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Manifolds with local quaternion Kähler structures

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Riassunto: *Si presentano alcuni risultati relativi a varietà M^{4n} localmente quaternionali kähleriane e localmente conformi quaternionali kähleriane. Si mostra che entrambe queste classi di varietà M^{4n} possono essere ottenute dallo studio delle azioni libere di gruppi finiti su varietà 3-sasakiane $\overline{P}$. A seconda che tali azioni cambino o preservino le foglie 3-dimensionali di una foliazione canonica, dal quoziente $P=\overline{P}/\Gamma$ si ottengono varietà proiettive 3-sasakiane o localmente 3-sasakiane. Le due classi sopra citate di varietà M^{4n} con struttura quaternionale kähleriana locale possono essere presentate rispettivamente come spazi delle foglie della foliazione indotta su P o come fibrati principali piatti in S^1 sopra P.*

Abstract: *We give a survey on locally quaternion Kähler and locally conformal quaternion Kähler manifolds M^{4n}. We show that the study of these two classes of Riemannian manifolds can be reported to the common framework of free actions of finite groups on 3-sasakian manifolds $\overline{P}$. According to whether such actions interchange or preserve the 3-dimensional leaves of a canonical foliation, then projective 3-sasakian or locally 3-sasakian manifolds are obtained as quotients $P=\overline{P}/\Gamma$. Then the two mentioned classes of manifolds M^{4n} with local quaternion Kähler structures can be presented as leaf spaces of the induced foliation on P or as flat principal S^1-bundles over P.*

1 – Introduction

The theory of quaternion Kähler manifolds is distinguished by their Einstein property into the three cases of positive, zero, and negative

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scalar curvature s. The intermediate situation $s = 0$ of Ricci-flat metrics corresponds to *locally hyperkähler* manifolds, whose reduced holonomy $\mathcal{H}_0$ is contained in the quaternionic unitary group $Sp(n)$. The full holonomy $\mathcal{H}$ of locally hyperkähler manifolds is thus a subgroup of $Sp(n) \cdot Sp(1)$, the normalizer of $Sp(n)$ in $O(4n)$. In terms of the rank 3 vector bundle $H \subset End\, TM$, locally spanned by compatible almost complex structures and that defines the *quaternionic structure* of M, the locally hyperkähler case corresponds to a flat induced Levi Civita connection on H.

The simplest examples of locally and non globally hyperkähler manifolds appear in dimension 4: the Enriques surfaces: $K/\mathbf{Z}_2$ and the Hitchin manifolds: $K/(\mathbf{Z}_2^+ \times \mathbf{Z}_2^-)$ - with full holonomy group respectively isomorphic to $Sp(1) \cdot \mathbf{Z}_4$ and to $Sp(1) \cdot Q_8$ - are the only finite quotients of hyperkähler K3-surfaces K [8]. Some quotients of flat 4-tori, that are not 4-tori themselves, show that also compact flat locally and non globally hyperkähler 4-dimensional Riemannian manifolds exist (cf. [4] for a classification). However, going to higher dimensions, it seems that very few examples of locally non globally hyperkähler manifolds are known. In particular, no examples of quotients of Beauville hyperkähler manifolds $K^{[r]}$ and K_r are known to the author, except for some $\mathbf{Z}_2$ and $\mathbf{Z}_2^+ \times \mathbf{Z}_2^-$-quotients of $K^{[2p+1]}$ with a factor by factor action, assured to be free by the odd quaternionic dimension (cf. [1] for a description of the manifolds $K^{[r]}$ and K_r, and [11] for these quotients).

In this paper we report on some recent work concerning two different classes of 4n-dimensional Riemannian manifolds (M^{4n}, g) admitting local and non global quaternion Kähler structures. Namely, we shall be concerned with:

A) *Locally quaternion Kähler manifolds* (M, g), i.e. complete Riemannian manifolds whose reduced holonomy group $\mathcal{H}_0(M)$ is a subgroup of $Sp(n) \cdot Sp(1)$.

B) *Locally conformal quaternion Kähler* (M, H, g), i.e. manifolds M equipped with a quaternionic structure $H \subset End\ TM$, and with local quaternion Kähler metrics g'_{U_i} defined on open neighborhoods U_i covering M, that are required to be conformally related to a unique global quaternion hermitian metric g.

Note that the first class includes the locally hyperkähler manifolds, and we shall see that the compact representatives of the second class turn out to be locally conformal to locally hyperkähler manifolds, that play therefore a special rôle in both situations. On the other hand, it is convenient not to include the globally quaternion Kähler and the globally conformal quaternion Kähler metrics in the descriptions of classes A) and B). Indeed, these two global counterparts of classes A) and B) are in several respects not so much subcases as opposite cases of the local ones. Accordingly, we will assume throughout this paper that *all the local quaternion Kähler structures considered are not global.*

Thus, by the term *locally quaternion Kähler* we shall understand that the full holonomy group $\mathcal{H}(M)$ is not allowed to be a subgroup of $Sp(n)\cdot Sp(1)$. Similarly, the term *locally conformal quaternion Kähler* will be reserved for the situation where no quaternion Kähler metric exists in the conformal class of g.

A geometric description of the positive representatives in the class A) and of the compact representatives in the class B) will be given in the following two paragraphs separately. References respectively [16] and [14] contain more informations. A common feature of these two classes will be pointed out in the last paragraph, where a general construction for the positive representatives in A) and the compact representatives in B) is described in the framework of 3-sasakian geometry.

2 – Positive locally quaternion Kähler manifolds

Let (M^{4n}, g) be a locally — and non globally — quaternion Kähler manifold, and let $(\overline{M}^{4n}, g)$ be its quaternion Kähler universal covering. We assume that the scalar curvature s of the Einstein metric g is positive as well as the completeness of M, referring to these manifolds by saying that (M^{4n}, g) is *positive* locally quaternion Kähler (cf. [10]).

A basic observation is now that $Sp(n)\cdot Sp(1)$, maximal subgroup of $SO(4n)$, cannot be the identity component of disconnected subgroups of $O(4n)$ [16]. This, together with Berger holonomy theorem for non locally symmetric irreducible Riemannian manifolds, proves the following fact.

Proposition 2.1. *Any positive locally quaternion Kähler manifold (M^{4n}, g) is compact, locally symmetric, and admits a finite covering $\overline{M} \to M$, where $\overline{M}$ is a quaternion Kähler Wolf space.*

Thus the problem of classifying positive locally quaternion Kähler manifolds is reduced to determine all the finite groups of isometries acting freely on the quaternion Kähler Wolf spaces. This is the so called "space form problem", discussed for most of compact irreducible symmetric spaces in Wolf's book [20]. Indeed, the quaternion Kähler Wolf spaces:

$$\mathbf{H}P^n, Gr_2(\mathbf{C}^{n+2}), \tilde{G}r_4(\mathbf{R}^{n+4}), G_2/SO(4)$$
$$F_4/Sp(3)\cdot Sp(1), E_6/SU(6)\cdot Sp(1), E_7/Spin(12)\cdot Sp(1), E_8/E_7\cdot Sp(1)$$

are not even mentioned in Wolf's book, although in its chapter 9 the classification of finite groups of isometries that act freely on them is implicitely contained. In particular, none of the exceptional quaternion Kähler Wolf spaces admits quotients. As for the three families $\mathbf{H}P^n$, $Gr_2(\mathbf{C}^{n+2})$, $\tilde{G}r_4(\mathbf{R}^{n+4})$, some restrictions to both n and the types of actions apply (cf. [20], p. 304). The following list of positive locally quaternion Kähler manifolds can thus be deduced.

Theorem 2.2. *The positive locally quaternion Kähler manifolds $M = \overline{M}/\Gamma$ are classified in the following table 1, where $\perp$, θ, σ_W, J denote the orthocomplementation, the change of orientation, the symmetry with respect to $W \subset \mathbf{C}^4$ or $\mathbf{R}^8$, and the standard complex structure, respectively.*

Table 1

$\overline{M}$	Γ	generators
$\mathbf{H}P^1$	$\mathbf{Z}_2$	$\perp$
$Gr_2(\mathbf{C}^4)$	$\mathbf{Z}_2$	$\perp$
$Gr_2(\mathbf{C}^4)$	$\mathbf{Z}_2$	$\perp\circ\sigma_{\mathbf{C}^3}$
$\tilde{G}r_4(\mathbf{R}^8)$	$\mathbf{Z}_2$	$\perp$
$\tilde{G}r_4(\mathbf{R}^8)$	$\mathbf{Z}_2$	$\perp\circ\sigma_{\mathbf{R}^6}$
$\tilde{G}r_4(\mathbf{R}^8)$	$\mathbf{Z}_2\times\mathbf{Z}_2$	$(\perp,\theta)$
$\tilde{G}r_4(\mathbf{R}^8)$	$\mathbf{Z}_2\times\mathbf{Z}_2$	$(\perp\circ\sigma_{\mathbf{R}^6},\theta)$
$\tilde{G}r_4(\mathbf{R}^8)$	$\mathbf{Z}_4$	$\perp\circ\sigma_{\mathbf{R}^7}$
$\tilde{G}r_4(\mathbf{R}^8)$	$\mathbf{Z}_4$	$\perp\circ\sigma_{\mathbf{R}^5}$
$\tilde{G}r_4(\mathbf{R}^m)$, $m\geq 7$	$\mathbf{Z}_2$	θ
$\tilde{G}r_4(\mathbf{R}^{2k})$, $k\geq 4$	$\mathbf{Z}_2$	$\theta\circ J$

Consider now the quaternionic structure H of the universal covering $\overline{M}$, i.e. the vector subbundle of $End\, T\overline{M}$ locally generated by compatible almost hypercomplex triples (I_1, I_2, I_3). Recall that the local (I_1, I_2, I_3) and (I'_1, I'_2, I'_3) are related in the intersection $U \cap U'$ of their trivializing open sets by $SO(3)$-valued functions. Then H induces on the finite quotients $M = \overline{M}/\Gamma$ a weaker structure that can be described as follows.

Definition 2.3. Let $\mathbf{P}(TM)$ be the projectified tangent bundle of a $4n$-dimensional manifold M. *A projective* (*almost*) *quaternionic structure* on M is a family of triples of local bundle maps $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$: $\mathbf{P}(TM) \to \mathbf{P}(TM)$, defined over open sets covering M, that are projective maps on each fiber and satisfy the following conditions:

i) $(\mathcal{I}_1)^2 = (\mathcal{I}_2)^2 = (\mathcal{I}_3)^2 = \mathrm{id}$; $\mathcal{I}_\alpha \circ \mathcal{I}_\beta = \mathcal{I}_\beta \circ \mathcal{I}_\alpha = \mathcal{I}_\gamma$ for all the cyclic permutations (α, β, γ) of $(1, 2, 3)$;

ii) in the intesections of the open sets U, U' where they are defined, the local maps $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ and $\mathcal{I}'_1$, $\mathcal{I}'_2$, $\mathcal{I}'_3$ are related by elements of $SO(3)$.

When the maps $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ are globally defined on $\mathbf{P}(TM)$, then the triple $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ is a *projective* (*almost*) *hypercomplex structure*. By looking at the actions of the finite groups Γ described in table 1, it is easy to recognize that a projective quaternionic structure is induced on all the manifolds $M = \overline{M}/\Gamma$ listed in table 1 (the prefix "almost" is here dropped with respect to the 1-integrable quaternionic structure of $\overline{M}$). Thus:

Proposition 2.4. *Any positive locally quaternion Kähler manifold admits a subordinated projective quaternionic structure.*

We want now to recognize that all the actions listed in table 1 can be lifted to both the twistor space $\overline{Z}$ and the simply connected 3-sasakian homogeneous manifold $\overline{P}$ over the quaternion Kähler Wolf spaces $\overline{M}^{4n}$. The fibrations:

$$\overline{P} \to \overline{Z} \to \overline{M}$$

from a 3-sasakian manifold $\overline{P}$ projecting to a positive quaternion Kähler $\overline{M}$ through its twistor space $\overline{Z}$, have been recently recovered to furnish a wide range of examples in both the manifolds' and the orbifolds' contexts [2]. We are here interested only in the situations arising from those

$\overline{M}$ admitting locally quaternion Kähler quotients. These are, up to finite coverings, the three fibrations:

$$S^7 \to \mathbf{C}P^3 \to \mathbf{H}P^1 \,,$$

$$F^1_{1,2}(\mathbf{C}^4) \to F_{1,2}(\mathbf{C}^4) \to Gr_2(\mathbf{C}^4) \,,$$

$$\frac{SO(m)}{SO(m-4)\times Sp(1)} \to \frac{SO(m)}{SO(m-4)\times U(2)} \to \tilde{G}r_4(\mathbf{R}^m) \,,$$

where $F_{1,2}(\mathbf{C}^4)$ and $F^1_{1,2}(\mathbf{C}^4)$ are the flag manifolds $\{\mathcal{L}_1 \subset \mathcal{L}_2 \subset \mathbf{C}^4\}$ and $\{real\ \mathcal{L}^1 \subset \mathcal{L}_1 \subset \mathcal{L}_2 \subset \mathbf{C}^4\}$, respectively twistor space and 3-sasakian homogeneous $SO(3)$-bundle over $Gr_2(\mathbf{C}^4)$. A similar description of the two bundles over $\tilde{G}r_4(\mathbf{R}^m)$ is the following:

$$\begin{aligned} \frac{SO(m)}{SO(m-4)\times U(2)} &= \big\{(\text{oriented 4-planes } \beta \text{ in } \mathbf{R}^m, \\ &\qquad \text{complex structures on } \beta)\big\} \\ \frac{SO(m)}{SO(m-4)\times Sp(1)} &= \big\{(\text{oriented 4-planes } \beta \text{ in } \mathbf{R}^m, \\ &\qquad \text{hypercomplex structures on } \beta)\big\} \,. \end{aligned}$$

PROPOSITION 2.5. *For all the free actions $\Gamma : \overline{M} \to \overline{M}$ listed in table 1, there is a corresponding free action of a finite group of isometries $\Gamma_{\overline{P}} : \overline{P} \to \overline{P}$ on the 3-sasakian manifold $\overline{P}$ inducing both the actions of Γ and of a group $\Gamma_{\overline{Z}} : \overline{Z} \to \overline{Z}$ on the twistor space.*

This can be checked by looking at all the Γ appearing in table 1. It is worth to note that all the lifted actions to the twistor space $\overline{Z}$ and to the 3-sasakian bundle $\overline{P}$ can be expressed by means of multiplication in the octonians' algebra $\mathbf{Ca}$. For example, the action of $\mathbf{Z}_2$ on $\mathbf{H}P^1$ is induced by an action of $\mathbf{Z}_4$ on S^7 given by $(q_1, q_2)\varepsilon = (-\bar{q}_2, \bar{q}_1)$, where ε is the unit octonian realizing $\mathbf{Ca} \cong \mathbf{H} \oplus \mathbf{H}\varepsilon$, and $(q_1, q_2) \in S^7 \subset \mathbf{H}^2$. In this way the first quotiented fibration to be considered is:

$$S^7/\mathbf{Z}_4 \to \mathbf{C}P^3/\mathbf{Z}_2 \to \mathbf{R}P^4 \,.$$

Similarly, one has fibrations:

$$F^1_{1,2}(\mathbf{C}^4)/\mathbf{Z}_2 \to F_{1,2}(\mathbf{C}^4)/\mathbf{Z}_2 \to Gr_2(\mathbf{C}^4)/\mathbf{Z}_2\,,$$

where both the actions of $\mathbf{Z}_2 = (\perp)$ and $\mathbf{Z}_2 = (\perp \circ\sigma_{\mathbf{C^3}})$ on the basis $\overline{M}$ can be lifted to the twistor and 3-sasakian flag manifolds through the right multiplication by ε.

The grassmannian $\tilde{G}r_4(\mathbf{R}^8)$ admits eight different group actions (two appearing also in higher dimensions): by looking at the descriptions of $\frac{SO(8)}{SO(4)\times U(2)}$ and of $\frac{SO(8)}{SO(4)\times Sp(1)}$ given above, all of them are seen to extend to the twistor and the 3-sasakian manifold. Thus again fibrations:

$$\frac{SO(8)}{SO(4)\times Sp(1)}/\Gamma_{\overline{P}} \to \frac{SO(8)}{SO(4)\times U(2)}/\Gamma_{\overline{Z}} \to \tilde{G}r_4(\mathbf{R}^8)/\Gamma$$

are obtained for all the listed $\Gamma \cong \mathbf{Z}_2, \mathbf{Z}_2\times\mathbf{Z}_2, \mathbf{Z}_4$. The remaining cases of θ or $\theta\circ J$ acting on $\tilde{G}r_4(\mathbf{R}^m)$ and $\tilde{G}r_4(\mathbf{R}^{2k})$ are similarly described: θ lifts to $\overline{P}$ as $(\beta; I_1, I_2, I_3) \to (\theta(\beta); -I_1, -I_2, -I_3)$, and $\theta\circ J$ as $(\beta; I_1, I_2, I_3) \to (\theta\circ J(\beta); -JI_1J, -JI_2J, -JI_3J)$.

Observe that the 3-sasakian structure of $\overline{P}$ is not invariant by any of the finite group actions. In fact the induced structure on the quotients P can be described as “projective” 3-sasakian. To formalize the definition, look at the simplest example, namely at the quotient $S^7/\mathbf{Z}_4$. Let $(1, i_1, i_2, i_3, \varepsilon, i_1\varepsilon, i_2\varepsilon, i_3\varepsilon)$ be the standard basis of the octonions $\mathbf{Ca}$, whose non-associativity gives $(i_\alpha x)\varepsilon \neq i_\alpha(x\varepsilon)$, but $(xi_\alpha)\varepsilon = -(x\varepsilon)i_\alpha$, for any $x \in \mathbf{Ca}$. We denote by $I_\alpha, E \in End\ TS^7$ the multiplication on the right of tangent vectors by i_α, ε, respectively. Thus neither the left nor the right natural 3-sasakian structure of S^7 project to the quotient $S^7/\mathbf{Z}_4$, $\mathbf{Z}_4 : x \to x\varepsilon$. This gives:

$$\begin{array}{ccccc} S^7 & \xrightarrow{s^1} & \mathbf{C}P^3 & \xrightarrow{S^2} & \mathbf{H}P^1 \\ \downarrow & & \downarrow & & \downarrow \\ S^7/\mathbf{Z}_4 & \xrightarrow{S^1} & \mathbf{C}P^3/\mathbf{Z}_2 & \longrightarrow & \mathbf{R}P^4 \end{array}$$

where only the line fields spanned by the vectors I_1x, I_2x, I_3x of S^7 project to the quotient $S^7/\mathbf{Z}_4$.

More generally, all the cases described in table 1 yield similar diagrams. Thus:

Corollary 2.6. *Let $M = \overline{M}/\Gamma$ be a positive locally quaternion Kähler manifold. Then there is a diagram:*

$$\begin{array}{ccccc} \overline{P} & \xrightarrow{S^1} & \overline{Z} & \xrightarrow{S^2} & \overline{M} \\ \downarrow & & \downarrow & & \downarrow \\ P & \xrightarrow{S^1} & Z & \xrightarrow{S^2} & M \end{array}$$

where the vertical maps are projections over finite quotients carrying the following structures:

i) $Z = \overline{Z}/\Gamma_{\overline{Z}}$ *is locally Kähler-Einstein with positive scalar curvature;*

ii) $P = \overline{P}/\Gamma_{\overline{P}}$ *is an Einstein manifold with positive scalar curvature, and carries three global mutually orthogonal distinguished tangent lines* k_1, k_2, k_3.

Of course, locally three unit Killing vector fields K_1, K_2, K_3 can be chosen on P defining a local 3-sasakian structure. We call the triple of distinguished lines k_1, k_2, k_3 a *projective 3-sasakian structure* on P, due to their correspondance, via the covariant derivative of the local vector fields K_1, K_2, K_3, with the projective quaternionic structure of the locally quaternion Kähler M (proposition 2.4).

Going back to the example $S^7/\mathbf{Z}_4$, it can be noted that there are still some global 3-sasakian structures on S^7 that project to $S^7/\mathbf{Z}_4$. These are given by the three Killing vector fields $I^*_\alpha x,\ E^*x, E^*(I^*_\alpha x)$, $\alpha = 1, 2, 3$, defined through the multiplication on the left of x looked at as a pair of quaternions. The basis $(1, i_\alpha, \varepsilon, i_\alpha\varepsilon)$ of quaternions over the reals is here understood. The generator of $\mathbf{Z}_4$, previously expressed as the multiplication on the right by ε, preserves in fact all the three Killing vector fields $I^*_\alpha x,\ E^*x,\ E^*(I^*_\alpha x)$, allowing them to induce a global 3-sasakian structure on $S^7/\mathbf{Z}_4$. On the other hand, global regular 3-sasakian structures can be supported only by simply connected manifolds or $\mathbf{R}P^{4n-1}$ [3]. Thus:

Remark 2.7 None of the projective 3-sasakian manifolds $P = \overline{P}/\Gamma_{\overline{P}}$ is allowed to carry an additional global regular 3-sasakian structure.

We compute now the Betti numbers of the projective 3-sasakian $P = \overline{P}/\Gamma_{\overline{P}}$ appearing in the extension 2.5 of table 1. Recall first that the map:

$$\pi^* : H^p(\overline{M}) \longrightarrow H^p(\overline{P}),$$

induced by the projection $\pi : \overline{P} \to \overline{M}$ from a $4n+3$-dimensional regular 3-sasakian $\overline{P}$ over its quaternion Kähler basis $\overline{M}$ is surjective for $p \leq 2n+1$, and $ker\ \pi^*$ is the intersection with the ideal generated by the Kähler 4-form Ω of $\overline{M}$ [5].

Table 2

	generators of H^2	generators of H^4	generators of H^8
$Gr_2(\mathbf{C}^4)$ $\tilde{G}r_4(\mathbf{R}^8)$	ω	ω^2, Ω $\Omega, \Omega', \Omega''$	ω^4, Ω^2 $\Omega^2, \Omega\Omega', \Omega\Omega'', \Omega'\Omega''$
$F^1_{1,2}(\mathbf{C}^4)$ $SO(8)/(SO(4) \times Sp(1))$	$\pi^*\omega$	$\pi^*\omega^2$ $\pi^*\Omega', \pi^*\Omega''$	 $\pi^*(\Omega'\Omega'')$

The Poincaré polynomials $\mathrm{Poin}_{\overline{\mathrm{P}}}(\mathrm{t})$ of most the cases involved in our discussion can be computed by choosing cohomology generators of the base manifold $\overline{M}$ as in table 2.

The notations ω, Ω in table 2 indicate the complex Kähler 2-form and the quaternion Kähler 4-form in $Gr_2(\mathbf{C}^4)$, and $\Omega, \Omega', \Omega''$ denote in $\tilde{G}r_4(\mathbf{R}^8)$ the Kähler 4-forms of three of the four different quaternion Kähler structures (cf. [18], [7]). The relations of their cohomology classes with the Euler and Pontrjagin classes $e, e^\perp, p_1$ of the tautological vector bundle V and its orthogonal complement $V^\perp$ are:

$$\Omega = 2e + p_1\ , \qquad \Omega' = -2e + p_1\ , \qquad \Omega'' = 2e^\perp - p_1$$

(the forth quaternion Kähler 4-form giving $\Omega''' = -2e^\perp - p_1 = -\Omega - \Omega' - \Omega''$). The cohomology generators for the corresponding $\overline{P}$ are then deduced as in table 2.

Thus the Poincaré polynomials of the 3-sasakian manifolds $\overline{P}$ are:

$$\mathrm{Poin}_{F^1_{1,2}(\mathbf{C}^4)}(t) = 1 + t^2 + t^4 + t^7 + t^9 + t^{11}\ ,$$
$$\mathrm{Poin}_{SO(8)/(SO(4)\times Sp(1))}(t) = 1 + 2t^4 + t^8 + t^{11} + 2t^{15} + t^{19}\ .$$

By looking at the description of the groups $\Gamma_{\overline{P}}$ given in the proof of proposition 2.5, the Poincaré polynomial of all the projective 3-sasakian manifolds $P \to Gr_2(\mathbf{C}^4)/\Gamma$ and $P \to \tilde{Gr}_4(\mathbf{R}^8)/\Gamma$ are then deduced:

$$
\begin{aligned}
\mathrm{Poin}_P(t) &= 1 + t^4 + t^7 + t^{11}, && P = \mathbf{Z}_2 \\
& && - \text{quotients of } \ F^1_{1,2}(\mathbf{C}^4), \\
\mathrm{Poin}_P(t) &= 1 + t^4 + t^8 + t^{11} + t^{15} + t^{19}, && P = \mathbf{Z}_2 \\
& && - \text{quotients of } \ \frac{SO(8)}{SO(4) \times Sp(1)}, \\
\mathrm{Poin}_P(t) &= 1 + t^8 + t^{11} + t^{19}, && P = \mathbf{Z}_2 \times \mathbf{Z}_2 \text{ or } \mathbf{Z}_4 \\
& && - \text{quotients of } \ \frac{SO(8)}{SO(4) \times Sp(1)}.
\end{aligned}
$$

Thus, a comparison with the constrains in [5] confirms that none of the projective 3-sasakian P considered so far can admit a regular 3-sasakian structure. The remaining cases are quotients P of $\overline{P} = \frac{SO(m)}{SO(m-4) \times Sp(1)}$, $n \geq 7$, 3-sasakian bundle over $\tilde{Gr}_4(\mathbf{R}^m)$. Here the allowed $\mathbf{Z}_2$ - actions are generated by θ or by $\theta \circ J$. But these actions, restricted to the submanifolds $\tilde{Gr}_4(\mathbf{R}^6)$, reduce to the ones just considered on $Gr_2(\mathbf{C}^4)$.

3 – Locally conformal quaternion Kähler manifolds

Let now (M^{4n}, H, g) be a quaternion hermitian manifold. As in the previous paragraph, $H \subset End\ TM$ denotes the quaternionic structure locally spanned by almost hypercomplex triples. If the quaternion hermitian metric g is locally conformal quaternion Kähler, the 1-integrability of H is assured by the *Weyl connection* D, obtained by glueing together the Levi Civita connections of the local quaternion Kähler metrics, and as such torsion free and preserving H.

An alternative description of these data is to say that $(M, H, [g], D)$ is a *quaternion hermitian Weyl* manifold, i.e. that besides H and the conformal class $[g]$, a torsion free connection D is given such that $DH \subset H$, $Dg = \omega \otimes g$, where the *Lee 1-form* ω is closed. In this respect and on account of the Einstein property of quaternion Kähler metrics, locally conformal quaternion Kähler manifolds can be viewed as examples of *Einstein Weyl manifolds*, cf. [6], [14], [15].

We assume now that g is locally and non globally quaternion Kähler. In the language of quaternion hermitian-Weyl structures, this is equivalent to require that the Lee form ω is not exact. If M is compact, this assumption enables to choose g in its conformal class such that ω is parallel with respect to the Levi Civita connection ∇ of g. This possibility follows from a result of P. Gauduchon [6], and a remarkable consequence of it is that the underlying Einstein Weyl structure is indeed *Ricci-flat Weyl*, i.e. the Ricci tensor of D vanishes. In fact the constancy of the scalar curvature s^D shows, after derivation of the local conformal requirement $s^D = e^{-f_i}[s^\nabla|_{U_i} - c]$, that $\omega|_{U_i} = df_i = dln[s^\nabla|_{U_i} - c]$, i.e. that ω is exact for $s^D \neq 0$, a condition equivalent to globally conformal Kähler.

On the other hand, the existence of a parallel Lee form shows that locally conformal quaternion Kähler manifolds are endowed with some canonical foliations. We are interested here with the 1-dimensional foliation $\mathcal{B}$, generated by the vector field B, the dual of ω, and with the 4-dimensional $\mathcal{D}$, given by the integrable quaternionic span of B.

The Ricci flatness of the Weyl connection D on compact locally conformal quaternion Kähler manifolds suggests that the geometry of such manifolds can be very close to that of their subclass of *locally conformal hyperkähler manifolds*. The following result draws in fact together these two locally conformal situations, in sharp contrast with their non locally conformal counterparts, namely quaternion Kähler and hyperkähler manifolds.

Theorem 3.1. *Let (M^{4n}, H, g) be a compact locally conformal quaternion Kähler manifold that is not quaternion Kähler. Assume that all the leaves of $\mathcal{B}$ and of $\mathcal{D}$ are compact. Then:*

(i) *M admits a finite covering $\overline{M}$ that is locally conformal hyperkähler and that enters into the commutative diagram:*

$$\begin{array}{ccccc}
\overline{M} & \xrightarrow{S^1} & \overline{P} & \xrightarrow{S^3/H} & \overline{N} \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
M & \xrightarrow{S^1} & P & \xrightarrow{S^3/G} & N
\end{array}$$

whose vertical arrows are finite coverings and horizonthal arrows are Riemannian submersions over orbifolds. The middle orbifolds $\overline{P}$ and P are *globally and locally 3-sasakian, respectively* (cf. *definition* 3.2)*, projecting over quaternion Kähler orbifolds with positive scalar curvature* $\overline{N}$*,* N*. The fibers of these latter maps are 3-dimensional spherical space forms* S^3/H*,* S^3/G*, homogeneous or generally inhomogeneous, in the two cases.*

(ii) *Any such* M *admits a global integrable compatible complex structure* $J \in H$*, making it a complex locally conformal Kähler manifold, that projects in 1-dimensional complex tori over the twistor space* Z *of* N*.*

It follows that the metric g is locally conformal locally hyperkähler. The proof is obtained by looking at both the geometry of the fibers in the horizontal arrows and the actions producing the vertical arrows. We give here a sketch of the arguments, referring for more details to [13], [14]. The following definition describes the structure produced by the middle vertical action in the diagram.

DEFINITION 3.2. Let (P, g) be a Riemannian manifold with Levi Civita connection ∇ and let $K \subset TP$ be a rank 3 vector subbundle of its tangent bundle. K is said to define a *locally 3-sasakian structure* on P if:

(i) K is locally spanned by orthonormal Killing vector fields X_1, X_2, X_3, defined over open sets $U \subset P$ so that: $[X_\alpha, X_\beta] = 2X_\gamma$ for all the cyclic permutations (α, β, γ) of $(1, 2, 3)$;

(ii) on the intersections $U \cap U'$ of such open sets $X'_\lambda = \sum_\mu f_{\lambda\mu} X_\mu$, with $(f_{\lambda\mu}) : U \cap U' \to SO(3)$;

(iii) the local tensor fields $F_\alpha = \nabla X_\alpha$, $\alpha = 1, 2, 3$, satisfy

$$(\nabla_Y F_\alpha) Z = \eta_\alpha(Z) Y - g(Y, Z) X_\alpha \,,$$

where $\eta_\alpha = X_\alpha^\natural$.

The globally 3-sasakian manifolds are obtained when K is globally trivialized by such Killing vector fields X_1, X_2, X_3. Since the local sasakian conditions (iii) assures the flatness of K for any locally 3-sasakian P, the pullback $\overline{K} \to \overline{P}$ to the universal covering $\overline{P}$ is a trivial vector bundle. To recognize that its triviality is specified by an induced globally 3-sasakian structure, two extra observations are needed. First, the integrability of K shows that locally 3-sasakian manifolds P are endowed with a canonical foliation $\mathcal{K}$. The computation of the curvature of its leaves identifies them with (generally inhomogeneous) 3-dimensional spherical space forms S^3/G. The classification of the allowed groups $G \subset SO(4)$ shows that all of them admit a global sasakian structure. Thus a global compatible sasakian structure exists on the manifold P. The second fact is that the analiticity of g, assured by the Einstein property of locally 3-sasakian metrics, allows to extend local Killing vector fields to global ones on the simply connected manifold $\overline{P}$. This result goes back to K. Nomizu [12], and his construction of the extended vector field can be pursued as well, in this locally 3-sasakian situation, including the condition (iii). Once two global orthonormal Killing-sasakian vector fields are obtained in these ways, their bracket completes the global 3-sasakian structure of $\overline{P}$. This discussion describes the middle vertical arrow $\overline{P} \to P$ in the diagram.

The two manifolds appearing in the left vertical arrow $\overline{M} \to M$ are flat principal S^1-bundles over $\overline{P}$ and P, respectively. The structure is here locally conformal hyperkähler on $\overline{M}$ and locally conformal quaternion Kähler on M, and relates with those of $\overline{P}$ and P in a way that is very similar to the complex case as described in [19]. Finally, the right vertical arrow is between the quaternion Kähler leaf spaces —generally orbifolds— of the foliation $\mathcal{D}$ mentioned in the statement. The distinction between the two groups H and G, both acting freely on S^3 and thus defining the fibers in the right horizonthal arrows, is that they are finite subgroups respectively of S^3 and of $SO(4)$. Accordingly, homogeneous or generally inhomogeneous fibers are obtained.

The existence of the compatible global $J \in H$ is a consequence of the structure of the (possibly) inhomogeneous fibers S^3/G. A basic result in the classification of 3-dimensional spherical space forms is in fact that the finite group $G \subset SO(4)$ is necessarily conjugate to a subgroup of either $Sp(1) \cdot U(1)$ or $U(1) \cdot Sp(1)$. Thus, as already mentioned, both the locally 3-sasakian manifolds S^3/G and P admit a global compatible sasakian structure, allowing to construct the complex structure J on M having the claimed property.

The next statement contains further similarities between compact locally conformal quaternion Kähler and compact locally conformal hyperkähler manifolds. We look here at the unicity of the induced Weyl structure and at the topology allowing it. Recall that the term *quaternionic structure* includes the existence of a torsion free connection preserving $H \subset End\, TM$.

THEOREM 3.3. *Let (M^{4n}, H) be a compact manifold equipped with a quaternionic structure.*

(i) *For each quaternion hermitian metric h on M there is at most one compatible quaternion hermitian-Weyl structure.*

(ii) *If a locally conformal quaternion Kähler metric g exists on M, making $\mathcal{D}$ a regular foliation, then the Betti numbers of M and $N = M/\mathcal{D}$ satisfy the following relations:*

$$b_{2p}(M) = b_{2p+1}(M) = b_{2p}(N) - b_{2p-4}(N), \quad (0 \leq 2p \leq 2n-2)\,,$$
$$b_{2n}(M) = 0\,.$$

Moreover:

$$\sum_{k=1}^{n-1} k(n-k+1)(n-2k+1) b_{2k}(M) = 0\,,$$

and all these constrains hold in particular when M^{4n} is locally conformal hyperkähler.

The unicity statement (i) reduces in the locally conformal hyperkähler case to the characterization of the Obata connection as the unique torsion free hypercomplex connection, thus coinciding with the Weyl connection associated to the metric. More generally, although the torsion free quaternionic connections have a structure of affine space modelled on the space of 1-forms, at most one of them can preserve a fixed conformal class $[h]$ of hermitian metrics. This can be recognized by the fact that the wedge multiplication by the Kähler 4-form of the metric maps the 1-forms injectively into the 5-forms.

The constrains (ii) on Betti numbers can be deduced from those of compact positive quaternion Kähler manifolds (cf. [1], pp. 417-419, [10]), by writing the Gysin sequences associated to the fibrations of theorem 3.1. The existence of a global compatible complex structure on M plays here a rôle, allowing to project M to the twistor space Z of N, and thus factorizing the projection $M \to N$ into three sphere bundles. In the locally conformal hyperkähler case, the constrains (ii) were observed in [13] and [5].

We say that M is a *locally conformal quaternion Kähler homogeneous* manifold if there exists a Lie group which acts transitively and effectively on the left on M by quaternionic isometries. This condition implies the regularity of all the foliations involved in the structure of M, and in fact the following classification statement can be deduced (cf. [2], [13]).

Theorem 3.4. *Any compact locally conformal quaternion Kähler homogeneous manifold M is finitely covered by the total space $\overline{M} \to \overline{P}$ of a flat principal S^1-bundles over one of the 3-sasakian homogeneous manifolds:*

$$S^{4n-1}, \mathbf{R}P^{4n-1}, SU(m)/S(U(m-2)\times U(1)), SO(k)/(SO(k-4)\times Sp(1)),$$
$$G_2/Sp(1), F_4/Sp(3), E_6/SU(6), E_7/Spin(12), E_8/E_7.$$

More precisely, the allowed $\overline{M}$ are the products of the listed $\overline{P}$ with the circle S^1 and the Möbius band over $\mathbf{R}P^{4n-1}$.

4 – Local quaternion Kähler and 3-sasakian geometry

The above two paragraphs suggest the existence of a unique framework to present the geometries of both positive locally quaternion Kähler manifolds and compact locally conformal quaternion Kähler manifolds. Both these two classes of manifolds appear in fact from constructions on compact 3-sasakian manifolds $\overline{P}$ acted on freely by finite groups Γ of isometries.

Namely, by looking at proposition 2.5, the following approach to positive locally quaternion Kähler manifolds is recognized. Consider on a compact 3-sasakian manifold $\overline{P}$ the canonical 3-dimensional foliation $\mathcal{K}$ generated by its structure Killing vector fields and the fibration $\overline{P} \to \overline{N}$ to the positive quaternion Kähler leaf space $\overline{N} = \overline{P}/\mathcal{K}$. If $\mathcal{K}$ is a regular foliation, $\overline{N}$ is a manifold, and this is certainly the case for all the 3-sasakian homogeneous manifolds. Any of the groups Γ described in the proof of proposition 2.5 and acting on the allowed 3-sasakian $\overline{P}$, interchanges the leaves of $\mathcal{K}$, so inducing a free action on the quaternion Kähler basis $\overline{N}$. The quotient N of these induced actions are thus the positive locally quaternion Kähler manifolds listed in table 1. Since positive locally quaternion Kähler manifolds are locally symmetric (proposition 2.1), the class of 3-sasakian manifolds $\overline{P}$ to be considered here is that of 3-sasakian homogeneous manifolds, classified in [2] (cf. also theorem 3.4 above).

We can also have a finite group Γ acting on the single leaves of the foliation $\mathcal{K}$. In this case, there is no induced action on the leaf space $\overline{N}$, and again the global 3-sasakian structure may be not preserved by the action. Examples of this situation are the spheres S^{4n+3}, acted on freely and diagonally by finite subgroups of $SO(4)$ that are not subgroups of S^3. The corresponding induced structure on $P = \overline{P}/\Gamma$ is now generally locally 3-sasakian.

Next, consider any flat principal S^1-bundle $\pi : M \to P$. If u is a closed 1-form on M defining a flat connection on π, define the metric $g_M = \pi^* g_P + u \otimes u$. Define also a subbundle $H \subset End\ TM$, by the requirement that H is locally generated by the local almost complex structures:

$$I_\alpha Y = -\nabla_Y X_\alpha - \eta_\alpha(Y)B, \quad I_\alpha B = X_\alpha\,.$$

(η_α is here the dual of the structure Killing vector field X_α, Y is any horizontal vector field, and B is the dual vector field of u, $\alpha = 1, 2, 3$). Then these data define a locally conformal quaternion Kähler structure on M.

This discussion can be summarized in the following statement.

Theorem 4.1. *Both the classes of positive locally quaternion Kähler manifolds and of compact locally quaternion Kähler manifolds can be obtained from free actions of finite groups Γ of isometries on compact 3-sasakian manifolds $\overline{P}$. Namely, in terms of the structure foliation $\mathcal{K}$ of $\overline{P}$:*

(i) *Positive locally quaternion Kähler manifolds are leaf spaces of the foliation induced by $\mathcal{K}$ on projective 3-sasakian manifolds $P = \overline{P}/\Gamma$, when $\overline{P}$ is 3-sasakian homogeneous and when the action of Γ is among the leaves of $\mathcal{K}$. Table* 1*, together with proposition* 2.5*, gives in this case the list of the allowed groups acting.*

(ii) *Compact locally quaternion Kähler manifolds are flat principal S^1-bundles over locally 3-sasakian manifolds, obtained again as $P = \overline{P}/\Gamma$, but when the group Γ acts on each leaf of $\mathcal{K}$.*

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