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On the Poisson kernel for the Kohn Laplacian

F. UGUZZONI – E. LANCONELLI

RIASSUNTO: *In questa nota otteniamo alcune stime della funzione di Green e del nucleo di Poisson per il Laplaciano di Kohn $\Delta_{\mathbb{H}^n}$ su domini limitati del gruppo di Heisenberg $\mathbb{H}^n$. Grazie ad esse otteniamo poi stime L^p delle funzioni $\Delta_{\mathbb{H}^n}$-armoniche in termini delle norme L^p dei loro valori al bordo.*

ABSTRACT: *In this note we obtain some estimates of the Green function and the Poisson kernel for the Kohn Laplacian $\Delta_{\mathbb{H}^n}$ on bounded domains of the Heisenberg group $\mathbb{H}^n$. As a consequence we are able to give L^p estimates for $\Delta_{\mathbb{H}^n}$-harmonic functions in terms of the L^p norms of their boundary values.*

1 – Introduction

We are concerned with the Kohn Laplacian $\Delta_{\mathbb{H}^n}$ on the Heisenberg group $\mathbb{H}^n$ and with the study of $\Delta_{\mathbb{H}^n}$-harmonic functions on a bounded domain Ω of $\mathbb{H}^n$. The aim of this paper is to establish some estimates of the Green function G and the Poisson kernel P for $\Delta_{\mathbb{H}^n}$ on Ω, in terms of the natural distance d on $\mathbb{H}^n$.

Indeed, under suitable boundary regularity assumptions on Ω, we prove the following inequalities:

$$G(\xi,\eta) \le cd(\eta,\partial\Omega)d(\xi,\eta)^{1-Q}, \quad |\nabla_{\mathbb{H}^n} G(\xi,\cdot)| \le cd(\xi,\cdot)^{1-Q}, \tag{1.1}$$

$$|P(\xi,\eta)| \le cd(\xi,\eta)^{1-Q}, \tag{1.2}$$

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where $Q = 2n + 2$ denotes the homogeneous dimension of $\mathbb{H}^n$ and c is a positive constant only depending on Ω and Q. As a consequence we easily obtain L^p estimates for $\Delta_{\mathbb{H}^n}$-harmonic functions in terms of the L^p norms of their boundary values. Our main results are theorem 3.6, theorem 4.3 and theorem 4.4.

For the classical Laplace operator in $\mathbb{R}^N$, $N \geq 3$, the estimates (1.1) and (1.2) are well known: for instance, a proof in the case of domains with Dini-Liapunov continuous boundary can be found in a paper by WIDMAN [13]. As a matter of fact, our procedure is partially inspired to the technique introduced in [13].

The paper is organized as follows. In section 2, after introducing some notation, we obtain some estimates for the derivatives of $\Delta_{\mathbb{H}^n}$-harmonic functions and we prove some Green-type representation formulas.

In section 3, theorem 3.6, we prove the inequalities in (1.1) for domains satisfying the uniform exterior ball condition (property (P), see definition 3.3). The proof is based on the maximum principle for $\Delta_{\mathbb{H}^n}$ and the estimates obtained in section 2.

In section 4 we show that, if Ω satisfies further regularity conditions, then the harmonic measures related to Ω are absolutely continuous with respect to the surface measure and have density functions given by the Poisson kernel

$$P(\xi, \cdot) = -\langle A\nabla(G(\xi, \cdot)), N\rangle \,.$$

Moreover (1.2) holds (theorem 4.3). From (1.2) L^p-estimates immediately follow: there exists a positive constant c, only depending on Ω and the homogeneous dimension Q, such that

$$\| \, u \, \|^p_{L^p(\Omega)} \leq c \, \| \, u \, \|^p_{L^p(\partial\Omega)}, \qquad 1 \leq p < +\infty \,,$$

for every $\Delta_{\mathbb{H}^n}$-harmonic function u in Ω, continuous up to the boundary (theorem 4.4).

Finally, in the Appendix, we prove that every convex open set satisfies condition (P) (see remark 3.5). In particular, such a condition is verified by the balls of the intrinsic distance d.

We would like to close this introduction by quoting two papers which contain results partially related to ours. In [7] GAVEAU and VAUTHIER showed an explicit representation formula for the Poisson kernelof a particular *halfspace* of $\mathbb{H}^1$. Very recently, in an article treating a statistical problem [10], Krylov has proved an L^p estimate for the Green's function of the intrinsic ball for Hormander's operators, sum of squares of vector fields.

2 – Notation and preliminary results

We denote the points of the Heisenberg group $\mathbb{H}^n$ with $\xi = (z,t) = (x,y,t)$. The group law on $\mathbb{H}^n$ is given by

$$\xi \circ \xi' = (z + z', t + t' + 2(\langle x', y\rangle - \langle x, y'\rangle))\,.$$

The Kohn Laplacian on $\mathbb{H}^n$ is the operator

$$\Delta_{\mathbb{H}^n} = \sum_{j=1}^{n} ({X_j}^2 + {Y_j}^2)$$

where for every $j \in \{1, \dots, n\}$

$$X_j = \partial_{x_j} + 2y_j\partial_t, \quad Y_j = \partial_{y_j} - 2x_j\partial_t\,.$$

We set

$$\nabla_{\mathbb{H}^n} = (X_1, \dots, X_n, Y_1, \dots, Y_n)\,.$$

We denote by $H(\Omega)$ the set of the $\Delta_{\mathbb{H}^n}$-harmonic functions in Ω. The operator $\Delta_{\mathbb{H}^n}$ has nonnegative characteristic form but it is not elliptic at any point of $\mathbb{H}^n$: however, for the $\Delta_{\mathbb{H}^n}$-harmonic functions the strong maximum principle holds (see [1]). It is also important to note that $\Delta_{\mathbb{H}^n}$ is a variational operator. Indeed

$$\Delta_{\mathbb{H}^n} = \operatorname{div}(A\nabla)$$

where

$$A = \begin{pmatrix} I_n & 0 & 2y \\ 0 & I_n & -2x \\ 2y & -2x & 4|z|^2 \end{pmatrix}\,.$$

A natural group of dilations on $\mathbb{H}^n$ is given by $\delta_\lambda(\xi) = (\lambda z, \lambda^2 t)$, for

$\lambda > 0$. The jacobian determinant of δ_λ is λ^Q where $Q = 2n+2$ is called the homogeneous dimension of $\mathbb{H}^n$. The operators $\nabla_{\mathbb{H}^n}$ and $\Delta_{\mathbb{H}^n}$ are invariant w.r.t. the left translations τ_ξ of $\mathbb{H}^n$ and homogeneous w.r.t. the dilations δ_λ of degree one and of degree two, respectively. A remarkable analogy between the Kohn Laplacian and the classical Laplace operator is that a fundamental solution of $-\Delta_{\mathbb{H}^n}$ with pole at zero is given by

$$\Gamma(\xi) = \frac{c_Q}{d(\xi)^{Q-2}} \ ,$$

where c_Q is a suitable positive constant and

$$d(\xi) = (|z|^4 + t^2)^{1/4}$$

(see [3]). Moreover, if we define $d(\xi, \xi') = d({\xi'}^{-1} \circ \xi)$, then d is a distance on $\mathbb{H}^n$ (see [2] for a complete proof of this statement). More details on the Heisenberg group and the Kohn Laplacian can be found, for example, in [4] and [6].

In our paper a basic role is played by the following mean value formulas, due to GAVEAU [5] and to GAROFALO and LANCONELLI ([6], theorem 2.1). Let u be a $\Delta_{\mathbb{H}^n}$-harmonic function in an open subset Ω of $\mathbb{H}^n$ and let $B_r = B_d(\xi, r)$ be a metric ball such that $\overline{B_r} \subseteq \Omega$. Then

$$u(\xi) = \frac{1}{Q\alpha_Q r^{Q-1}} \int_{\partial B_r} \frac{\psi(\xi,\xi')}{|\nabla d(\xi,\xi')|} u(\xi') d\sigma(\xi') \tag{2.1}$$

and

$$u(\xi) = \frac{1}{\beta_Q r^{Q+2}} \int_{B_r} |z - z'|^2 u(\xi') d\xi' \tag{2.2}$$

where we have set $\psi(\xi) = \frac{|z|^2}{d(\xi)^2}$ and $\psi(\xi,\xi') = \psi({\xi'}^{-1} \circ \xi)$. We remark that the solid mean value formula (2.2) is different from the one in [6], but it can be easily obtained from (2.1) by integration. Indeed, by using coarea formula and keeping in mind the definition of ψ,

$$u(\xi) = \frac{Q+2}{r^{Q+2}} \int_0^r \varrho^{Q+1} u(\xi) d\varrho = \frac{1}{\beta_Q r^{Q+2}} \int_0^r \varrho^2 \int_{\partial B_\varrho} \frac{\psi(\xi,\xi')}{|\nabla d(\xi,\xi')|} u(\xi') d\sigma(\xi') d\varrho =$$

$$= \frac{1}{\beta_Q r^{Q+2}} \int_0^r \int_{\partial B_\varrho} \frac{d(\xi,\xi')^2 \psi(\xi,\xi')}{|\nabla d(\xi,\xi')|} u(\xi') d\sigma(\xi') d\varrho =$$

$$= \frac{1}{\beta_Q r^{Q+2}} \int_{B_r} |z - z'|^2 u(\xi') d\xi'.$$

We will use (2.2) in order to obtain a priori estimates of the derivatives of u along the left-invariant fields X_j and Y_j.

Proposition 2.1. *Let Ω be an open subset of $\mathbb{H}^n$, $u \in H(\Omega)$ and $\overline{B_d(\xi,r)} \subseteq \Omega$. Then for every $Z_1, \dots, Z_k \in \{X_1, \dots, X_n, Y_1, \dots, Y_n\}$ we have*

$$|Z_1 \dots Z_k u(\xi)| \le \frac{c}{r^k} \sup_{B_d(\xi,r)} |u| \tag{2.3}$$

where $c = c(k,Q)$.

Proof. Let $\varphi \in C_0^\infty(]0,1[, [0,+\infty[)$ be such that $\int \varphi = 1$. For every $\varepsilon > 0$, we set $\varphi_\varepsilon = \frac{1}{\varepsilon}\varphi(\frac{\cdot}{\varepsilon})$ and define $\Phi : \mathbb{H}^n \to \mathbb{R}$,

$$\Phi(\xi) = |z|^2 \int_{d(\xi)}^{+\infty} \frac{\varphi(t)}{\beta_Q t^{Q+2}} dt\,.$$

We also set $\Phi_\varepsilon = \varepsilon^{-Q} \Phi \circ \delta_{\varepsilon^{-1}}$. It is easy to see that for every $\xi \in \mathbb{H}^n$

$$\Phi_\varepsilon(\xi) = |z|^2 \int_{d(\xi)}^{+\infty} \frac{\varphi_\varepsilon(t)}{\beta_Q t^{Q+2}} dt\,.$$

Moreover, for every $\xi \in \Omega_\varepsilon := \{\xi \mid \overline{B_d(\xi,\varepsilon)} \subseteq \Omega\}$, we have

$$u(\xi) = (\Phi_\varepsilon * u)(\xi) := \int \Phi_\varepsilon({\xi'}^{-1} \circ \xi) u(\xi') d\xi'\,. \tag{2.4}$$

Indeed, by (2.2), we get

$$u(\xi)=\int_0^{+\infty}\varphi_\varepsilon(\varrho)u(\xi)d\varrho=\int_0^{\varepsilon}\frac{\varphi_\varepsilon(\varrho)}{\beta_Q\varrho^{Q+2}}\int_{d(\xi,\xi')<\varrho}|z-z'|^2u(\xi')d\xi'd\varrho=$$
$$=\int_{\mathbb{H}^n}|z-z'|^2u(\xi')\int_{d(\xi,\xi')}^{+\infty}\frac{\varphi_\varepsilon(\varrho)}{\beta_Q\varrho^{Q+2}}d\varrho d\xi'=(\Phi_\varepsilon * u)(\xi)\,.$$

Let us denote the k-th order operator $Z_1\dots Z_k$ by Z. By differentiating (2.4) and keeping in mind that every Z_j is invariant w.r.t. the left translations on $\mathbb{H}^n$, for every $\xi\in\Omega_\varepsilon$ we obtain

$$|Zu(\xi)|=|((Z\Phi_\varepsilon)*u)(\xi)|\le\Big(\sup_{B_d(\xi,\varepsilon)}|u|\Big)\int|Z\Phi_\varepsilon|=$$
$$=\Big(\sup_{B_d(\xi,\varepsilon)}|u|\Big)\varepsilon^{-k}\int|Z\Phi|\le\frac{c}{\varepsilon^k}\sup_{B_d(\xi,\varepsilon)}|u|\,.\qquad\square$$

Recalling that $\Delta_{\mathbb{H}^n}=\mathrm{div}A\nabla$, by using the divergence theorem in a standard way we obtain the following representation formula:

$$\begin{aligned}u(\xi)=&\int_{\partial D}(\Gamma(\xi,\cdot)\langle A\nabla u,N\rangle-u\langle A\nabla(\Gamma(\xi,\cdot)),N\rangle)d\sigma+\\&-\int_D\Gamma(\xi,\cdot)\Delta_{\mathbb{H}^n}u,\quad\forall\xi\in D\end{aligned}\tag{2.5}$$

for every $u\in C^\infty(\overline{D})$. Here D denotes a bounded open set with boundary sufficiently smooth and N is the outer unit normal to D. As usual, we will say that a bounded open set $\Omega\subseteq\mathbb{H}^n$ has a Green function, if for every $\xi\in\Omega$ there exists the classical solution h_ξ of the Dirichlet problem

$$\begin{cases}\Delta_{\mathbb{H}^n}h_\xi=0 & \text{in }\ \Omega\\ h_\xi=\Gamma(\xi,\cdot) & \text{in }\ \partial\Omega\end{cases}\tag{2.6}$$

In such a case the Green function is defined by

$$G(\xi,\xi')=\Gamma(\xi,\xi')-h_\xi(\xi')\qquad\forall\xi,\xi'\in\Omega\,.\tag{2.7}$$

Let now D be a smooth domain with closure $\overline{D}$ contained in a bounded open set Ω having a Green function G. If $u\in H(\Omega)$ we have, by the divergence theorem,

$$0 = \int_D (u\Delta_{\mathbb{H}^n} h_\xi - h_\xi \Delta_{\mathbb{H}^n} u) = \int_{\partial D} (u\langle A\nabla h_\xi, N\rangle - h_\xi \langle A\nabla u, N\rangle) d\sigma.$$

Adding this identity to (2.5) and keeping in mind (2.7), we obtain

$$u(\xi) = \int_{\partial D} (G(\xi,\cdot)\langle A\nabla u, N\rangle - u\langle A\nabla(G(\xi,\cdot)), N\rangle) d\sigma\,. \tag{2.8}$$

3 – Estimates of the Green function

We start by proving this simple lemma.

LEMMA 3.1. *Let $r > 0$ and $\xi, \xi' \in \mathbb{H}^n \setminus B_d(0,r)$. Then*

$$|\Gamma(\xi) - \Gamma(\xi')| \le c r^{1-Q} d(\xi,\xi') \tag{3.1}$$

where $c = c(Q)$.

PROOF. We set $t = d(\xi)$, $t' = d(\xi')$ and we apply the mean value theorem to the function t^{2-Q}. Since $t, t' \ge r$, there exists $t_0 \ge r$ such that

$$\begin{aligned}|\Gamma(\xi) - \Gamma(\xi')| &= c|t^{2-Q} - {t'}^{2-Q}| = c|(2-Q){t_0}^{1-Q}(t - t')| \le \\ &\le c r^{1-Q}|d(\xi) - d(\xi')| \le c r^{1-Q} d(\xi,\xi')\,.\end{aligned}$$

□

The following lemma shows a behavior of $\Delta_{\mathbb{H}^n}$-harmonic functions near the boundary points where they locally vanish.

LEMMA 3.2. *Let D be a bounded open subset of $\mathbb{H}^n$, $\xi_0 \in \partial D$ and $u \in H(D) \cap C(\overline{D})$. We set $\varphi = u|_{\partial D}$. If there exist $r > 0$ and $\eta_0 \in \mathbb{H}^n$ such that*
(i) *$B_d(\eta_0, r) \cap D = \emptyset$, $\xi_0 \in \partial B_d(\eta_0, r)$,*
(ii) *$\varphi \equiv 0$ in $\partial D \cap B_d(\eta_0, 2r)$,*
then

$$|u(\xi)| \le c(\max_{\partial D} |\varphi|)\frac{d(\xi,\xi_0)}{r} \qquad \forall \xi \in D \tag{3.2}$$

where $c = c(Q)$ only depends on Q.

PROOF. We set $M = \max_{\partial D} |\varphi|$ and define

$$w(\xi) = M \frac{\Gamma(r) - \Gamma(\xi, \eta_0)}{\Gamma(r) - \Gamma(2r)}, \qquad \xi \in D .$$

Since w is $\Delta_{\mathbb{H}^n}$-harmonic in D and, due to (ii), $w \geq |\varphi|$ in ∂D, by the maximum principle we obtain $w \geq |u|$ in D. Then, using (3.1)

$$|u(\xi)| \leq M \frac{\Gamma(\xi_0, \eta_0) - \Gamma(\xi, \eta_0)}{c r^{2-Q}} \leq cM \frac{r^{1-Q} d(\xi, \xi_0)}{r^{2-Q}} \qquad \forall \xi \in D$$

where all the constants c only depend on Q. ∎

DEFINITION 3.3. We say that an open subset Ω of $\mathbb{H}^n$ verifies the uniform exterior ball property (or, in short, that it verifies (P)) if there exists $r_0 > 0$ such that

$$(\mathrm{P})_{r_0} \qquad \begin{aligned} &\forall \xi \in \partial\Omega \quad \forall r \in]0, r_0] \ \exists \eta \in \mathbb{H}^n \\ &\text{such that } B_d(\eta, r) \cap \Omega = \emptyset \text{ and } \xi \in \partial B_d(\eta, r) . \end{aligned}$$

REMARK 3.4. If Ω is a bounded open set satisfying (P) then, for every $\varphi \in C(\partial\Omega)$ the Dirichlet problem

$$\begin{cases} \Delta_{\mathbb{H}^n} u = 0 & \text{in } \ \Omega \\ u = \varphi & \text{in } \ \partial\Omega \end{cases}$$

has a classical solution $u \in H(\Omega) \cap C(\overline{\Omega})$ (see [2], see also [8] and [12]). In particular Ω has the Green function.

REMARK 3.5. Every convex open set (in particular every d-ball) verifies $(\mathrm{P})_{r_0}$ for every $r_0 > 0$.

For the proof (not immediate) of the last statement we refer to the appendix (corollary A.3).

THEOREM 3.6. *Let Ω be a bounded and connected open subset of $\mathbb{H}^n$ verifying* (P) *and let G be its Green function. Then for every $\xi, \eta \in \Omega$,*

$\xi \neq \eta$, we have

$$0 < G(\xi,\eta) \leq c d(\xi,\eta)^{2-Q}, \tag{3.3}$$

$$G(\xi,\eta) \leq c d(\eta,\partial\Omega) d(\xi,\eta)^{1-Q}, \tag{3.4}$$

$$|\nabla_{\mathbb{H}^n} G(\xi,\cdot)|(\eta) \leq c d(\xi,\eta)^{1-Q}, \tag{3.5}$$

where $c = c(\Omega, Q)$.

Proof. From the maximum principle we immediately obtain $0 < h_\xi < \Gamma(\xi,\cdot)$ in Ω. Then (3.3) holds.

We now prove (3.4). There exists $r_0 = r_0(\Omega) > 0$ such that Ω verifies $(\mathrm{P})_{r_0}$. We fix $\xi_0, \eta_0 \in \Omega$ ($\xi_0 \neq \eta_0$) and we set

$$\delta = d(\eta_0, \partial\Omega), \ \ \varrho = d(\xi_0, \eta_0)\,.$$

We define also

$$r = \min\{\frac{\varrho}{8}, r_0\}\,. \tag{3.6}$$

If either $\delta \geq 2r_0$ or $\varrho \leq 4\delta$, then (3.4) easily follows from (3.3), since Ω is bounded. Suppose $\delta < 2r_0$ and $\varrho > 4\delta$, i.e

$$\delta < 2r. \tag{3.7}$$

There exists $\eta_1 \in \partial\Omega$ such that

$$d(\eta_0, \eta_1) = \delta\,.$$

Moreover, since Ω verifies $(\mathrm{P})_{r_0}$ and $r \leq r_0$, there exists $\eta_2 \in \mathbb{H}^n$ such that

$$B_d(\eta_2, r) \cap \Omega = \emptyset, \ \ \eta_1 \in \partial B_d(\eta_2, r)\,. \tag{3.8}$$

We define

$$B_1 = B_d(\eta_2, r), \quad B_2 = B_d(\eta_2, 2r), \qquad B_4 = B_d(\eta_2, 4r)\,,$$

$$D = \Omega \cap B_4, \qquad \partial_0 \ = (\partial\Omega) \cap B_2, \qquad \partial_1 \ = (\partial B_4) \cap \Omega\,.$$

We now choose $\varphi \in C(\partial D)$ such that

$$\varphi \equiv 0 \;\text{ in } \partial_0, \quad \varphi \equiv 1 \;\text{ in } \partial_1, \quad 0 \leq \varphi \leq 1\,. \tag{3.9}$$

Since both Ω and B_4 verify (P) (see remark 3.5) also D does. Then, by remark 3.4, there exists $u \in H(D) \cap C(\overline{D})$ such that $u = \varphi$ in ∂D. Moreover, from the maximum principle and from (3.9), $0 \leq u \leq 1$. Thanks to (3.8) and (3.9) we can now use lemma 3.2 and obtain

$$|u(\xi)| \leq c\frac{d(\xi, \eta_1)}{r} \qquad \forall \xi \in D$$

where $c = c(Q)$. In particular, since $\eta_0 \in D$ by (3.7), we get

$$|u(\eta_0)| \leq c\frac{\delta}{r}\,. \tag{3.10}$$

We now compare $v := G(\xi_0, \cdot)$ and u in D. By (3.6), (3.7) and (3.8) we have

$$\begin{aligned} d(\xi_0, \eta_2) &\geq d(\xi_0, \eta_0) - d(\eta_0, \eta_2) \geq \varrho - d(\eta_0, \eta_1) - d(\eta_1, \eta_2) = \\ &= \varrho - \delta - r > \varrho - 3r \geq \varrho - \frac{3}{8}\varrho = \frac{5}{8}\varrho \end{aligned}$$

so that

$$d(\xi_0, \overline{D}) \geq d(\xi_0, \overline{B_4}) \geq d(\xi_0, \eta_2) - 4r \geq \frac{5}{8}\varrho - \frac{\varrho}{2} = \frac{\varrho}{8}\,. \tag{3.11}$$

Recalling (3.3), for every $\xi \in \overline{D}$ we finally obtain

$$v(\xi) \leq c d(\xi, \xi_0)^{2-Q} \leq c_0 \varrho^{2-Q}\,.$$

Thereby, if we set

$$w = \frac{v}{c_0 \varrho^{2-Q}}$$

we have $w \leq 1$ in $\overline{D}$. As a consequence, using (3.9),

$$w \leq 1 = u \;\text{ in }\; \partial_1\,.$$

Since $G(\xi_0, \xi) = 0$ for every $\xi \in \partial\Omega$, we also have

$$w = 0 \le u \;\text{ in }\; \partial\Omega \cap \partial D = \partial D \setminus \partial_1 .$$

Therefore $w \le u$ in ∂D. Moreover, since $\xi_0 \notin \overline{D}$ (see (3.11)), $w \in H(D)$. By the maximum principle we then obtain $w \le u$ in D, i.e. $v \le c_0 \varrho^{2-Q} u$ in D. This inequality, together with (3.10), yields

$$\begin{aligned} G(\xi_0, \eta_0) = v(\eta_0) \le c_0 \varrho^{2-Q} u(\eta_0) \le c\frac{\delta}{r}\varrho^{2-Q} = \\ = c\delta\Big(\frac{\varrho}{r}\Big)\varrho^{1-Q} = c\Big(\frac{\varrho}{r}\Big) d(\eta_0, \partial\Omega) d(\xi_0, \eta_0)^{1-Q} .\end{aligned}$$

This proves (3.4) since $\frac{\varrho}{r} \le c(\Omega)$ (see (3.6)).

Let us prove (3.5). While we will keep the previous notation ($\delta = d(\eta_0, \partial\Omega)$, $\varrho = d(\xi_0, \eta_0)$ and $v = G(\xi_0, \cdot)$) we define

$$\begin{aligned} \varepsilon &= \frac{1}{2}\min\{\varrho, \delta\} , \\ B &= B_d(\eta_0, \varepsilon),\; B_0 = B_d(\eta_0, 2\varepsilon) . \end{aligned}$$

It results $\overline{B} \subseteq B_0 \subseteq \Omega$ and $v \in H(B_0)$. Moreover, for every $\eta \in B$,

$$\left\{ \begin{aligned} & d(\xi_0, \eta) \ge d(\xi_0, \eta_0) - d(\eta, \eta_0) \ge \varrho - \varepsilon \ge \tfrac{\varrho}{2} \\ & d(\eta, \partial\Omega) \le d(\eta_0, \partial\Omega) + d(\eta, \eta_0) \le \delta + \varepsilon \le \tfrac{3}{2}\delta \end{aligned} \right. \tag{3.12}$$

We now use the estimate (2.3) for $|\nabla_{\mathbb{H}^n} v|$ in B:

$$|\nabla_{\mathbb{H}^n} v(\eta_0)| \le \frac{c}{\varepsilon} \sup_B |v| .$$

Then if $\delta \le \varrho$, i.e. $\varepsilon = \frac{\delta}{2}$, using (3.4) and (3.12) we get

$$|\nabla_{\mathbb{H}^n} v(\eta_0)| \le \frac{c}{\varepsilon} \sup_{\eta \in B} \big(d(\eta, \partial\Omega) d(\xi_0, \eta)^{1-Q}\big) \le \frac{c}{\varepsilon}\delta\varrho^{1-Q} = c\varrho^{1-Q} .$$

On the other hand, if $\delta \ge \varrho$, i.e. $\varepsilon = \frac{\varrho}{2}$, using (3.3) and (3.12) we obtain

$$|\nabla_{\mathbb{H}^n} v(\eta_0)| \le \frac{c}{\varepsilon} \sup_{\eta \in B} d(\xi_0, \eta)^{2-Q} \le \frac{c}{\varepsilon}\varrho^{2-Q} = c\varrho^{1-Q} .$$

Keeping in mind the meaning of v and ϱ, this proves (3.5). $\square$

4 – The Poisson kernel and L^p estimates of $\Delta_{\mathbb{H}^n}$-harmonic functions

Let Ω be an open subset of $\mathbb{H}^n$. We will say that Ω satisfies (I_1) if

$$(I_1) \qquad \partial\Omega \quad \text{is smooth.}$$

Moreover, we will say that Ω satisfies (I_2) if

$$(I_2) \qquad \sigma(K_\Omega) = 0$$

where σ denotes the surface measure and

$$\begin{aligned} K_\Omega &= \{\text{characteristic points of } \partial\Omega\} \\ &= \{\xi \in \partial\Omega \mid N_j + 2y_j N_{2n+1} = 0 = N_{n+j} - 2x_j N_{2n+1} \ \forall j \in \{1, \dots, n\}\}. \end{aligned}$$

Here $N = N(\xi)$ denotes, as usual, the outer unit normal to Ω. The following almost obvious remark will be useful to our purposes.

Remark 4.1. If Ω is bounded then K_Ω is compact.

Let Ω be a bounded open subset of $\mathbb{H}^n$ satisfying (P), (I_1) and (I_2). Let G be the Green function of Ω. For every $\xi \in \Omega$ and $\eta \in \partial\Omega \setminus K_\Omega$ we define

$$P(\xi, \eta) = -\langle A\nabla(G(\xi, \cdot)), N\rangle(\eta). \tag{4.1}$$

We want to stress that, due to Kohn and Nirenberg results [11] (see also [9], theorem 7.1) the function $G(\xi, \cdot)$ is smooth up to $\partial\Omega \setminus K_\Omega$. Then, P in (4.1) is well defined.

Lemma 4.2. *Let Ω be a bounded open subset of $\mathbb{H}^n$ verifying* (P), (I_1) *and* (I_2) *and let $\varphi \in C^\infty(\partial\Omega)$. Suppose that φ is constant in a neighborhood of $K = K_\Omega$. Then if we denote by u the classical solution to*

$$\begin{cases} \Delta_{\mathbb{H}^n} u = 0 & in \ \ \Omega \\ u = \varphi & in \ \ \partial\Omega \end{cases}$$

we have

$$|\nabla_{\mathbb{H}^n} u| \in L^\infty(\Omega) \tag{4.2}$$

and

$$u(\xi) = \int_{\partial\Omega} P(\xi, \cdot)\varphi d\sigma \quad \forall \xi \in \Omega\,. \tag{4.3}$$

PROOF. Without loss of generality we can suppose $\varphi = 0$ in $V \cap \partial\Omega$ where V is an open subset of $\mathbb{H}^n$ containing K. Since K is compact and Ω verifies (P), there exist another neighborhood V_1 of K, $V_1 \subseteq V$, and a radius $r_1 > 0$ such that: for every $\eta \in (\partial\Omega) \cap V_1$ there is an exterior ball of radius r_1 touching Ω in η and such that the corresponding double radius ball is contained in V. Then, by lemma 3.2, for every $\xi \in \Omega$ and $\eta \in (\partial\Omega) \cap V_1$,

$$|u(\xi)| \leq c \max_{\partial\Omega} |\varphi| \frac{d(\xi, \eta)}{r_1} = cd(\xi, \eta)\,.$$

Moreover there exists another neighborhood V_2 of K, $V_2 \subseteq V_1$, such that for every $\xi \in V_2 \cap \Omega$ and for every $\xi' \in B_\xi = B_d(\xi, \frac{d(\xi,\partial\Omega)}{2})$, there exists $\eta' \in (\partial\Omega) \cap V_1$ such that $d(\xi', \partial\Omega) = d(\xi', \eta')$. Hence,

$$|u(\xi')| \leq cd(\xi', \partial\Omega) \qquad \forall \xi' \in B_\xi\,.$$

We also have $d(\xi', \partial\Omega) \leq d(\xi', \xi) + d(\xi, \partial\Omega) \leq \frac{3}{2} d(\xi, \partial\Omega)$. Therefore, using the estimate (2.3) for $|\nabla_{\mathbb{H}^n} u|$ in B_ξ, we obtain for every $\xi \in V_2 \cap \Omega$

$$|\nabla_{\mathbb{H}^n} u(\xi)| \leq \frac{c}{d(\xi, \partial\Omega)} \sup_{B_\xi} |u| \leq \frac{c}{d(\xi, \partial\Omega)} \sup_{\xi' \in B_\xi} d(\xi', \partial\Omega) \leq c.$$

This proves that $|\nabla_{\mathbb{H}^n} u|$ is bounded in $V_2 \cap \Omega$. On the other hand, by means of the results in [11], $|\nabla_{\mathbb{H}^n} u| \in C^\infty(\overline{\Omega \setminus V_2})$. Then (4.2) holds.

We now show that (4.3) follows from (4.2). Let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a family of smooth open sets such that $\overline{\Omega_\varepsilon} \subseteq \Omega$ and $\bigcup_{\varepsilon>0} \Omega_\varepsilon = \Omega$. For every $\varepsilon > 0$ the representation formula (2.8) holds for $D = \Omega_\varepsilon$. As ε goes to zero, from these representation formulas we obtain (4.3). Indeed, for every fixed $\xi \in \Omega$, $G(\xi, \eta)$ goes to zero as η approaches the boundary of Ω. Moreover (by (3.3) and (4.2))

$$\begin{aligned} |\langle A\nabla u, N\rangle G(\xi, \cdot)| &\leq \langle A\nabla u, \nabla u\rangle^{1/2} \langle AN, N\rangle^{1/2} G(\xi, \cdot) \leq \\ &\leq |\nabla_{\mathbb{H}^n} u| \ \parallel A \parallel^{\frac{1}{2}} G(\xi, \cdot) \leq cd(\xi, \cdot)^{2-Q} \leq c_\xi \end{aligned}$$

and, by (3.5),

$$|u\langle A\nabla G(\xi,\cdot),N\rangle| \le |u\nabla_{\mathbb{H}^n}G(\xi,\cdot)| \ \parallel A \parallel^{\frac{1}{2}} \le cd(\xi,\cdot)^{1-Q} \le c_\xi$$

in $\partial\Omega_\varepsilon$, if ε is small enough. $\square$

We are now in position to prove the main results of this paper.

THEOREM 4.3. *Let Ω be a bounded open subset of $\mathbb{H}^n$ verifying (P), (I_1) and (I_2) and let $\xi \in \Omega$. Let us denote by μ^ξ the $\Delta_{\mathbb{H}^n}$-harmonic measure of Ω with respect to ξ. Then*

$$d\mu^\xi = P(\xi,\cdot)d\sigma \tag{4.4}$$

where σ denotes the surface measure. Moreover, the following estimate of the Poisson kernel P holds:

$$0 \le P(\xi,\eta) \le cd(\xi,\eta)^{1-Q} \qquad \forall\xi\in\Omega,\ \forall\eta\in\partial\Omega \tag{4.5}$$

where $c = c(\Omega, Q)$.

PROOF. We first prove (4.5). From the maximum principle and from (4.3) it follows that $P \ge 0$. Moreover, using the estimate (3.5) we get

$$\begin{aligned} P(\xi,\eta) &\le (\langle A\nabla G(\xi,\cdot),\nabla G(\xi,\cdot)\rangle^{1/2}\langle AN,N\rangle^{1/2})(\eta) \le \\ &\le |\nabla_{\mathbb{H}^n}G(\xi,\cdot)|(\eta) \ \parallel A(\eta)\parallel^{\frac{1}{2}} \le c_\Omega d(\xi,\eta)^{1-Q}\,. \end{aligned}$$

Then (4.5) holds.

We now prove (4.4). We fix $\xi \in \Omega$ and we set, for every measurable subset E of $\partial\Omega$,

$$P^\xi(E) = \int_E P(\xi,\cdot)d\sigma\,.$$

Then, since $P \ge 0$, P^ξ is a positive measure on $\partial\Omega$. We have to prove that $P^\xi = \mu^\xi$. For sake of brevity we set $P = P^\xi$, $\mu = \mu^\xi$, $K = K_\Omega$. Thanks to lemma 4.2 we know that

$$\int_{\partial\Omega}\varphi d\mu = \int_{\partial\Omega}\varphi dP$$

for every function $\varphi \in C_0^\infty(\partial\Omega)$ constant in a neighborhood of K. Let E be a closed subset of $\partial\Omega$. Then for every $\varepsilon > 0$ there exists an open subset A of $\partial\Omega$ such that $E \subseteq A$, $P(A \setminus E) < \varepsilon$. Moreover, since $\sigma(K) = 0$, there exist U_0, U_1, U_2, open subsets of $\partial\Omega$, such that $K \subseteq U_0 \subseteq \overline{U_0} \subseteq U_1 \subseteq \overline{U_1} \subseteq U_2$, $P(U_2) < \varepsilon$. Let now $\Phi, \varphi, \psi \in C_0^\infty(\partial\Omega)$ be such that

$$0 \le \Phi \le 1 \ , \ \Phi \equiv 1 \ \text{ in } \ \partial\Omega \setminus U_1 \ , \ \Phi \equiv 0 \ \text{ in } \ U_0 \, ,$$

$$0 \le \varphi \le 1 \ , \ \varphi \equiv 0 \ \text{ in } \ \partial\Omega \setminus U_2 \ , \ \varphi \equiv 1 \ \text{ in } \ U_1 \, ,$$

$$0 \le \psi \le 1 \ , \ \psi \equiv 0 \ \text{ in } \ \partial\Omega \setminus A \ , \ \psi \equiv 1 \ \text{ in } \ E \, .$$

We have

$$\mu(U_1) \le \int \varphi d\mu = \int \varphi dP \le P(U_2) < \varepsilon \, ,$$

then

$$\begin{aligned} \mu(E) &\le \mu(U_1) + \mu(E \setminus U_1) < \varepsilon + \mu(E \setminus U_1) \le \\ &\le \varepsilon + \int \psi\Phi d\mu = \varepsilon + \int \psi \Phi dP \le \varepsilon + P(A) = \\ &= \varepsilon + P(E) + P(A \setminus E) < P(E) + 2\varepsilon \, . \end{aligned}$$

As $\varepsilon \to 0$ we obtain

$$\mu(E) \le P(E)$$

for every closed subset E of $\partial\Omega$. This estimate can be extended to the open subsets of $\partial\Omega$ by a standard argument. In particular, keeping the previous notation, $\mu(A \setminus E) \le P(A \setminus E)$. Hence for every $\varepsilon > 0$ we get

$$\begin{aligned} P(E) &\le P(U_1) + P(E \setminus U_1) < \varepsilon + P(E \setminus U_1) \le \\ &\le \varepsilon + \int \psi \Phi dP = \varepsilon + \int \psi\Phi d\mu \le \varepsilon + \mu(A) = \\ &= \varepsilon + \mu(E) + \mu(A \setminus E) \le \varepsilon + \mu(E) + P(A \setminus E) < \\ &< \mu(E) + 2\varepsilon \, , \end{aligned}$$

then $P(E) \le \mu(E)$. Therefore $P(E) = \mu(E)$ for every closed subset E of $\partial\Omega$. This implies that $P = \mu$. $\square$

THEOREM 4.4. *Let Ω be a bounded open subset of $\mathbb{H}^n$ verifying* (P), (I_1) *and* (I_2). *Let $\varphi \in C(\partial\Omega)$ and define*

$$u(\xi) = \int_{\partial\Omega} P(\xi, \cdot)\varphi d\sigma, \qquad \xi \in \Omega\,.$$

Then u is the (*unique*) *classical solution of*

$$\begin{cases} \Delta_{\mathbb{H}^n} u = 0 & in\ \Omega \\ u = \varphi & in\ \partial\Omega \end{cases}$$

Moreover, for every $p \in [1, +\infty[$,

$$\| u \|^p_{L^p(\Omega)} \leq c \| \varphi \|^p_{L^p(\partial\Omega)}$$

where c only depends on Ω and Q.

PROOF. The first part of the theorem is a straightforward consequence of (4.4). The L^p estimate can be easily proved by using (4.5). Indeed, remarking that $\int_{\partial\Omega} P(\xi, \cdot)d\sigma = 1$, by the Hölder inequality we get

$$\begin{aligned} \| u \|^p_{L^p(\Omega)} &\leq \int_\Omega \Big| \int_{\partial\Omega} P(\xi, \cdot)d\sigma \Big|^{\frac{p}{p'}} \Big| \int_{\partial\Omega} P(\xi, \cdot)|\varphi|^p d\sigma \Big| d\xi = \\ &= \int_{\partial\Omega} |\varphi|^p(\eta) \int_\Omega P(\xi, \eta) d\xi d\sigma(\eta) \leq \\ &\leq c_\Omega \int_{\partial\Omega} |\varphi|^p(\eta) \int_\Omega d(\xi, \eta)^{1-Q} d\xi d\sigma(\eta) \leq c'_\Omega \| \varphi \|^p_{L^p(\partial\Omega)} \end{aligned}$$

since

$$\sup_{\eta \in \partial\Omega} \int_\Omega d(\xi, \eta)^{1-Q} d\xi < +\infty. \qquad \square$$

– Appendix

PROPOSITION A.1. *Let π be an hyperplane of $\mathbb{H}^n$ and let $\xi_0 \in \pi$; then for every $R > 0$ there exist two metric balls B and $\widetilde{B}$ of radius R, lying on opposite sides of π, such that*

$$\overline{B} \cap \pi = \{\xi_0\} = \overline{\widetilde{B}} \cap \pi\,.$$

Proof. Since τ_{ξ_0} is a bijective affine transformation mapping balls into balls, without loss of generality we can suppose $\xi_0 = 0$.

If the hyperplane π includes the t axis, then

$$\pi = \{\xi = (z,t) \mid \langle A, z\rangle = 0\}$$

where $A = (a,b) \in \mathbb{R}^{2n} \setminus \{0\}$. By means of a direct computation we easily verify that the metric balls of radius R centered at $(R\frac{A}{|A|}, 0)$ and $-(R\frac{A}{|A|}, 0)$ have the required properties.

Let us now suppose that π does not include the t axis. In this case

$$\pi = \{\xi' = (x', y', t') \mid t' = \langle a, x'\rangle + \langle b, y'\rangle\}$$

where $a, b \in \mathbb{R}^n$. We want to find two points $\xi, \widetilde{\xi} \in \partial B_d(0,R)$, lying on opposite sides with respect to π, such that the origin is a minimum point of the functions $d(\xi,\cdot), d(\widetilde{\xi},\cdot) : \pi \to \mathbb{R}$. We set $\varrho = |a|^2 + |b|^2$ and $r = R^4$. If $\varrho = 0$, then the points $\xi = (0, R^2)$ and $\widetilde{\xi} = -\xi$ satisfy our requirement.

If $\varrho \neq 0$ we set

$$t = \frac{1}{\varrho}(-8r^2 + 4\sqrt{4r^4 + \varrho^2 r^3})^{1/2}. \tag{A.1}$$

Then t is a solution of the equation

$$\varrho^2 t^4 + 16r^2t^2 - 16r^3 = 0. \tag{A.2}$$

In particular $t^2 < r$. We set $\gamma = \sqrt{r - t^2}$ and

$$\begin{cases} x = -\dfrac{t}{2r}(tb + \gamma a) \\ y = \dfrac{t}{2r}(ta - \gamma b) \\ \xi = (x, y, t) \end{cases} \tag{A.3}$$

Using (A.2) a simple computation yields

$$d(\xi) = R. \tag{A.4}$$

We now set $F_\xi : \mathbb{R}^{2n} \to \mathbb{R}$,

$$\begin{aligned} F_\xi(z') &= d(\xi, (z', \langle a, x'\rangle + \langle b, y'\rangle))^4 = \\ &= |z - z'|^4 + (\langle a - 2y, x'\rangle + \langle b + 2x, y'\rangle - t)^2. \end{aligned}$$

By a direct computation one can verify that (since (A.3) and (A.4) hold) it is

$$\nabla F_\xi(0) = 0\,.$$

Hence the origin is a critical point for F_ξ. Moreover F_ξ has hessian

$$H(z') = H_{F_\xi}(z') = (8\zeta_j\zeta_k + 4|\zeta|^2\delta_{j,k} + 2\alpha_j\alpha_k)_{j,k\in\{1,\dots,2n\}}$$

(where we have set for shortness $\zeta = z' - z$, $\alpha = (a-2y, b+2x)$) which is positive semidefinite for every $z' \in \mathbb{R}^{2n}$, since for every $w \in \mathbb{R}^{2n}$ we have

$$\langle H(z')w, w\rangle = 8\langle\zeta, w\rangle^2 + 4|\zeta|^2|w|^2 + 2\langle\alpha, w\rangle^2 \geq 0.$$

In addition, H is positive definite in the origin, since $|z| \neq 0$ (being $t^2 < r$) and

$$\langle H(0)w, w\rangle \geq 4|z|^2|w|^2\,.$$

Therefore F_ξ is convex and 0 is its strong absolute minimum point.

Since also $\widetilde{t} = -t$ is a solution of (A.2), if we define $\widetilde{\xi}$ replacing $\widetilde{t}$ to t in (A.3), $F_{\widetilde{\xi}}$ has the same properties of F_ξ. To conclude the proof it is now enough to observe that ξ and $\widetilde{\xi}$ belong to different connected components of $\mathbb{H}^n \setminus \pi$ since $\frac{\xi+\widetilde{\xi}}{2} \in \pi$, being

$$\langle (a, b, -1), \xi + \widetilde{\xi}\rangle = \Big\langle (a, b, -1), \Big(-\frac{t^2}{r}b, \frac{t^2}{r}a, 0\Big)\Big\rangle = 0\,. \qquad \square$$

Corollary A.2. *If Ω is an halfspace of $\mathbb{H}^n$ then for every $\xi_0 \in \partial\Omega$ and for every $R > 0$ there exists a metric ball B of radius R such that $\overline{B} \cap \overline{\Omega} = \{\xi_0\}$.*

Corollary A.3. *If Ω is a convex subset of $\mathbb{H}^n$ (in particular if Ω is a d-ball) then for every $\xi_0 \in \partial\Omega$ and for every $R > 0$ there exists a metric ball B of radius R such that $\overline{B} \cap \overline{\Omega} = \{\xi_0\}$.*

We explicitly remark that every d-ball centered at the origin is a convex set since d is a convex function. As a consequence every d-ball $B_d(\xi, r)$ is convex because $B_d(\xi, r) = \tau_\xi(B_d(0, r))$ and τ_ξ is an affine mapping.

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Indirizzo degli Autori:

F. Uguzzoni – E. Lanconelli – Dipartimento di Matematica, Università di Bologna – Piazza di Porta S. Donato 5 – 40127 Bologna, Italy
E-mail: uguzzoni@dm.unibo.it
lanconel@dm.unibo.it