singular holomorphic foliations are defined, called **H-foliations** in the following, using so called locally simple integrals (see examples 1.9).

The following proposition uses proposition 1.5.(2) to produce examples of coherent foliations:

Proposition 1.8. *If $f : X \to Z$ is an open holomorphic mapping from X onto the normal complex space Z, then*

(1) *there exists a nowhere dense analytic subset $A \subset X$ such that $f(X\backslash A) \subset Z\backslash \mathrm{Sing} Z$ and such that $\tilde{f} := f|_{X\backslash A} : X\backslash A \to Z\backslash \mathrm{Sing} Z$ is a submersion,* i.e. *$\tilde{f}$ defines a regular foliation on $X\backslash A$,*

(2) *f defines a (coherent) foliation $\mathcal{F}$ on X,*

(3) *f is a (global) integral of $\mathcal{F}$.*

Examples 1.9. $f_k : \mathbb{C}^2 \to \mathbb{C}$, $f_k(z_1, z_2) := z_1^k z_2$, $k = 1,2$, are open surjective mappings, submersions on $\mathbb{C}^2\backslash A_k$, where $A_1 = \{(0,0)\}$, $A_2 := \{0\} \times \mathbb{C}$, defining regular holomorphic foliations $\mathcal{F}_{A_k}$ on $X\backslash A_k$, representing coherent foliations $\mathcal{F}_k$ on $\mathbb{C}^2$. The coherent sheaves $\Omega_{\mathcal{F}_k}$ and $\Theta_{\mathcal{F}_k}$ are generated by

$$\omega_k = k\, z_2\, dz_1 + z_1\, dz_2, \quad \text{resp.} \quad \theta_k = z_1 \frac{\partial}{\partial z_1} - k\, z_2 \frac{\partial}{\partial z_2}\,.$$

By remark 1.6, we get $S(\mathcal{F}_k) = \{(0,0)\}$ for $k = 1,2$.

One can describe $\mathcal{F}_k$ also through the holomorphic $\mathbb{C}^*$-action

$$\Phi_k : \mathbb{C}^* \times \mathbb{C}^2 \to \mathbb{C}^2, \quad \Phi_k(t, z_1, z_2) := (t z_1, t^{-k} z_2)\,.$$

The decompositions of $\mathbb{C}^2$ into Φ_k-orbits define regular holomorphic foliations on $\mathbb{C}^2\backslash\{(0,0)\}$ for $k = 1,2$ which coincide with $\mathcal{F}_{A_k}$ on $\mathbb{C}^2\backslash A_k$. Consequently again we obtain $S(\mathcal{F}_k) = \{(0,0)\}$ for $k = 1,2$, $f_k : \mathbb{C}^2 \to \mathbb{C}$ are global integrals where f_1 is locally simple but f_2 not, $\mathcal{F}_1$ is an H-foliation but $\mathcal{F}_2$ not.

In general foliations are not integrable (compare 2.8; many examples can be found in [4, §8]).

The theorem of Malgrange-Frobenius([14], [15]) gives a sufficient condition for the existence of integrals; these are integrals in the sense of our definition 1.7 if $\mathrm{codim}\mathcal{F} = 1$, and in the more general sense of [19] if $\mathrm{codim}\mathcal{F} \geq 2$.

2 – Leaves and leaf spaces

Let $\mathcal{F}$ be a (coherent) foliation on a complex manifold X as in section 1. We give a survey of the theory of leaves of $\mathcal{F}$ following [19]. All complex spaces are assumed to be reduced.

DEFINITION 2.1. A holomorphic mapping $\iota : Y \to X$ (Y a complex space) is called an **injective holomorphic immersion**, if ι is injective and $\iota_y^* : \mathcal{O}_{X,\iota(y)} \to \mathcal{O}_{Y,y}$ surjective for all $y \in Y$.

DEFINITION 2.2. Let Y be a connected complex space and $\phi : Y \to X$ an injective holomorphic immersion. ϕ is called an **integral variety of** $\mathcal{F}$ iff for every $y \in Y$ and for every $f \in \left(\mathcal{O}_{\mathcal{F}}\right)_{\phi(y)}$ the germ $f \circ \phi$ is constant. An integral variety $\phi : Y \to X$ is called **big** iff $\dim_y Y \geq \dim \mathcal{F}$ for all $y \in Y$.

DEFINITION 2.3. Let $Y \subset U$ be a connected analytic subset of an open subset U of X. We call Y a **local leaf** (plaque) **of** $\mathcal{F}$ iff the following holds:

(1) The inclusion $Y \hookrightarrow X$ is a big integral variety of $\mathcal{F}$.

(2) For every $y \in Y$ and for every germ $(Z,y) \hookrightarrow (X,y)$ (where Z is a big integral variety, locally analytic in U with $y \in Z$) we have $(Z,y) \hookrightarrow (Y,y)$.

PROPOSITION AND DEFINITION 2.4. *Let $\Sigma = \Sigma(\mathcal{F})$ be the set of all $x \in X$ such that there exists no local leaf passing through x. The local leaves form a base of a topology $\mathcal{T} = \mathcal{T}_{\mathcal{F}}$ on $X \backslash \Sigma$, the so-called* **leaf topology of** $\mathcal{F}$. *$(X \backslash \Sigma, \mathcal{T})$ is a complex space in a natural way, the canonical inclusion $(X \backslash \Sigma, \mathcal{T}) \to X$ is an injective immersion. The connected components of $(X \backslash \Sigma, \mathcal{T})$ are called the* **leaves of** $\mathcal{F}$.

In the following remark we collect some properties of leaves:

REMARK 2.5. Let L be a leaf of $\mathcal{F}$.

- L has two topologies which do not coincide in general: first the relative topology induced by X, second the $\mathcal{T}$-topology (sometimes we write $(L, \mathcal{T})$ in this case).
- $(L, \mathcal{T})$ is a connected complex space, the inclusion $L \subset X$ defines an injective holomorphic immersion $\iota : (L, \mathcal{T}) \to X$.
- $(L, \mathcal{T})$ is paracompact due to the Poincaré-Volterra theorem ([5]) since X is paracompact.

– In general, L is not an analytic subset of X.
– If L is an analytic subset of X, then the inclusion $L \subset X$ is a biholomorphic mapping between the complex spaces $(L, \mathcal{T})$ and L (equipped with the complex structure induced by X) (compare 2.12 and 2.13). Hence, in this case, we may identify the complex spaces $(L, \mathcal{T})$ and the analytic subspace $L \subset X$.

When we say that a subset of X is "open" or "closed" or "compact", then we mean open with respect to the usual topology etc. If we want to express that a subset of X is open or closed or compact with respect to the topology $\mathcal{T}$, then we say that the subset is **$\mathcal{T}$-open** resp. **$\mathcal{T}$-closed** resp. **$\mathcal{T}$-compact**.

The following proposition (compare [19]) is a sufficient condition for the existence of leaves:

PROPOSITION 2.6. *Let $f : U \to Z$ be an integral of $\mathcal{F}$, U connected. Then $U \subset X \backslash \Sigma$ and the leaves of $\mathcal{F}|_U$ are the connected components of the fibres of f.*

By 2.6 we obtain the following

(ADDENDUM TO PROPOSITION 1.8) 2.7.
(4) *The leaves of $\mathcal{F}$ are the connected components of the fibres of f.*

EXAMPLE 2.8. For the foliations $\mathcal{F}_k$, $k = 1, 2$ of $\mathbb{C}^2$ in example 1.9 the leaf passing through the origin is both times $L(0) = \{(z_1, z_2) \in \mathbb{C}^2 : z_1 z_2 = 0\}$, i.e. the union of the three orbits $\{(0,0)\}$, $\mathbb{C}^* \times \{0\}$ and $\{0\} \times \mathbb{C}^*$ of the $\mathbb{C}^*$-actions Φ_k, therefore $\Sigma(\mathcal{F}_k)$ is empty for $k = 1, 2$. — The vectorfield $\theta := z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}$ and the differential form $\omega = z_2\, dz_1 - z_1\, dz_2$ define the same foliation $\mathcal{F}$ on $\mathbb{C}^2$ with singular locus $S(\mathcal{F}) = \Sigma(\mathcal{F}) = \{(0,0)\}$. There are too many big integral varieties of $\mathcal{F}$ passing through $(0,0)$, e.g. any finite union of complex lines through $(0,0)$. Condition (2) of definition 2.3 is not satisfied, i.e. there is no local leaf passing through $(0,0)$. Especially $\mathcal{F}$ is not integrable in $(0,0)$.

DEFINITION 2.9. We call $\mathcal{F}$ a foliation with **leaves everywhere** iff $\Sigma(\mathcal{F}) = \emptyset$.

By the proposition 2.6 we get

PROPOSITION 2.10. *If $\mathcal{F}$ is integrable then $\mathcal{F}$ is a foliation with leaves everywhere.*

From now on we consider foliations with leaves everywhere only.

For injective holomorphic immersions, one has the following result:

REMARK 2.11. Let $\phi : Y \to X$ be an injective holomorphic immersion between complex spaces. Then the following statements are equivalent:

(1) ϕ is proper.

(2) $\phi(Y)$ is an analytic subset of X and $\phi : Y \to \phi(Y)$ is biholomorphic.

For leaves one can sharpen this result considerably:

PROPOSITION AND DEFINITION 2.12. *Let $L \subset X$ be a leaf of $\mathcal{F}$. Then the following statements are equivalent:*

(1) *The inclusion $\iota : (L, \mathcal{T}) \to X$ is proper.*

(2) *L is an analytic subset of X.*

Under these conditions (*remark that the first condition is purely topological*), *$\iota : (L, \mathcal{T}) \to L$ is an isomorphism of complex spaces by* 2.11 *and L shall be called a* **proper leaf**.

For the proof one uses that leaves are paracompact (see remark 2.5). As a consequence of 2.12 we obtain

COROLLARY 2.13. *A leaf L is $\mathcal{T}$-compact iff L is a compact analytic subset of X.*

For H-foliations one can sharpen 2.12 (compare [6, Satz 3.1]):

PROPOSITION 2.14. *For a leaf L of an H-foliation, the natural embedding $\iota : (L, \mathcal{T}) \to X$ is proper iff L is a closed subset of X.*

The following lemma follows easily from the definition of a local leaf; it is similar to the Thullen-Remmert-Stein singularity theorem:

LEMMA 2.15. *Suppose $L \subset X$ is a proper leaf of $\mathcal{F}$ and $F \subset X \backslash L$ is a proper leaf of $\mathcal{F}|_{X \backslash L}$. Then the closure $\overline{F}$ of F in X is not analytic in any point $x \in \overline{F} \cap L$ (of course, $\overline{F} \cap L$ may be empty).*

Definition and Remarks 2.16. We denote by $X/\mathcal{F}$ the set of all leaves of $\mathcal{F}$ and by $\pi : X \to X/\mathcal{F}$ the natural projection. For $x \in X$ we denote by $L(x)$ the leaf of $\mathcal{F}$ passing through x. We equip $X/\mathcal{F}$ with the quotient topology and the canonical ringed structure. Then $\pi : X \to X/\mathcal{F}$ is a surjective morphism of ringed spaces.

$\mathcal{F}$ defines an equivalence relation $R = R_{\mathcal{F}}$ on X satisfying $X/\mathcal{F} = X/R$ in a natural way:

$$(x, y) \in R \subset X \times X \iff L(x) = L(y).$$

Definition 2.17. Let $M \subset X$. We call $R(M) = \bigcup_{x\in M} L(x)$ the **$\mathcal{F}$-saturation of** M. M is called **$\mathcal{F}$-saturated** iff $M = R(M)$.

In our investigations the cases, where R is an open resp. analytic equivalence relation (i.e. R is an analytic subset of $X \times X$), play an important role. Basic results about the theory of analytic equivalence relations one finds in [12].

Definition 2.18. Let $(L_\nu)_{\nu\in\mathbb{N}}$ be a sequence of leaves. Then $y \in X$ is called a **weak limit point** of $(L_\nu)_{\nu\in\mathbb{N}}$ iff there exists a sequence $(x_\nu)_{\nu\in\mathbb{N}}$ in X such that $x_\nu \in L_\nu$ for all ν and $y = \lim_{\nu\to\infty} x_\nu$. By $\operatorname{Lim} L_\nu$ we denote the set of all weak limit points of the sequence $(L_\nu)_{\nu\in\mathbb{N}}$.

One sees easily that $\operatorname{Lim} L_\nu$ is a closed subset of X. If $L_\nu = L$ for all ν, we get

$$\operatorname{Lim} L_\nu = \operatorname{Lim} L = \overline{L}\,. \tag{1}$$

Remark 2.19. The following statements are equivalent:
(1) R is an open equivalence relation.
(2) For every sequence $(L_\nu)_{\nu\in\mathbb{N}}$ of leaves the set $\operatorname{Lim} L_\nu$ is $\mathcal{F}$-saturated.
(3) For every sequence $(L_\nu)_{\nu\in\mathbb{N}}$ of leaves the set $\operatorname{Lim} L_\nu$ is $\mathcal{T}$-open.

Proposition 2.20. *If $\mathcal{F}$ is integrable by integrals with connected fibres, then R is open.*

Proof. Let L and L_ν, $\nu \in \mathbb{N}$, be leaves of $\mathcal{F}$. We have to show that $L \cap \operatorname{Lim} L_\nu$ is $\mathcal{T}$-open in L. Suppose $y \in L \cap \operatorname{Lim} L_\nu$. We choose a sequence $(x_\nu)_{\nu \in \mathbb{N}}$ as in definition 2.18 and an integral $f : U \to Z$ with connected fibres on an open connected neighborhood U of y. We can assume that $x_\nu \in U$ for all ν. Because of 2.7 the leaves of $\mathcal{F}|_U$ are the fibres of f, i.e. $f^{-1}(f(x_\nu)) \subset L_\nu$ and $f^{-1}(f(y)) \subset L$ is a $\mathcal{T}$-open neighborhood of y. Since f is open we have $f^{-1}(f(y)) \subset \operatorname{Lim} f^{-1}(f(x_\nu)) \subset \operatorname{Lim} L_\nu$, i.e. $L \cap \operatorname{Lim} L_\nu$ is $\mathcal{T}$-open. □

For H-foliations the equivalence relation R is open. Using the addendum in [17] we get

Proposition 2.21. *If $\mathcal{F}$ is of codimension 1 and integrable, then R is open.*

In [3] Bohnhorst has shown: The mapping $f : \mathbb{C}^3 \to \mathbb{C}^2$, $f(z) := (z_1 z_2, (z_1 + z_2) z_3)$ defines a foliation $\mathcal{F}$ of codimension 2 with f as an integral; f is open but R is not open.

Proposition 2.22. *The following statements are equivalent:*
(1) *R is an open and analytic equivalence relation.*
(2) *$X/\mathcal{F}$ is a complex space and π is open.*
(3) *There exists a global integral $f : X \to Z$ with connected fibres.*

Proof. For (1) $\iff$ (2) compare [11]. For (2) $\iff$ (3) compare [19]. □

If $\mathcal{F}$ is an H-foliation we have a stronger result (compare [6, Satz 3.4]):

Proposition 2.23. *For H-foliations $\mathcal{F}$ the following statements are equivalent:*
(1) *R is an analytic equivalence relation.*
(2) *$X/\mathcal{F}$ is a Hausdorff space.*
(3) *$X/\mathcal{F}$ is a complex space.*

Remark 2.24. In 2.22.(2) and 2.23.(3) the quotient $X/\mathcal{F}$ is a normal complex space automatically. In 2.22.(3) we may assume that f is surjective and Z normal. Then Z and $X/\mathcal{F}$ are biholomorphic

3 – Ends and limit sets

For locally compact topological spaces Y one can define its end compactification Y^+ and its number of ends $e(Y) :=$ number of elements of $Y^+ \backslash Y$, compare [1, §2].

In the following let Y be a connected, paracompact, reduced complex space. Then one can define the number $e(Y)$ of ends of Y in the following equivalent way:

Definition 3.1. For a compact subset $K \subset Y$, let $h(K)$ be the (finite) number of connected components of $Y \backslash K$ which are not relatively compact in Y. Then

$$e(Y) := \sup\{h(K);\ K \subset Y \text{compact}\}.$$

Remark 3.2.

(1) Let $K \subset Y$ be compact and let U be the union of those connected components of $Y \backslash K$ which are relatively compact in Y. Then the **completion** $\tilde{K} := K \cup U$ of K is compact too and $h(K) = h(\tilde{K}) < \infty$; K is called **complete** iff $K = \tilde{K}$.

(2) Let $K \subset L \subset Y$ be complete compact subsets. Then each connected component of $Y \backslash L$ is contained in a connected component of $Y \backslash K$, hence $h(K) \leq h(L)$.

(3) If $(K_\nu)_{\nu \in \mathbb{N}}$ is a sequence of complete compact subsets of Y such that $K_\nu \uparrow Y$ $\Big($i.e. $K_{\nu-1}$ is contained in the interior $\overset{\circ}{K}_\nu$ of K_ν for all ν and $\bigcup K_\nu = Y\Big)$, then $e(Y) = \lim h(K_\nu)$.

Now let $\mathcal{F}$ be a foliation (with leaves everywhere) on X as in sections 1 and 2. We shall use the notations of these sections.

Definition 3.3. A point $y \in X$ is called a **limit point** of a leaf L of $\mathcal{F}$ iff there exists a sequence $(x_\nu)_{\nu \in \mathbb{N}}$ in L such that $y = \lim x_\nu$ and $(x_\nu)_{\nu \in \mathbb{N}}$ has no $\mathcal{T}$-accumulation point. The **limit set** of L is defined by $\lim L := \{x \in X;\ x \text{ is limit point of } L\}$.

Remark 3.4. For leaves L of $\mathcal{F}$ the following holds:

(1) $\lim L \subset \overline{L}$.

(2) If L is a proper leaf (for example if L is $\mathcal{T}$-compact), then $\lim L = \emptyset$.

(3) If L is not $\mathcal{T}$-compact but relatively compact in X, then $\lim L \neq \emptyset$.

LEMMA 3.5. *Let L be a leaf of $\mathcal{F}$ and $(K_\nu)_{\nu\in\mathbb{N}}$ a sequence of $\mathcal{T}$-complete $\mathcal{T}$-compact subsets of L such that $K_\nu \uparrow L$ (see remark* 3.2). *If we denote by $\overline{L\backslash K_\nu}$ the closure of $L\backslash K_\nu$ in X, then*

$$\lim L = \bigcap_{\nu\in\mathbb{N}} \overline{L\backslash K_\nu}.$$

As a consequence we obtain

COROLLARY 3.6. *The limit set* $\lim L$ *of a leaf L of $\mathcal{F}$ is closed in X.*

DEFINITION 3.7. A subset $A \subset X$ is called a **limit cycle** of a leaf L of $\mathcal{F}$, if A is a non-void union of connected components of $\lim L$ such that $A \cap L = \emptyset$.

PROPOSITION 3.8. $(\lim L)\backslash L = \overline{L}\backslash L$ *for leaves L of $\mathcal{F}$.*

PROPOSITION 3.9. *If the equivalence relation R corresponding to $\mathcal{F}$ is open then every limit cycle of a leaf L of $\mathcal{F}$ is $\mathcal{F}$-saturated.*

PROOF. Let A be a connected limit cycle of L. Then A is a connected component of $\overline{L}\backslash L$. Now apply remark 2.19. □

By modifying the proof of 2.20 one gets

PROPOSITION 3.10. *If $\mathcal{F}$ is integrable by integrals with connected fibres, then* $\lim L$ *is $\mathcal{F}$-saturated for every leaf L of $\mathcal{F}$.*

DEFINITION 3.11. For a leaf L of $\mathcal{F}$ we set

$$l(L) := \text{ number of connected components of } \lim L\,.$$

PROPOSITION 3.12. *If the leaf L of $\mathcal{F}$ is relatively compact in X, then $l(L)\leq e(L)$.*

Proof. We may assume that $r := e(L) < \infty$. There exists a sequence $(K_\nu)_{\nu\in\mathbb{N}}$ of non-void $\mathcal{T}$-complete $\mathcal{T}$-compact subsets of L with $K_\nu \uparrow L$ and $h(K_\nu) = e(L)$ (see remark 3.2). Let C_j^ν, $j = 1, \dots, r$ be the connected components of $L\backslash K_\nu$. We can assume that $C_j^\nu \supset C_j^{\nu+1}$ for $j = 1, \dots, r$ and all $\nu \in \mathbb{N}$. Now let $\overline{C_j^\nu}$ denote the closure of C_j^ν in X. Then $R_j := \bigcap_{\nu\in\mathbb{N}} \overline{C_j^\nu}$ is a non-void connected and compact subset of X (see [21, §2.B, Aufgabe 25]) and because of lemma 3.5 we obtain $\lim L = R_1 \cup \ldots \cup R_r$. $\square$

Proposition 3.13. *If a leaf L of $\mathcal{F}$ satisfies the following conditions:*

(1) $\lim L$ *is a limit cycle of L,*

(2) *The connected components of* $\lim L$ *are compact and form a locally finite family,*

(3) *L is a proper leaf of $\mathcal{F}|_{X\setminus \lim L}$,*

then $l(L) \leq e(L)$.

Proof. We denote the connected components of $\lim L$ by $\Lambda_1, \Lambda_2, \dots$. There exists a system $U_1, U_2, \dots$ of relatively compact open neighborhoods of $\Lambda_1, \Lambda_2, \dots$ in X such that $\overline{U_j} \cap \overline{U_k} = \emptyset$ for $j \neq k$. We choose an $r \in \mathbb{N}$ with $1 \leq r \leq l(L)$ and set $Q := \bigcup_{j=1}^r \partial U_j$. Then Q is a compact subset of $X \setminus \lim L$ and $Q \cap L$ is a compact subset of X by (3), i.e. $\mathcal{T}$-compact in L. We consider the completion K of $Q \cap L$ in L. Let $C_1, \dots, C_s$ denote the connected components of $L\backslash K$. By definition of $\lim L$ one can find points $z_j \in U_j \cap (L\backslash K)$, $j = 1, \dots, r$. Since each z_j lies in exactly one C_k we can conclude: $r \leq s \leq h(L) = e(L)$. $\square$

4 – Attractive leaves

In this section we study first equivalence relations on complex spaces with only one compact equivalence class und apply the results later to the equivalence relation $R_{\mathcal{F}}$ associated to a holomorphic foliation $\mathcal{F}$ on a complex manifold X.

We begin with a paracompact n-dimensional irreducible reduced complex space X with an equivalence relation $R \subset X \times X$. For any $A \subset X$ (resp. $X \times X$) by $\overline{A}$ we always understand the closure of A in X (resp. $X \times X$).

We obtain interesting results under the following

ASSUMPTIONS 4.1.

(a) *Every equivalence class* $R(x)$ *is a connected locally analytic subset of* X *of the same pure dimension* $p < n$.

(b) R *admits exactly one compact equivalence class which we denote by* Γ.

(c) *On* $X' := X \backslash \Gamma$ *the restriction* $R' := R \cap (X' \times X')$ *of* R *is an open analytic equivalence relation; especially, all equivalence classes* $R(x) \subset X'$ *are analytic subsets of* X'.

(d) *For every class* $R(x) \subset X'$, *the closure* $\overline{R(x)}$ *of* $R(x)$ *in* X *is not analytic in any point* $y \in \Gamma \cap \overline{R(x)}$, i.e. *every point of* Γ, *which is an accumulation point of* $R(x)$, *is a singularity for* $R(x)$.

Some simple consequences of our assumptions are

REMARK 4.2.

(1) X' is an open R-saturated subset of X and again irreducible.

(2) Γ is a connected, compact analytic subset of X with a finite number of irreducible components which we denote by $\gamma_1, \dots, \gamma_r$.

(3) R' is an analytic subset of $X' \times X'$ of pure dimension $n + p$.

(4) For every $x \in X'$ there exist only two alternatives: either $\overline{R(x)} \cap \Gamma = \emptyset$ or $\overline{R(x)}$ is the union of $R(x)$ with some (at least one) irreducible components of Γ.

By the THULLEN-REMMERT-STEIN theorem (compare [20] and [22]), remark 4.2.(4) is a consequence of our assumptions (c) and (d).

LEMMA 4.3. $R = R' \cup (\Gamma \times \Gamma)$ *is not closed in* $X \times X$.

PROOF. It is sufficient to find a sequence $(x_k, y_k)_{k \in \mathbb{N}}$ in R' converging to a point $(x, y) \in \Gamma \times X'$. If $\overline{R(y)} \cap \Gamma = \emptyset$ for all $y \in X'$, we choose a compact connected neighborhood U of Γ in X, a point $x \in \Gamma$ and a sequence $X' \cap U \ni x_k \to x$. Since all $R(x_k)$ are not compact, there exists a point $y_k \in R(x_k) \cap \partial U$. We may assume that $y_k \to y \in \partial U$. Then the sequence $(x_k, y_k)_{k \in \mathbb{N}}$ has the desired properties. The case $\overline{R(y)} \cap \Gamma \neq \emptyset$ for a point $y \in X'$ is trivial. □

Definition 4.4. For subsets U, C of X, where U is open and C is compact, we define

$$\mathcal{B}_U(C) := \{x \in U : \ \overline{R(x)} \cap C \neq \emptyset\}.$$

Instead of $\mathcal{B}_X(C)$ resp. $\mathcal{B}_{X'}(C)$ we shall write simply $\mathcal{B}(C)$ resp. $\mathcal{B}'(C)$.

With these notations we get for $C := \Gamma$:

$$\mathcal{B}(\Gamma) = \mathcal{B}'(\Gamma) \cup \Gamma, \ \text{with} \ \mathcal{B}'(\Gamma) \cap \Gamma = \emptyset. \tag{2}$$

When we study $\mathcal{B}(\Gamma)$ and $\mathcal{B}'(\Gamma)$, we have to work with the closure $\overline{R'}$ of R' in $X \times X$. We observe that $\overline{R'}$ is a symmetric subset of $X \times X$, but in general not an equivalence relation.

Definition 4.5. Let $\pi_1, \pi_2 : \overline{R'} \to X$ be the canonical projections $\pi_k(x_1, x_2) := x_k$. Then, for subsets A of X, we define

$$\overline{R'}(A) := \pi_2(\pi_1^{-1}(A)) = \pi_2((A \times X) \cap \overline{R'}) = \pi_1(\pi_2^{-1}(A)) = \pi_1((X \times A) \cap \overline{R'}).$$

Especially we have

$$\overline{R'}(\Gamma) = \pi_2((\Gamma \times X) \cap \overline{R'}) = \pi_1((X \times \Gamma) \cap \overline{R'}). \tag{3}$$

Lemma 4.6. $\mathcal{B}'(\Gamma) = \overline{R'}(\Gamma) \backslash \Gamma$.

Proof. "$\supset$" For $x \in \overline{R'}(\Gamma) \backslash \Gamma$ we have to show that $\overline{R(x)} \cap \Gamma \neq \emptyset$. It is sufficient to prove that $R(x) \cap U \neq \emptyset$ for every compact connected neighborhood U of Γ in X. We may assume that $x \notin U$. By (3), there exists $y \in \Gamma$ such that $(y, x) \in (\Gamma \times X) \cap \overline{R'}$. We choose a sequence $R' \ni (y_\nu, x_\nu) \to (y, x)$ such that $x_\nu \notin U$ and $y_\nu \in U$ for all ν. Then $R(x_\nu) \cap \partial U \neq \emptyset$ for all ν; hence we may assume that there exists a sequence $R(x_\nu) \cap \partial U \ni z_\nu \to z \in \partial U \subset X'$. Then $R' \ni (x_\nu, z_\nu) \to (x, z) \in X' \times X'$; since R' is closed in $X' \times X'$ by assumption 4.1.(c), we have $(x, z) \in R'$, i.e. $z \in R(x) \cap \partial U$. "$\subset$" is trivial. $\square$

Corollary 4.7. $\mathcal{B}'(\Gamma) \neq \emptyset$.

PROOF. Suppose $\mathcal{B}'(\Gamma) = \emptyset$. Then, by (3) and lemma 4.6, we obtain $\overline{R'}(\Gamma) \subset \Gamma$, $\overline{R'} \subset R' \cup (\Gamma \times \Gamma) = R$. Hence $R = \overline{R'} \cup (\Gamma \times \Gamma)$ is closed in contradiction to lemma 4.3. ∎

We define (compare remark 4.2.(4))

$$\Xi_\rho := (X \times \gamma_\rho) \cup (\gamma_\rho \times X) \quad (1 \leq \rho \leq r)$$

and

$$\Xi := (X \times \Gamma) \cup (\Gamma \times X) = \bigcup_{\rho=1}^{r} \Xi_\rho = (X \times X) \backslash (X' \times X') .$$

All Ξ_ρ and Ξ are analytic subsets of $X \times X$ of pure dimension $n + p = \dim R'$. Therefore, by the Thullen-Remmert-Stein theorem, we have the following two alternatives:

PROPOSITION 4.8. *Either: there exists an irreducible component γ of Γ such that $\overline{R'} \supset \gamma \times X$, in which case $\overline{R'}$ is not analytic in $X \times X$* **(case I)**, *or: $\overline{R'}$ is analytic in $X \times X$* **(case II)**.

PROOF. If $\overline{R'}$ is analytic, then $\overline{R'}$ cannot contain an analytic set of the form $\gamma \times X$, since $\dim(\gamma \times X) = n + p = \dim R'$. ∎

Now we can prove

THEOREM 4.9.

(I) *If $\overline{R'}$ is not analytic in $X \times X$ then $\mathcal{B}'(\Gamma) = X \backslash \Gamma$,* i.e. $\overline{R(x)} \cap \Gamma \neq \emptyset$ *for all $x \in X$.*

(II) *If $\overline{R'}$ is analytic in $X \times X$ then $\mathcal{B}'(\Gamma)$ is analytic in X' of dimension $> p$ everywhere and $\overline{\mathcal{B}'(\Gamma)}$ and $\mathcal{B}(\Gamma)$ are analytic in X of dimension $> p$ everywhere.*

PROOF. **Ad (I)** If $\overline{R'}$ is not analytic, then there is at least one irreducible component γ of Γ such that $\gamma \times X \subset \overline{R'}$. Hence, by (3) and lemma 4.6, we obtain

$$\mathcal{B}'(\Gamma) \supset \pi_2(\gamma \times X) \backslash \Gamma = X \backslash \Gamma .$$

Ad (II). The projection $\pi_2 : \Gamma \times X \to X$ is proper, therefore $\overline{R'}(\Gamma) = \pi_2\big(\overline{R'} \cap (\Gamma \times X)\big)$ is analytic in X. By lemma 4.6, $\mathcal{B}'(\Gamma)$ is analytic in X' and $\overline{\mathcal{B}'(\Gamma)} \subset \overline{R'}(\Gamma)$ is analytic in X. Therefore $\mathcal{B}(\Gamma) = \mathcal{B}'(\Gamma) \cup \Gamma = \overline{\mathcal{B}'(\Gamma)} \cup \Gamma$ is analytic in X too. We still have to show, that $\mathcal{B}'(\Gamma)$ has dimension $> p$ everywhere. Since $\mathcal{B}'(\Gamma)$ is R-saturated, we have $\dim \mathcal{B}'(\Gamma) \geq p$ everywhere. If $\mathcal{B}'(\Gamma)$ has an irreducible component Z of dimension p, then by remark 4.2.(4) for every point $x \in Z$ we get $\gamma \subset \overline{R(x)} \subset \overline{Z}$ for a certain component γ of Γ. But this is impossible since $\overline{Z}$ is analytic in X and $\dim \overline{Z} = \dim \gamma = p$. ∎

Corollary 4.10. *If $p = n - 1$ then $\mathcal{B}(\Gamma) = X$.*

Proof. $\mathcal{B}'(\Gamma)$ is analytic in X' of dimension bigger than $p = n - 1$, hence $\mathcal{B}'(\Gamma) = X'$. ∎

If R is an **open** equivalence relation, than we can sharpen theorem 4.9:

Proposition 4.11. *Let R be an open equivalence relation. Then $\overline{R(x)} \supset \Gamma$ for every $x \in X'$*, i.e. $\mathcal{B}(\Gamma) = X$.

Proof. It is sufficient to show that $\overline{R'}$ is not analytic in $X \times X$ (see theorem 4.9). By corollary 4.7, we can choose an $x \in \mathcal{B}'(\Gamma)$. Because of the subsequent proposition 4.12 we have $\overline{R(x)} = R(x) \cup \Gamma$. Since $\overline{R'} \subset R' \cup (X \times \Gamma) \cup (\Gamma \times X)$, we have $\overline{R'}(x) = \pi_2\Big(\overline{R'} \cap (\{x\} \times X)\Big) \subset R(x) \cup \Gamma$. Since $\overline{R(x)} \subset \overline{R'}(x)$, we have

$$\overline{R'}(x) = R(x) \cup \Gamma \quad \forall\, x \in \mathcal{B}'(\Gamma)\,. \tag{4}$$

Suppose $\overline{R'}$ is analytic in $X \times X$, then $\overline{R'}(x) = R(x) \cup \Gamma$ is analytic in X, but this is impossible because of assumption 4.1.(d). ∎

Proposition 4.12. *If R is an open equivalence relation, then $\overline{R(x)} \supset \Gamma$ for all $x \in \mathcal{B}'(\Gamma)$, hence $\overline{\mathcal{B}'(\Gamma)} \supset \Gamma$.*

Proof. Fix $x \in \mathcal{B}'(\Gamma)$ which is not empty by corollary 4.7. Because of remark 4.2.(4), there exists an irreducible component γ of Γ such that $\overline{R(x)} \supset \gamma$. Fix $y \in \Gamma$ and $y' \in \gamma$. Then there exists a sequence $(y'_\nu)_{\nu \in \mathbb{N}}$ in $R(x)$ such that $y'_\nu \to y'$. Since R is an open equivalence relation, there exists a sequence (y_ν) in $R(x)$ such that $y_\nu \to y$. Therefore $\Gamma \subset \overline{R(x)} \subset \overline{\mathcal{B}'(\Gamma)}$. ∎

Now we have a very simple description of $\overline{R'}$ if R is open:

Proposition 4.13. *If R is open, then $\overline{R'} = R' \cup \Xi = R' \cup (X \times \Gamma) \cup (\Gamma \times X)$.*

The conclusion of proposition 4.11 is very strong. This is shown by the following

Proposition 4.14. *The following statements are equivalent:*
(1) *$\overline{R(x)} \supset \Gamma$ for every $x \in X'$.*
(2) *R is an open equivalence relation.*
(3) *For every $y \in \Gamma$ and for every neighborhood U of y in X the R-saturated hull $R(U)$ of U is equal to X.*

As mentioned in the beginning of this section, we now shall apply our results to equivalence relations $R = R_{\mathcal{F}}$ induced by a holomorphic foliation $\mathcal{F}$ on an n-dimensional complex manifold X. We shall use the notions and the conditions for $\mathcal{F}$ of the previous sections.

Definition 4.15. Let M be a non-void closed $\mathcal{F}$-saturated subset of X. The set $\mathcal{A}(M) := M \cup \{x \in X : \lim L(x) \neq \emptyset$ and a connected component of $\lim L(x)$ is contained in $M\}$ is called the **basin of attraction of** M.

Obviously $\mathcal{A}(M)$ is $\mathcal{F}$-saturated.

Definition 4.16. Let M be as in definition 4.15.
M is called **attractive** iff $\mathcal{A}(M)$ is a neighborhood of M.
M is called **globally attractive** iff $\mathcal{A}(M) = X$.
M is called **almost attractive** resp. **almost globally attractive** iff there exists a meager subset N of X such that $\mathcal{A}(M) \cup N$ is a neighborhood of M resp. $\mathcal{A}(M) \cup N = X$.

In the rest of this section we shall always work under the following

Assumptions 4.17.

(a) *Every leaf L of $\mathcal{F}$ is a locally analytic subset of X of the same pure dimension $p < n$.*

(b) *The compact leaves form a non-void locally finite family $(\Gamma_j)_{j\in J}$. (In our case, compact leaves are compact analytic subsets of X, proper leaves and also $\mathcal{T}$-compact, compare corollary* 2.13.)

(c) *On $X' := X\backslash\mathbb{\Gamma}$ with $\mathbb{\Gamma} := \bigcup_{j\in J}\Gamma_j$, the restriction $R' := R\cap(X'\times X')$ of R is an open analytic equivalence relation on X' (R' is the equivalence relation corresponding to $\mathcal{F}' := \mathcal{F}|_{X'}$).*

For $X_j := X'\cup\Gamma_j$ and $R_j := R_{\mathcal{F}}\cap\big(X_j\times X_j\big)$, $j\in J$, the assumptions 4.1 are satisfied. Only 4.1.(d) needs some special attention, since it has no analogue in the assumptions 4.17. It follows, however, directly from lemma 2.15.

Remark 4.18. If the singular locus S of $\mathcal{F}$ is contained in $\mathbb{\Gamma}$ then (compare proposition 2.23) the assumption (4.17).(c) is equivalent to : $X'/\mathcal{F}'$ is a Hausdorff space.

As a consequence of remark 4.2.(4) we obtain

Proposition 4.19. *For every $x\in X'$ with $\lim L(x)\neq\emptyset$ we get*

(1) $\lim L(x)$ *is a limit cycle of* $L(x)$.

(2) *The connected components of* $\lim L(x)$ *are unions of certain irreducible components of* $\mathbb{\Gamma}$ *and therefore they are compact and form a locally finite family.*

(3) $L(x)$ *is a proper leaf of* $\mathcal{F}|_{X\backslash\lim L(x)}$.

Hence the conditions of proposition 3.13 *hold.*

From point (2) of the proposition above follows directly

Proposition 4.20. *For all compact leaves Γ of $\mathcal{F}$, we have*

$$\mathcal{A}(\Gamma) = \Gamma\cup\{x\in X:\ \lim L(x)\cap\Gamma\neq\emptyset\} = \mathcal{B}(\Gamma) = \mathcal{B}'(\Gamma)\cup\Gamma.$$

Hence the results of the first part of this section concerning $\mathcal{B}(\Gamma)$ can be applied to $\mathcal{A}(\Gamma)$. Using the following abbreviations

$c\,(\mathcal{F}) :=$ number of compact leaves of $\mathcal{F}$ (where $c(\mathcal{F}) = \infty$ is allowed)
$c'(\mathcal{F}) :=$ number of compact almost attractive leaves of $\mathcal{F}$ (where $c'(\mathcal{F}) = \infty$ is allowed)
$l\,(L) :=$ number of connected components of $\lim L$
$e\,(L) :=$ number of ends of L

we get:

THEOREM 4.21.
(1) *For every compact leaf* Γ *of* $\mathcal{F}$ *the basin of attraction* $\mathcal{A}(\Gamma)$ *is an analytic subset of* $X' \cup \Gamma$ *of dimension* $> p$ *everywhere.*
(2) *If a compact leaf* Γ *of* $\mathcal{F}$ *is almost attractive then it is attractive and almost globally attractive, even* $\mathcal{A}(\Gamma) = X' \cup \Gamma$.
(3) *For every non compact leaf* L *of* $\mathcal{F}$ *we have* $c'(\mathcal{F}) \leq l(L) \leq e(L)$.
(4) *If* codim$\mathcal{F} = 1$, *then* $\mathcal{A}(\Gamma) = X' \cup \Gamma$ *for all compact leaves of* $\mathcal{F}$, i.e. $c'(\mathcal{F}) = c(\mathcal{F})$.
(5) *If* X *is compact, then there exists at least one almost attractive compact leaf, i.e.* $c'(\mathcal{F}) \geq 1$ (*compare* (2)).

PROOF. For (1) compare proposition 4.20 and theorem 4.9. (2) follows from (1). For (3) compare the propositions 4.19 and 3.13. ∎

For the proof of the following theorem compare proposition 4.11:

THEOREM 4.22. *If* $R_{\mathcal{F}}$ *is an open equivalence relation, then*
(1) *For every non-compact leaf* L *of* $\mathcal{F}$ *we get* $\lim L = \amalg\Gamma$.
(2) *For every compact leaf* Γ *of* $\mathcal{F}$ *we have* $\mathcal{A}(\Gamma) = X' \cup \Gamma$.
(3) $c'(\mathcal{F}) = c(\mathcal{F}) = l(L) \leq e(L)$ *for all noncompact leaves* L *of* $\mathcal{F}$.

5 – $\mathbb{C}$-actions

In this section we shall study holomorphic $\mathbb{C}$-actions on connected paracompact complex manifolds X of dimension $n \geq 2$. By this we understand a holomorphic mapping $\Phi : \mathbb{C} \times X \to X$ with $\Phi(t, (\Phi(s,x)) = \Phi(t+s,x)$ and $\Phi(0,x) = x$.

Φ defines biholomorphic mappings $\Phi_t : X \to X$ and holomorphic mappings $\Phi^x : \mathbb{C} \to X$ by $\Phi_t(x) := \Phi(t,x) =: \Phi^x(t)$ for $t \in \mathbb{C}$ and $x \in X$. Φ is called **effective** if $\Phi_t = \mathrm{Id}_X$ implies $t = 0$. The set $\Phi^x(\mathbb{C}) =: \mathbb{C} \cdot x$ is called the **Φ-orbit** through $x \in X$. By the **isotropy group** I_x of the point $x \in X$ we understand the closed subgroup $I_x := \{t \in \mathbb{C} : \Phi_t(x) = x\}$ of $\mathbb{C}$. We observe that two Φ-orbits coincide or are disjoint, that $I_x = I_y$ if x and y lie on the same Φ-orbit and that Φ^x induces an injective holomorphic immersion $\phi^x : \mathbb{C}/I_x \to X$.

One can define the **rank** of I_x and $\mathbb{C} \cdot x$ as follows:

(0)	$\operatorname{rank} I_x$	$:=$	0 if $I_x = \{0\}$, i.e. $\mathbb{C}/I_x = \mathbb{C}$
(1)	$\operatorname{rank} I_x$	$:=$	1 if $I_x = \mathbb{Z} \cdot \omega_x$ with $\omega_x \in \mathbb{C}^*$, i.e. $\mathbb{C}/I_x \cong \mathbb{C}^*$
(2)	$\operatorname{rank} I_x$	$:=$	2 if $I_x = \mathbb{Z} \cdot \omega_x + \mathbb{Z} \cdot \eta_x$ with $\omega_x, \eta_x \in \mathbb{C}^*$ linearly independent over $\mathbb{R}$, i.e. $\mathbb{C}/I_x$ is a torus.
(∞)	$\operatorname{rank} I_x$	$:=$	∞ if $I_x = \mathbb{C}$, i.e. x is a fixed point.
	$\operatorname{rank}(\mathbb{C} \cdot x)$	$:=$	$\operatorname{rank} I_x$

NOTATION 5.1. *For* $k = 0, 1, 2, \infty$ *we set* $X_k := \{x \in X;\ \operatorname{rank} I_x = k.\}$.

The $\mathbb{C}$-action Φ induces the following holomorphic vector field on X:

$$\theta := \frac{\partial \Phi}{\partial t}\Big|_{t=0}.$$

Conversely, every holomorphic vectorfield on X defines a holomorphic $\mathbb{C}$-action if X is compact. The set

$$\mathrm{Fix}(\Phi) = X_\infty = \{x \in X;\ \theta(x) = 0\}$$

of fixed points of Φ is an analytic subset of X. In the following we assume that $\mathrm{Fix}(\Phi) \neq X$, i.e. $\theta \not\equiv 0$.

The holomorphic $\mathbb{C}$-action defines an 1-dimensional regular holomorphic foliation on $X \setminus \mathrm{Fix}(\Phi)$, i.e. a (singular) holomorphic foliation $\mathcal{F} = \mathcal{F}_\Phi$ on X which is coherent because of 1.5.(4), since $\Theta_{\mathcal{F}}$ and $\mathcal{O} \cdot \theta$ coincide on $X \setminus \mathrm{Fix}(\Phi)$. The singular locus $S = S(\mathcal{F})$ is contained in $\mathrm{Fix}(\Phi)$. Without additional assumptions $\mathcal{F}$ may not have leaves everywhere, i.e. $\Sigma(\mathcal{F}) \neq \emptyset$ is possible (compare example 2.8). A leaf $L(x)$ of $\mathcal{F}$ may contain the orbit $\mathbb{C} \cdot x$ as a proper subset (compare examples 2.8 and 5.3).

By proposition 1.5.(5) and the second Riemann extension theorem follows

PROPOSITION 5.2. *$S = \mathrm{Fix}(\Phi)$ iff $\operatorname{codim} \mathrm{Fix}(\Phi) \geq 2$.*

Φ is called **regular** if $\mathrm{Fix}(\Phi) = \emptyset$. In this case $\mathcal{F}$ is regular too. In general the opposite conclusion is wrong:

EXAMPLE 5.3. Consider the holomorphic action $\Phi : \mathbb{C} \times \mathbb{C}^2 \to \mathbb{C}^2$, $(t, z) \mapsto (e^t z_1, z_2)$. Since

$$\mathbb{C} \cdot z = \begin{cases} \mathbb{C}^* \times \{z_2\} & \text{if } z_1 \neq 0 \\ \{(0, z_2)\} & \text{if } z_1 = 0 \end{cases} \quad \text{but } L(z) = \mathbb{C} \times \{z_2\} \text{ for every} z \in \mathbb{C}^2,$$

the foliation $\mathcal{F}$ is regular but the operation Φ is not regular since $\mathrm{Fix}(\Phi) = \{0\} \times \mathbb{C} \neq \emptyset$. The foliation $\mathcal{F}$ is generated by the vectorfield $\partial/\partial z_1$, but the vectorfield corresponding to Φ is $\theta = z_1 \, \partial/\partial z_1$.

REMARK 5.4. If Φ is regular, i.e. $\mathrm{Fix}(\Phi) = \emptyset$, then

(1) $\mathcal{F}$ has leaves everywhere and $L(x) = \mathbb{C} \cdot x$ for all $x \in X$.
(2) A leaf is proper iff it is a closed subset of X (compare proposition 2.14).
(3) A leaf $\mathbb{C} \cdot x$ is compact iff $\operatorname{rank} I_x = 2$.

LEMMA 5.5. *Let Φ be regular and $X/\mathcal{F}$ a Hausdorff space. Then for every $y \in X \setminus X_0$ there exists an open neighborhood U of y and a holomorphic function $\omega : U \to \mathbb{C}$ such that $\omega(x) \in I_x^* := I_x \setminus \{0\}$ $\forall\, x \in U$.*

Proof. By [9, Satz 15] resp. proposition 2.23 the canonical projection $f : X \to X/\mathcal{F} =: Z$ is a surjective integral and Z is a normal complex space. We consider an open flow box neighborhood $W = D \times V$ of y where $D = \{s \in \mathbb{C} : |s| < 1\}$ and $V \subset \mathbb{C}^{n-1}$ is an open neighborhood of $0 \in \mathbb{C}^{n-1}$, such that $y = (0,0) \in \mathbb{C} \times \mathbb{C}^{n-1}$ and

$$\Phi(s, \Phi(t,0,v)) = \Phi(s,t,v) = (s+t, v)$$

for s, t, v sufficiently small; we observe, that $\mathcal{F}|_W$ is given by the projection $W \to V$. Then, for every $\alpha \in I_y^*$, there exist open neighborhoods T_α of α in $\mathbb{C}$, D_α of 0 in D and V_α of 0 in V such that

$$\Phi\Big(T_\alpha \times (D_\alpha \times V_\alpha)\Big) \subset D \times V\,.$$

Since the restriction of f to $V \cong \{0\} \times V \subset X$ is holomorphic and discrete, we can assume that it is a finite branched covering of its image. For each $\alpha \in I_y^*$ we have $\Phi(\alpha, 0, 0) = (0,0)$, and for any v small, we can write

$$\Phi(\alpha, 0, v) = \big(\mu_\alpha(v), \phi_\alpha(v)\big)$$

with a holomorphic function $\mu_\alpha : V_\alpha \to D$ with $\mu_\alpha(0) = 0$ and a (holomorphic) deck transformation $\phi_\alpha : V_\alpha \to V_\alpha$ if V_α is properly chosen. Hence

$$\lambda_\alpha(v) := \alpha - \mu_\alpha(v)$$

is holomorphic in a neighborhood of $0 \in V$ with values in a neighborhood of α and $\lambda_\alpha(0) = \alpha$. It satisfies

$$\begin{aligned}\Phi(\lambda_\alpha(v), 0, v) &= \Phi\big(-\mu_\alpha(v), \Phi(\alpha, 0, v)\big) = \\ &= \Phi\big(-\mu_\alpha(v), \mu_\alpha(v), \phi_\alpha(v)\big) = \big(0, \phi_\alpha(v)\big)\,.\end{aligned} \tag{5}$$

Since ϕ_α is of finite order, there exists a positive integer l such that $\phi_\alpha^l = \mathrm{id}_{V_\alpha}$. Using 5 one verifies that $\phi_\beta = \mathrm{id}_{V_\beta}$ for $\beta := l \cdot \alpha \in I_y^*$. Consequently $\lambda_\beta(v) \in I_{(0,v)} = I_{(s,v)}$ for all $(s,v) \in D_\beta \times V_\beta$, especially $\lambda_\beta(0) = \beta$. Hence $U := D_\beta \times V_\beta$ and $\omega : U \to T_\beta$, defined by $\omega(s,v) := \lambda_\beta(v)$, have the desired properties. ∎

PROPOSITION 5.6. *Let Φ be regular and $X/\mathcal{F}$ a Hausdorff space. Then $X\backslash X_0 = X_1 \cup X_2$ is either empty or an open and dense subset of X.*

PROOF. Assume that $X\backslash X_0 \neq \emptyset$. Then $X\backslash X_0$ is an open subset of X by lemma 5.5. The set

$$I := \{(t,x) \in \mathbb{C} \times X;\ \Phi(t,x) = x\} = \bigcup_{x \in X} (I_x \times \{x\})$$

is analytic in $\mathbb{C} \times X$ and the projection $I \to X$ is discrete. For a point $y \in X\backslash X_0$ we choose a holomorphic function $\omega : U \to \mathbb{C}$, U connected, satisfying lemma 5.5. The irreducible component B of I containing the graph $\{(\omega(x), x) :\ x \in U\}$ of ω is a purely n-dimensional analytic subset of I. Let $\overline{A}$ be the closure of $A := B\backslash\Big(\{0\}\times X\Big)$ in $\overline{\mathbb{C}}\times X$, where $\overline{\mathbb{C}} = \mathbb{P}^1$ is the Riemann sphere. Because of the Thullen-Remmert-Stein singularity theorem there are only two possibilities:

$$\text{(I) } \overline{A} \text{ is an analytic subset of } \overline{\mathbb{C}} \times X.$$

$$\text{(II) } \overline{A} \supset \{\infty\} \times X \text{ or } \overline{A} \supset \{0\} \times X\,.$$

Let $p_2 : \overline{\mathbb{C}} \times X \to X$ denote the canonical projection. In case (I) the image $p_2(\overline{A})$ is an analytic subset of X because p_2 is proper. Since $U \subset p_2(A)$, we get $X = p_2(\overline{A}) \subset \overline{p_2(A)}$. In case (II) the same conclusion holds trivially. ∎

In the following we assume that $\mathrm{Fix}(\Phi)$ **consists of isolated points**, i.e. $\dim \mathrm{Fix}(\Phi) = 0$. In this case, since $n \geq 2$, we get $\mathrm{Fix}(\Phi) = S$ (see proposition 5.2), and $X\backslash \mathrm{Fix}(\Phi)$ is the regular locus X^r of the foliation $\mathcal{F}$.

Any big integral variety $\phi : Y \to X$ of $\mathcal{F}$ is called an **integral curve**. It is called **proper** if ϕ is proper. We may interpret Y as an analytic subset of X in this case. As in the preceding sections we call the integral curve **compact** iff Y is compact (compare 2.13).

Now we formulate assumptions for Φ which are similar to the assumptions 4.17 in section 4:

Assumptions 5.7.

(a) $\mathrm{Fix}(\Phi)$ *consists of isolated points only; there exists a non-void locally finite family* $(\Gamma_j)_{j\in J}$ *of mutually disjoint compact integral curves of* $\mathcal{F} = \mathcal{F}_\Phi$ *such that* $\mathrm{Fix}(\Phi) \subset \mathrm{I\!\Gamma} := \bigcup_{j\in J}\Gamma_j$.

(b) $X'/\mathcal{F}'$ *is Hausdorff, where* $X' := X\backslash\mathrm{I\!\Gamma}$ *and* $\mathcal{F}' = \mathcal{F}|_{X'}$.

(c) *No leaf* L *of* $\mathcal{F}'$ *is compact.*

(d) *For every leaf* L *of* $\mathcal{F}'$ *the points* $y \in \mathrm{Fix}(\Phi)\cap\overline{L}$ *are singularities for* L *in the following sense: there is no open neighborhood* U *of* y *with an analytic subset* $A \subset U$ *of pure dimension 1 passing through* y *such that* $A\backslash\{y\} \subset L$.

We formulate some simple consequences of the assumptions 5.7:

Remarks 5.8. Because of (a) the Γ_j are Φ-invariant since they are compact. Hence, X' is Φ-invariant too and the leaves of $\mathcal{F}'$ are Φ-orbits. Because of (b) the leaves of $\mathcal{F}'$ are analytic subsets of X' and the equivalence relation R' corresponding to $\mathcal{F}'$ is open and analytic.
The leaves L of $\mathcal{F}'$ in X' and the connected components of $\mathrm{I\!\Gamma}\backslash\mathrm{Fix}(\Phi)$ are the leaves of $\mathcal{F}^r$.
Because of (c) and (d), $\mathcal{F}$ has leaves everywhere, and $(\Gamma_j)_{j\in J}$ is the family of all compact leaves of $\mathcal{F}$.

For $\mathcal{F} = \mathcal{F}_\Phi$ the assumptions 4.17 and all its conclusions hold (compare the theorems 4.21 and 4.22). So we get

Proposition 5.9.

(1) *For every* Γ_j *the basin of attraction* $\mathcal{A}(\Gamma_j)$ *is an analytic subset of* $X' \cup \Gamma_j$ *of dimension* > 1 *everywhere.*

(2) *If* Γ_j *is almost attractive then* $\mathcal{A}(\Gamma_j) = X' \cup \Gamma_j$.

(3) *For all* Φ*-orbits* L *in* X', *we have* $c'(\mathcal{F}) \le l(L) \le e(L) \le 2$, *in particular* $c'(\mathcal{F})$ *is bounded by 2.*

(4) *If* $\dim X = 2$ *or if the equivalence relation* $R = R_{\mathcal{F}}$ *is open, then* $\mathcal{A}(\Gamma_j) = X' \cup \Gamma_j$ *for all* Γ_j, i.e. $c'(\mathcal{F}) = c(\mathcal{F})$.

(5) *If* R *is open, then* $\lim L = \mathrm{I\!\Gamma}$ *for all* Φ*-orbits* L *in* X' *and* $1 \le c'(\mathcal{F}) = c(\mathcal{F}) = l(L) \le e(L) \le 2$.

Using proposition 5.6 and lemma 5.5, we get

THEOREM 5.10. *If X is compact and Φ operates effectively, then:*
(1) $X_0 = X'$, i.e. $\operatorname{rank} I_x = 0$ *for all* $x \in X'$.
(2) *For all Φ-orbits L in X' we have* $1 = c'(\mathcal{F}) = c(\mathcal{F}) = l(L) = e(L)$, i.e. *there exists exactly one compact leaf and this is a global attractor.*

PROOF. (2) follows from (1) and proposition 5.9.

Ad (1) Let us assume that there exists a point $y \in X'$ with $\operatorname{rank} I_y = 1$. We follow the proof of proposition 5.6 and make use of a holomorphic mapping $\omega : U \to \mathbb{C}$ on an open, connected neighborhood U of y with $0 \neq \omega(x) \in I_x$ for all $x \in U$, given by lemma 5.5. Let B be the irreducible component of $I = \bigcup_{x \in X}(I_x \times \{x\})$ containing $\{(\omega(x), x);\ x \in U\}$. We shall prove that there exists an $\varepsilon > 0$ such that $|t| \geq \varepsilon$ for all $(t, x) \in B$. Since X is compact, $\operatorname{Fix}(\Phi)$ consists of finitely many points $P_1, \dots, P_N$ and $\mathbb{\Gamma}$ of finitely many compact leaves $\Gamma_1, \dots, \Gamma_K$. We choose neighborhoods U_ν of P_ν such that $\overline{U_\nu} \cap \overline{U_\mu} = \emptyset$ for $\nu \neq \mu$ and $\Gamma_\kappa \not\subset U_\nu$ for all κ and ν. We set $I^*_A := \bigcup_{y \in A}\left(I^*_y \times \{y\}\right)$ for subsets A of X. For the compact set $Y := X \setminus \bigcup_{\nu=1}^N U_\nu$ there exists an $\varepsilon > 0$ such that $|t| \geq \varepsilon$ for all $(t, y) \in I^*_Y$ (otherwise one could find a point $y \in Y$ with non discrete I_y). This holds also for all $(t, y) \in I^*_{X^r}$ since each Φ-orbit in $X^r = X \setminus \operatorname{Fix}(\Phi)$ intersects Y because of remark 5.8. Since $B \cap (\{0\} \times X^r) = \emptyset$, i.e. $B^r := B \cap (\mathbb{C} \times X^r) \subset I^*_{X^r}$, we get $|t| \geq \varepsilon$ for all $(t, y) \in B^r$, and also for all $(t, y) \in B$ since $\overline{B^r} = B$.

Now we consider the projection $p_1 : \mathbb{C} \times X \to \mathbb{C}$. Since X is compact, the image $p_1(B)$ is an analytic subset of $\mathbb{C}$ and $p_1(B) \subset \{t \in \mathbb{C};\ |t| \geq \varepsilon\}$. Therefore $p_1(B)$ is discrete. We conclude $\omega(x) = c\ \forall\, x \in U$ for a constant $c \in \mathbb{C}$ with $|c| \geq \varepsilon$, hence $B = \{c\} \times X$, i.e. $c \in I_x$ for all $x \in X$. This contradicts our assumption that Φ acts effectively. ∎

PROPOSITION 5.11. *If Φ satisfies the assumptions* 5.7.(*a*) *and* (*c*), *then*
(1) *If* $\Gamma_j \cap \operatorname{Fix}(\Phi) = \emptyset$ *then* Γ_j *is a torus.*
(2) *If* $\Gamma_j \cap \operatorname{Fix}(\Phi) \neq \emptyset$ *then the normalization* $\widehat{\Gamma_j}$ *of* Γ_j *is a disjoint union of Riemann spheres.*

(3) *If* $L = L(x)$, $x \in X' = X \backslash \Gamma$, *is a proper leaf of* X' *with* $\mathrm{rank} L = 0$ *and if the number of elements of* $\lim L$ *is at least* 2, *then all points* $y \in \mathrm{Fix}(\Phi) \cap \overline{L}$ *are singularities for* L.

There are examples (compare section 6) for which the Hausdorff condition 5.7.(b) is not satisfied, but where the following assumptions are valid (the Hausdorff assumption is weakened but a certain rank condition added):

ASSUMPTIONS 5.12.

(a′) $\mathrm{Fix}(\Phi)$ *consists of isolated points; there exists a non-void locally finite family* $(\Gamma_j)_{j \in J}$ *of compact mutually disjoint integral curves of* $\mathcal{F}$ *such that* $\mathrm{Fix}(\Phi) \subset \Gamma := \bigcup_{j \in J} \Gamma_j$.

(b′) *There exists an open connected dense and* $\mathcal{F}$*-saturated subset* $\tilde{X}$ *of* $X' = X \backslash \Gamma$ *such that* $\tilde{X}/\tilde{\mathcal{F}}$ *is Hausdorff, where* $\tilde{\mathcal{F}} := \mathcal{F}|_{\tilde{X}}$.

(c′) *Every leaf* L *in* X' *is proper and of rank 0.*

(d′) *For every leaf* L *in* X' *the number of elements of* $\lim L$ *is at least* 2.

LEMMA 5.13. *Under the assumptions* 5.12 *the points* (*a*), (c) *and* (d) *of the assumptions* 5.7 *hold and* $\mathcal{F}_\Phi$ *has leaves everywhere.*

PROOF. (c) follows from (c′) and (d) is a consequence of (d′) and proposition 5.11. Because of the remarks 5.8 the foliation has leaves everywhere. ∎

So we get under the assumptions 5.12:

PROPOSITION 5.14. *If* J *is finite,then there exists precisely one* Γ_j *which is almost attractive. We have* $\mathcal{A}(\Gamma_j) \supset \tilde{X}$.

PROOF. Every leaf L of $\tilde{\mathcal{F}}$ satisfies the conditions of proposition 3.13, hence there exists precisely one $j \in J$ such that $\overline{L} \cap \Gamma_j \neq \emptyset$. Let $\lambda : \tilde{X} \to J$ be the mapping $\lambda(x) := j$ iff $\overline{L(x)} \cap \Gamma_j \neq \emptyset$. Using arguments similar to those in the proofs of 4.3 and 4.6 one shows that λ is locally constant. ∎

6 – Examples of $\mathbb{C}$-actions on $\mathbb{P}^n$

In this section we study examples of nontrivial holomorphic vector fields θ on the n-dimensional complex projective space $\mathbb{P}^n = (\mathbb{C}^{n+1}\backslash\{0\}) /\mathbb{C}^*$, $n\geq 2$, and the associated holomorphic $\mathbb{C}$-actions $\Phi : \mathbb{C}\times\mathbb{P}^n \to \mathbb{P}^n$. These define holomorphic foliations $\mathcal{F} = \mathcal{F}_\Phi$ which necessarily have singularities. The vectorfields θ and the corresponding $\mathbb{C}$-actions Φ can be lifted to holomorphic vectorfields η on $\mathbb{C}^{n+1}\backslash\{0\}$ of the form

$$\eta = \sum_{\nu=0}^{n} \alpha_\nu \frac{\partial}{\partial z_\nu}$$

with linear coefficients α_ν resp. to holomorphic $\mathbb{C}$-actions

$$\Psi : \mathbb{C}\times(\mathbb{C}^{n+1}\backslash\{0\}) \to \mathbb{C}^{n+1}\backslash\{0\}\,,$$

which commute with the canonical projection $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{P}^n$, i.e. $\theta = \pi_*\eta$ and $\pi\circ\Psi = \Phi(\operatorname{Id}_{\mathbb{C}}\times\pi)$. For similar constructions in a more general setting compare [13].

Definition 6.1 θ is called **diagonal** iff one can choose linear coordinates $t_0,\dots,t_n$ of $\mathbb{C}^{n+1}$ such that

$$\alpha_\nu(t) = \lambda_\nu t_\nu\,,\quad \lambda_\nu\in\mathbb{C},\ \text{for } \nu = 0,\dots,n\,.$$

If no three of the λ_ν are $\mathbb{R}$-collinear in $\mathbb{C}$, one says that θ is **diagonal of general type**. Φ is called **diagonal** resp. **diagonal of general type** iff θ has this property. In this case the lifted $\mathbb{C}$-action Ψ is of the form

$$\Psi(\tau,(t_0,\dots,t_n)) = (t_0e^{\lambda_0\tau},\dots,t_ne^{\lambda_n\tau})\,.$$

In the following, $\mathbb{C}$-actions Φ are always diagonal of general type.

Notations 6.2. *If $(e_0,\dots,e_n)$ denotes the standard basis of $\mathbb{C}^{n+1}$, we set*

$$\begin{aligned} 0_\nu &:= \pi(e_\nu),\ \nu = 0,\dots,n,\\ A_{\nu\mu} &:= \pi\{t_\nu e_\nu + t_\mu e_\mu;\ (t_\nu,t_\mu)\in\mathbb{C}^2\backslash\{0\}\},\ 0\leq\nu<\mu\leq n,\\ \Gamma &:= \bigcup_{0\leq\nu<\mu\leq n} A_{\nu\mu}\,. \end{aligned}$$

REMARK 6.3.
(1) $\mathrm{Fix}(\Phi) = \{0_0, \dots, 0_n\}$.
(2) $A_{\nu\mu} \setminus \mathrm{Fix}(\Phi)$ is a Φ-orbit of rank 1 for $0 \le \nu < \mu \le n$.
(3) $\mathcal{F} = \mathcal{F}_\Phi$ is a singular holomorphic foliation with Γ as (the only) compact leaf; it contains the singular locus of $\mathcal{F}$.

PROPOSITION 6.4. *All orbits in* $X' := \mathbb{P}^n \setminus \Gamma$ *have rank 0 and are proper leaves of* $\mathcal{F}' := \mathcal{F}|_{X'}$.

PROOF. We have to show that for every point $x^0 \in X'$ the mapping

$$\phi : \mathbb{C} \to X', \quad \phi(\tau) := \Phi(\tau, x^0)$$

is injective and proper, i.e. for each sequence $(\tau_\nu)_{\nu \in \mathbb{N}}$ in $\mathbb{C}$ with $\lim_{\nu\to\infty} \phi(\tau_\nu) = y^0 \in X'$ the limit $\lim \tau_\nu$ exists in $\mathbb{C}$. There are points $t^0 = (t_0, \dots, t_n)$ and $s^0 = (s_0, \dots, s_n)$ in $\mathbb{C}^{n+1}$ with at least three components different from 0, such that $\pi(t^0) = x^0$ and $\pi(s^0) = y^0$. We can assume that $t_i^0 \neq 0$ and $s_i^0 \neq 0$ for $i = 0, 1, 2$ and that $t_0^0 = s_0^0 = 1$. We can write $\phi(\tau)$ as follows:

$$\begin{aligned}\phi(\tau) &= \Phi\big(\tau, \pi(t^0)\big) = \pi\big(e^{\lambda_0\tau}, e^{\lambda_1\tau} t_1^0, \dots, e^{\lambda_n\tau} t_n^0\big) \\ &= \pi\big(1, e^{(\lambda_1-\lambda_0)\tau} t_1^0, \dots, e^{(\lambda_n-\lambda_0)\tau} t_n^0\big).\end{aligned}$$

We can assume $\lambda_1 - \lambda_0 = 1$ $\big($otherwise we replace $\tau(\lambda_1 - \lambda_0)$ by $\tau^*\big)$ and set $\lambda := \lambda_2 - \lambda_0 = \alpha + i\beta$ with $\beta \neq 0$. Since

$$\lim_{\nu\to\infty} \phi(\tau_\nu) = y^0 = \pi(s^0) = \pi(1, s_1^0, \dots, s_n^0),$$

we get

$$\lim_{\nu\to\infty} e^{\tau_\nu} t_1^0 = s_1^0, \quad \lim_{\nu\to\infty} e^{\lambda\tau_\nu} t_2^0 = s_2^0.$$

Setting $\tau_\nu = \alpha_\nu + i\beta_\nu$ we obtain

$$\begin{aligned}\lim_{\nu\to\infty} e^{\alpha_\nu} &= \left|\frac{s_1^0}{t_1^0}\right|, \quad \lim_{\nu\to\infty} \alpha_\nu = \log\left|\frac{s_1^0}{t_1^0}\right| =: \rho, \\ \lim_{\nu\to\infty} e^{\alpha\alpha_\nu - \beta\beta_\nu} &= \left|\frac{s_2^0}{t_2^0}\right|, \quad \lim_{\nu\to\infty} \beta_\nu = \frac{\alpha\rho}{\beta} - \frac{1}{\beta}\log\left|\frac{s_2^0}{t_2^0}\right| =: \sigma,\end{aligned}$$

i.e.

$$\lim \tau_\nu = \rho + i\sigma.$$

NOTATIONS 6.5. *For* $0 \leq \kappa < \nu < \mu \leq n$ *we set*

$$\begin{aligned} X_{\kappa\nu\mu} &:= \pi\{t \in \mathbb{C}^{n+1}\backslash\{0\};\ t_\kappa \cdot t_\nu \cdot t_\mu \neq 0.\}, \\ A_{\kappa\nu\mu} &:= \mathbb{P}^n \backslash X_{\kappa\nu\mu} \text{ (union of 3 hypersurfaces)} \\ \mathcal{F}_{\kappa\nu\mu} &:= \mathcal{F}|_{X_{\kappa\nu\mu}}. \end{aligned}$$

By arguments similar to those applied in the proof of proposition 6.4 one obtains that the equivalence relation defined by the foliation $\mathcal{F}_{\kappa\nu\mu}$ on $X_{\kappa\nu\mu}$ is a closed subset of $X_{\kappa\nu\mu} \times X_{\kappa\nu\mu}$. Consequently we get

PROPOSITION 6.6. *The quotients* $X_{\kappa\nu\mu}/\mathcal{F}_{\kappa\nu\mu}$, $0 \leq \kappa < \nu < \mu \leq n$, *are all hausdorff. Since*

$$\bigcup_{0 \leq \kappa < \nu < \mu \leq n} X_{\kappa\nu\mu} = X',$$

each point $x^0 \in X'$ *lies in such an open* $\mathcal{F}$*-saturated subset* $X_{\kappa\nu\mu} = \mathbb{P}^n \backslash A_{\kappa\nu\mu}$ *of* $\mathbb{P}^n$.

DEFINITION 6.7. An axis $A_{\nu\mu}$, $0 \leq \nu < \mu \leq n$, is called **extreme** iff the segment $[\lambda_\nu, \lambda_\mu]$ connecting λ_ν and λ_μ in $\mathbb{C}$ is a side of the convex hull of the points $\lambda_0, \ldots, \lambda_n$ in $\mathbb{C}$.

Again by arguments similar to those applied in the proof of proposition 6.4 one obtains

PROPOSITION 6.8. *For each leaf* L *in the open* $\mathcal{F}$*-saturated subset* $\tilde{X} := \pi\big((\mathbb{C}^*)^{n+1}\big)$ *the limit set* $\lim L$ *is the union of all extreme axes. There are at least three of them.*

From results above follows

THEOREM 6.9. *For diagonal* $\mathbb{C}$*-actions of general type the assumptions* 5.12 *are valid. Assumption* 5.12.(b′) *holds in the stronger version given by proposition* 6.6. *Assumption* 5.12.(d′) *holds in the stronger version* (d″)*: for each leaf* $L \subset X'$ *the limit set* $\lim L$ *contains at least three axes.*

PROOF. (d″) One can apply proposition 6.8, possibly only after restriction to a lower dimensional projective subspace of $\mathbb{P}^n$ containing L. ∎

In the special case $n = 2$, the stronger assumptions 5.7 are satisfied, and the conditions of theorem 5.10 and of proposition 5.9. (4) and (5) hold.

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Lavoro pervenuto alla redazione il 5 marzo 1997
ed accettato per la pubblicazione il 9 luglio 1997.
Bozze licenziate il 22 settembre 1997

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