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Moduli and twistor spaces

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RIASSUNTO: *Si studiano la geometria e la topologia di certi spazi di moduli di fibrati stabili, di dimensione pari (arbitraria), su una superficie di Riemann iperellittica, usando descrizioni di spazi twistor. Se ne deducono relazioni con la geometria quaternionale di grassmaniane reali.*

ABSTRACT: *We study the geometry and topology of certain moduli spaces of stable bundles of (arbitrary) even rank on a hyperelliptic Riemann surface, by using a description involving twistor spaces. We show that there are interesting relations to the quaternionic geometry of real grassmannians.*

1 – Introduction

In this note we discuss some aspects of the geometry of certain moduli spaces $\mathcal{M}_{g,n}$ of orthogonal vector bundles of even rank over a hyperelliptic Riemann surface $\sum_g$ of genus g. These spaces may be described as complex submanifolds of partial flag manifolds

$$\mathcal{F}_{g,n} = \frac{SO(2g+2)}{U(g+1-n) \times SO(2n)}$$

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[19], and it is from this description that we derive our knowledge about $\mathcal{M}_{g,n}$. Moreover, these flag manifolds are twistor spaces for the real grassmannians

$$\mathcal{G}_{g,n} = \frac{SO(2g+2)}{SO(2g+2-2n)\times SO(2n)}$$

in the sense of [21]. They project onto the real grassmannians with fibre the hermitian symmetric space $SO(2g+2-2n)/U(g+1-n)$.

In § 4, we restrict ourselves to studying the cases when $n=2$ and $n=g-1$ for $g\geq 2$ (see fig. 1), since with these values the real grassmannians

$$\mathcal{G}_{g,2} = \frac{SO(2g+2)}{SO(2g-2)\times SO(4)} = \mathcal{G}_{g,g-1}$$

are quaternionic Kähler symmetric spaces [29], and we are able to establish links with quaternionic geometry.

The note is organised as follows. In § 2 we define the moduli spaces of orthogonal bundles. In § 3 we quote Ramanan's result on $\mathcal{M}_{g,n}$ (see [19]) and explain some of its implications. In § 4 we describe some of our results on the cohomology of these spaces [9], [8], including relations to quaternionic geometry.

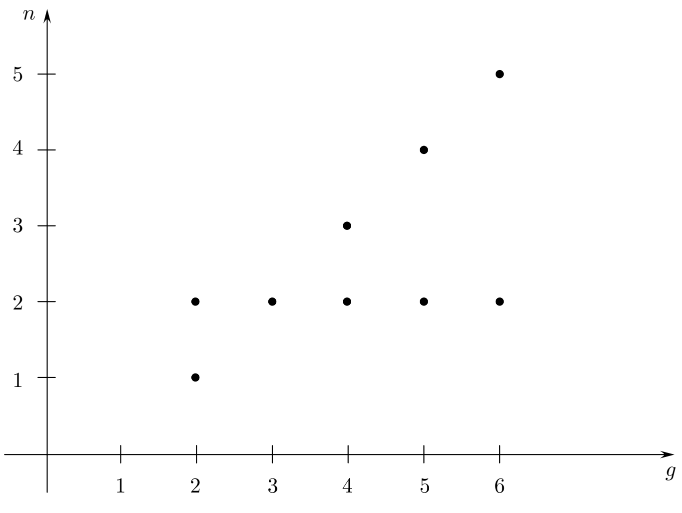

Fig. 1

2 – Moduli spaces of orthogonal vector bundles

Moduli spaces of vector bundles over algebraic varieties and differentiable manifolds have been studied intensely in the last few decades, from the points of view of Algebraic Geometry [15], [18], Differential Geometry and Topology [1], [4], and Theoretical Physics [27].

Take for example, the moduli space $\mathcal{M}(2,1)$ of rank 2 stable holomorphic vector bundles over a Riemann surface $\sum_g$ (of genus g) with fixed and odd determinant. This space has been largely studied and very much is known about its topology [1], [2], [4], [3], [6], [15], [16], [23], [24], [25], [26], [27], [30]. It can also be described in the following ways:

- $\mathcal{M}(2,1)$ is the space of isomorphism classes of flat $SU(2)$ connections on $\sum - \{p\}$ with holonomy -1 around the point p.
- Let $f\colon SU(2)^{2g} \longrightarrow SU(2)$ be given by

$$(a_1, b_1, \dots, a_g, b_g) \mapsto \prod_i a_i b_i a_i^{-1} b_i^{-1}\,.$$

Then $\mathcal{M}(2,1)$ is isomorphic to $f^{-1}(-1)/SO(3)$ where $SO(3)$ acts by conjugation on each entry. This is actually the space of representations of $\pi_1(\sum)$ in $SU(2)$.

Replacing the structure group $SU(2)$ with another algebraic/Lie group G produces other moduli spaces of vector bundles over a Riemann surface whose features depend very much on the group G. For example, when $G = SU(n)$ with $n \geq 2$, the moduli spaces $\mathcal{SU}(n,d)$ of rank n semistable holomorphic vector bundles over $\sum$ with fixed determinant of degree d, have been studied by Jeffrey-Kirwan [10] and by Witten [28].

In this note we take G to be the special Clifford group $SC_m = \mathbb{C}^* \times_{\mathbf{Z}_2} \mathrm{Spin}(m)$, and we also impose some conditions on the vector bundles and on $\sum$. We first recall some Clifford algebra.

2.1 – Clifford algebra

We consider the standard quadratic form Q_0 on $\mathbb{C}^m$, and make the following conventions:

Cl_m denotes the Clifford algebra;
Cl_m^+ denotes the even Clifford algebra;
Cl_m^* denotes the group of units;

$C_m = \{s \in Cl_m^* \mid a(s)\mathbb{C}^m s^{-1} \subset \mathbb{C}^m\}$ is the *Clifford group*, where a is the involution of Cl_m induced by the map $x \mapsto -x$ for $x \in \mathbb{C}^m$;
$SC_m = C_m \cap Cl_m^+$ is the *special Clifford group*.

Since for $s \in SC_m$ the transformation $\pi_s : x \mapsto a(s)xs^{-1}$ is orthogonal ($x \in \mathbb{C}^m$), we get an orthogonal representation of C_m

$$\pi : C_m \longrightarrow O(m)$$

where $\ker(\pi) = \mathbb{C}^*$, $\pi(C_m) = O(m)$ and $\pi(SC_m) = SO(m)$. Furthermore, SC_m is a connected reductive algebraic group. The *spinor norm* is the homomorphism

$$\begin{gathered} \mathrm{Nm} : SC_m \longrightarrow \mathbb{C}^* \\ x_1 \cdots x_r \mapsto Q_0(x_1) \cdots Q_0(x_r) \end{gathered}$$

where $x_j \in \mathbb{C}^m$. Thus, $\mathrm{Spin}(m) = \ker \mathrm{Nm}$. Therefore, multiplication by scalars induces a double cover

$$\{\pm(1,1)\} \longrightarrow \mathbb{C}^* \times \mathrm{Spin}(m) \longrightarrow SC_m\,,$$

ie. $SC_m = \mathbb{C}^* \times_{\mathbf{Z}_2} \mathrm{Spin}(m)$, and we also have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{C}^* & \longrightarrow & SC_m & \longrightarrow & SO(m) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & || & & \\ 1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \mathrm{Spin}(m) & \longrightarrow & SO(m) & \longrightarrow & 1 \end{array}$$

2.2 – Moduli spaces

We shall not be considering the full moduli space of semistable SC_m-bundles over $\sum$ (see [17]); instead, we take a subvariety described as follows.

Let $m = 2n$ and $\sum$ be a hyperelliptic Riemann surface with involution $\imath : \sum \longrightarrow \sum$ and Weierstrass set $\{\omega_1, \ldots, \omega_{2g+2}\}$. Let $E \longrightarrow \sum$ be a vector bundle with structure group SC_{2n}. Via the orthogonal representation $\pi : SC_{2n} \longrightarrow SO(2n)$, E can be considered as an orthogonal vector bundle of rank $2n$. Suppose that it is $\imath$-invariant, ie. that there is a lift of $\imath$ to E such that $E \cong \imath^* E$. Thus, we have the restriction of the lift of $\imath$ to the fibres over the Weierstrass points, $\imath : E_{\omega_j} \longrightarrow E_{\omega_j}$; since $\imath^2 = 1$, the eigenvalues of $\imath$ on these fibres are ± 1, and we denote $E_{\omega_j}^{\pm}$ the corresponding eigenspaces. Let Λ be an $\imath$-invariant line bundle over $\sum$ of degree $2g-1$.

Definition 2.1. Let $\mathcal{M}_{g,n}$ denote the moduli space of rank $2n$, $\imath$-invariant holomorphic semistable orthogonal vector bundles $E \to \sum$ such that $\dim((E \otimes \Lambda)^-_{\omega_j}) = 1$ for all $j = 1, \dots, 2g+2$.

Although the definition of $\mathcal{M}_{g,n}$ may look rather cumbersome, it does generalise two well known moduli spaces of bundles.

Examples.
$n = 1$. Since $SO(2) \cong U(1)$, $\mathcal{M}_{g,1}$ is the Jacobian $J(\sum)$ of $\sum$ (see [19]).
$n = 2$. The special Clifford group is

$$SC_4 = \{(A, B) \in Gl(2) \times Gl(2) \,|\, \det(A) \cdot \det(B) = 1\}$$

and the homomorphism $SC_4 \longrightarrow SO(4)$ is given by $(A, B) \longrightarrow A \otimes B$. Thus a SC_4-bundle is essentially a pair of $Gl(2)$-bundles M, N with $\det(M) \otimes \det(N) = 1$ a trivial bundle. Since C_4 does not distinguish between M and N we have that $\mathcal{M}_{g,2} = \mathcal{M}(2,1)$ is the moduli space of (stable) vector bundles of rank 2 and fixed odd determinant (see [19]).

3 – $\mathcal{M}_{g,n}$ as a submanifold of a twistor space

Let $\sum$ be as in the previous section. Ramanan proved in [19] the following theorem.

Theorem 3.1 [19, theorem 3]. *$\mathcal{M}_{g,n}$ is isomorphic to the variety of $(g+1-n)$-dimensional subspaces of $\mathbb{C}^{2g+2}$ which are isotropic with respect to the two quadratic forms*

$$\sum_{i=1}^{2g+2} y_i^2, \qquad \sum_{i=1}^{2g+2} \omega_i y_i^2 \,. \tag{1}$$

Therefore we have a holomorphic embedding of $\mathcal{M}_{g,n}$ into the complex partial flag manifold

$$\mathcal{F}_{g,n} = \frac{SO(2g+2)}{U(g+1-n)\times SO(2n)}$$

which clearly parametrises the $(g+1-n)$-dimensional subspaces of $\mathbb{C}^{2g+2}$ which are isotropic with respect to the fist quadratic form. $\mathcal{F}_{g,n}$ is a twistor space for $\mathcal{G}_{g,n}$ since the fibre $SO(2g+2-2n)/U(g+1-n)$ parametrises orthogonal almost complex structures on the real oriented $(2g+2-2n)$-dimensional subspaces of $\mathbb{R}^{2g+2}$, and which are compatible with the orientation.

Let Q, W denote the duals of the tautological complex vector bundles over $\mathcal{F}_{g,n}$ with fibres $\mathbb{C}^{g+1-n}, \mathbb{C}^{2n}$ and structure groups $U(g+1-n), SO(2n)$ respectively. The second quadratic form in theorem 3.1 determines a holomorphic non-degenerate section of the second symmetric tensor power S^2Q of Q, whose zero-set is precisely $\mathcal{M}_{g,n}$. In this way we know that $\mathcal{M}_{g,n}$ is a smooth manifold with complex dimension $(2n-1)(g+1-n)$.

The standard representation of $SO(2g+2)$ on $\mathbb{C}^{2g+2}$ splits under $U(g+1-n)\times SO(2n)$ as

$$Q^*\oplus Q\oplus W = 2g+2. \tag{2}$$

This implies that

$$\mathfrak{so}(2g+2)_c \cong (\mathfrak{u}(g+1-n)\oplus\mathfrak{so}(2n))_c \oplus (\wedge^2 Q\oplus Q\otimes W)\oplus\overline{(\wedge^2 Q\oplus Q\otimes W)}\,,$$

where $\wedge^2 Q\oplus Q\otimes W$ corresponds to the holomorphic tangent bundle $T^{1,0}\mathcal{F}_{g,n}$ of $\mathcal{F}_{g,n}$. Here $\wedge^2 Q$ is the holomorphic tangent bundle to the hermitian fibres $SO(2g+2-2n)/U(g+1-n)$ of $\mathcal{F}_{g,n}\longrightarrow\mathcal{G}_{g,n}$ and its complement $Q\otimes W$ is a holomorphic horizontal bundle.

On the other hand we have that

$$T^{1,0}\mathcal{F}_{g,n}|_{\mathcal{M}_{g,n}} = T^{1,0}\mathcal{M}_{g,n}\oplus S^2Q|_{\mathcal{M}_{g,n}}\,,$$

so that

$$T = T^{1,0}\mathcal{M}_{g,n} = \wedge^2 Q\oplus Q\otimes W - S^2 Q\,,$$

and we get the following

Proposition 3.1.

$$T = Q \otimes W - \psi^2 Q\,,$$

where $\psi^2 = S^2 - \wedge^2$ in K-theory. ∎

The operator ψ^2 is one of the series of Adams operators, defined by the formula

$$\sum_{p\geq 0}(\psi^p E)t^p = r - t\frac{d}{dt}\log \Lambda_{-t}E,$$

where $E \in K(\mathcal{M})$ has virtual rank r and $\Lambda_t E = \sum_{i\geq 0}(\wedge^i E)t^i$ [5]. Each ψ^p is a ring homomorphism in K-theory, and is characterised by the property that

$$\mathrm{ch}_k(\psi^p E) = p^k \mathrm{ch}_k(E)\,, \tag{3}$$

where $\mathrm{ch}_k(E)$ denotes the term of dimension $2k$ in the Chern character.

4 – Some intersection numbers and cohomology

From now on, we shall restrict ourselves to the cases $n = 2$ and $n = g-1$ for $g \geq 2$, in which the real grassmannians $\mathcal{G}_g = \mathcal{G}_{g,2} = \mathcal{G}_{g,g-1}$ are quaternionic Kähler symmetric spaces [29], and we will recall some of their quaternionic features.

These real grassmannians parametrise real oriented 4-dimensional subspaces of $\mathbb{R}^{2g+2}$. Consider g to be fixed. Let V be the rank 4 tautological vector bundle over $\mathcal{G}_g$ whose complexification pulls back to W on $\mathcal{F}_{g,2}$ and to $Q \oplus Q^*$ on $\mathcal{F}_{g,g-1}$. Let $V^\perp$ be its orthogonal complement with respect to the standard equivariant metric. Thus

$$T\mathcal{G}_g = V \otimes V^\perp.$$

Lifting the $SO(4)$ structure of V to $\mathrm{Spin}(4) \cong SU(2)\times SU(2) \cong Sp(1)\times Sp(1)$ on a suitable open dense subset $\mathcal{G}_g{}' \subset \mathcal{G}_g$ implies that

$$V_c = U_1 \otimes U_2$$

where U_1, U_2 are rank 2 complex vector bundles over $\mathcal{G}_g{}'$ corresponding to each one of the $SU(2)$'s (the subscript $_c$ denotes complexification). Therefore

$$(T\mathcal{G}_g)_c \cong U_1 \otimes (U_2 \otimes V_c^{\perp})$$

which shows that $\mathcal{G}_g$ is a quaternionic Kähler manifold [29], [20] since $Sp(1)SO(2g-2) \subset Sp(2g-2)$ and the holonomy is then contained in $Sp(2g-2)Sp(1)$. Also, given that U_1 can be thought of as a locally defined quaternionic line bundle, the quaternionic structure is characterised by a 4-dimensional cohomology class $u = -c_2(U_1)$, called a *quaternionic class*. Analogously, we have for U_2, a 4-class $v = -c_2(U_2)$, which by symmetry gives rise to another quaternionic structure. The bundles S^2U_1, S^2U_2 are globally defined, thus the classes $4u$, $4v \in H^4(\mathcal{G}_g, \mathbb{Z})$. Finally, note that the form $4u$ (resp. $4v$) is non-degenerate.

Now we will study the two cases $n = 2$ and $n = g-1$.

4.1 – Case $n = 2$. In this case

$$\mathcal{F}_{g,2} = \frac{SO(2g+2)}{U(g-1) \times SO(4)} \longrightarrow \mathcal{G}_{g,2} = \frac{SO(2g+2)}{SO(2g-2) \times SO(4)}$$

with fibre $SO(2g-2)/U(g-1)$, Q has rank $g-1$ and W has rank 4. Let $L = \det(Q)$ be the ample line bundle on $\mathcal{F}_{g,2}$ which pulls back to the ample generator of $\mathrm{Pic}(\mathcal{M}_{g,2})$.

Universal cohomology classes

$$\alpha \in H^2(\mathcal{M}_{g,2}, \mathbb{Z}), \quad \beta \in H^4(\mathcal{M}_{g,2}, \mathbb{Z}), \quad \gamma \in H^6(\mathcal{M}_{g,2}, \mathbb{Z}) \tag{4}$$

were introduced by Newstead [15], [1]. They are obtained from the Künneth components of the characteristic class $c_2(\mathbb{V})$, where $\mathbb{V}$ is a universal $SO(3)$ bundle over $\mathcal{M}_{g,2}$, and generate the ring $H_I^*(\mathcal{M}_{g,2})$ of cohomology classes of $\mathcal{M}_{g,2}$ invariant by the action of the mapping class group on $H^3(\mathcal{M}_{g,2})$. From our point of view these classes are characterised as follows:

$\alpha = c_1(L)$;

$\beta = p_1(W)$ (see below);

γ is Poincaré dual to $2g$ copies of $\mathcal{M}_{g-1,2}$ in $\mathcal{M}_{g,2}$ [25].

By expressing $T = T^{1,0}\mathcal{M}_{g,2}$ in terms of a push-forward of $\mathbb{V}$, Newstead obtained the following result, which we take as given and is effectively the definition of (4) for our purposes:

THEOREM 4.1 [16, THEOREM 2].

$$\operatorname{ch}(T)=3g-3+2\alpha+\sum_{k\geq 2}\frac{\operatorname{ch}_k}{k!},\ \text{where}\begin{cases}\operatorname{ch}_{2k-1}=2\alpha\beta^{k-1}-8(k-1)\gamma\beta^{k-2},\\ \operatorname{ch}_{2k}\quad=2(g-1)\beta^k.\end{cases}\qquad\square$$

Applying lemma 3.1 and (2), we see that the complexification of the real tangent bundle of $\mathcal{M}_{g,n}$ is

$$\begin{aligned}T+T^* &= (Q^*+Q)W-\psi^2(Q^*+Q),\\ &= (2g+2-W)W-(2g+2-\psi^2W)\\ &= (2g+2)(W-1)-W^2+\psi^2W.\end{aligned}\tag{5}$$

which implies for $n=2$

$$p_1(W)=\beta,\qquad p_2(W)=0,$$

$$\operatorname{ch}(W)=2+e^{\sqrt{\beta}}+e^{-\sqrt{\beta}}\tag{6}$$

on $\mathcal{M}_{g,2}$.

THEOREM 4.2 [16,Conjecture (a)].

$$\beta^g=0.$$

PROOF. Given that $Q^*+Q=2g+2-W$ is a genuine complex vector bundle of rank $2g-2$, the top dimensional component of its Chern class is of dimension $4g-4$ in

$$c(Q\oplus Q^*)=c(W)^{-1}=\sum_{k=0}^{\infty}\beta^k\qquad\square$$

This was first proved in [13] and later in [11], [27], [26]

The fact that $p_1(\mathcal{M}_{g,2}) = 2(g-1)\beta$ generates the Pontrjagin ring of $\mathcal{M}_{g,2}$ can be linked to the geometry of the real grassmannian $\mathcal{G}_{g,2}$. For W and $Q+Q^*$ are (complexifications of) the pullbacks of the real vector bundles V, $V^\perp$ over $\mathcal{G}_g = \mathcal{G}_{g,2}$, and the real tangent bundle of $\mathcal{G}_g$ is isomorphic to $V\otimes V^\perp$. The choice of an orientation of V gives the manifold $\mathcal{G}_g$ a quaternion-Kähler structure with quaternionic form $\Omega = 4u \in H^4(\mathcal{G}_g, \mathbb{Z})$ arising from the curvature of the locally-defined quaternionic line bundle U_1, ie. $u = -c_2(U_1)$ [14].

Proposition 4.1. *For $g \geq 3$, β is the pull-back of the class $4u$ by means of the mapping $\mathcal{M}_{g,2} \hookrightarrow \mathcal{F}_{g,2} \to \mathcal{G}_{g,2}$.*

Proof. β is the pull-back to $\mathcal{M}_{g,2}$ of $\hat{\beta} = p_1(V)$. A calculation from [22] shows that

$$p_2(V) = (\hat{\beta} - 4u)^2 \in H^8(\mathcal{G}_{g,2}, \mathbb{Z})\,.$$

Assuming that $g \geq 3$, $b_4(\mathcal{M}_{g,2}) = 2$ and so $\hat{\beta} - 4u$ must pull back to $a\alpha^2 + b\beta$ on $\mathcal{M}_{g,2}$ for some $a, b \in \mathbb{Z}$; from (6), $(a\alpha^2 + b\beta)^2 = 0$. Siebert and Tian [24] provided a minimal set of relations on the subring of $H^*(\mathcal{M}_{g,2})$ generated by α, β, γ (see corollary 4.1 and comments after it). This implies that there are no non-trivial relations involving α^4, $\alpha^2\beta$, β^2 in $H^8(\mathcal{M}_{g,2})$ except that

$$0 = -8\,(c_4(Q) - \alpha\, c_3(Q)) = \alpha^4 + 2\alpha^2\beta - 3\beta^2$$

in genus 3. (There are actually four distinct quaternion-Kähler structures on $\mathcal{G}_{3,2} = SO(8)/(SO(4)\times SO(4))$, and proposition 4.1 holds for only two of them. See remarks after proposition 4.3). It follows that in all cases $\beta = 4u$ in $H^4(\mathcal{M}_{g,2})$. ∎

On the other hand, the quaternionic volume of $\mathcal{G}_g$ is

$$v(\mathcal{G}_g) = \left\langle (4u)^{2g-2}, [\mathcal{G}_g] \right\rangle = \frac{2}{g}\binom{4g-3}{2g-1}\,,$$

which shows the contrast to the non-degenerate nature of the 4-form $\Omega = 4u$ over $\mathcal{G}_g$, and reflects the failure of $\mathcal{M}_{g,2}$ to map onto a quaternionic subvariety of $\mathcal{G}_g$. Analogous results hold also for $4v$.

On the other hand, from lemma 3.1 and theorem 4.1 one may readily compute the Chern character of Q in terms of the classes (4).

THEOREM 4.3.

$$\mathrm{ch}\,(Q) = g - 1 + \alpha + \sum_{k\geq 2} \frac{s_k}{k!}, \quad \text{where} \quad \begin{cases} s_{2k-1} = \alpha\beta^{k-1} + 2\gamma\beta^{k-2}, \\ s_{2k} \quad = -\beta^k. \end{cases}$$

PROOF. Let ch_k, s_k denote the components of $\mathrm{ch}\,(T)$, $\mathrm{ch}\,(Q)$, respectively, in dimension $2k$. Using lemma 3.1, (3) and (6),

$$3g - 3 + \sum_{k\geq 1} \frac{\mathrm{ch}_k}{k!} = 2\Big(g - 1 + \sum_{k\geq 1} \frac{s_k}{k!}\Big)\Big(2 + \sum_{k\geq 1} \frac{\beta^k}{(2k)!}\Big) - \Big(g - 1 + \sum_{k\geq 1} \frac{2^k s_k}{k!}\Big).$$

The result now follows from theorem 4.1 by induction on k. ∎

An analogue of the last equation can be found in [2], though the authors were led to it by the paper of SIEBERT and TIAN [24], who give an equivalent expression for $\mathrm{ch}\,(Q)$. Using a standard trick [30], theorem 4.3 leads to their recurrence relation for the Chern classes of Q.

$$\begin{aligned} c(t) &= \exp\Big[\sum_{k\geq 1} \frac{(-1)^{k-1} s_k t^k}{k}\Big] = \\ &= \exp\Big[\alpha t + \sum_{n\geq 2} (\alpha\beta^{n-1} + 2\gamma\beta^{n-2}) \frac{t^{2n-1}}{2n-1} + \sum_{n\geq 1} \beta^n \frac{t^{2n}}{2n}\Big]. \end{aligned} \tag{7}$$

Thus the relation [24, proposition 25], namely

$$(1 - \beta t^2) c'(t) = (\alpha + \beta t + 2\gamma t^2) c(t).$$

Whence

COROLLARY 4.1. *The Chern classes of the rank $g-1$ bundle Q on $\mathcal{M}_g$ satisfy*

$$(k+1) c_{k+1} = \alpha c_k + k\beta c_{k-1} + 2\gamma c_{k-2}\,.$$

The identities in α, β, γ arising from the equations $c_k = 0$ for $k = g, g{+}1, g{+}2$ provide a minimal set of relations which completely determine the cohomology ring $H_I^*(\mathcal{M}_{g,2})$ [30], [2], [12], [24]. It is worth pointing out that corollary 4.1 is analogous, but simpler, to the recurrence relation for the Chern classes of T given at the end of [16].

4.2 – Case $n = g-1$. In this case

$$q : \mathcal{F}_{g,2} = \frac{SO(2g+2)}{U(2) \times SO(2g-2)} \longrightarrow \mathcal{G}_{g,2} = \frac{SO(2g+2)}{SO(4) \times SO(2g-2)}$$

with fibre $SO(4)/U(2) \cong \mathbb{CP}^1$, Q has rank 2 and W has rank $2g-2$. Let $L = \det(Q)$ be the ample line bundle on $\mathcal{F}_{g,2}$ which pulls back to an ample element of $\mathrm{Pic}(\mathcal{M}_{g,g-1})$.

This time the complex $(4g{-}3)$-dimensional homogeneous space $\mathcal{F}_{g,g-1}$ is the usual twistor space of $\mathcal{G}_{g,g-1}$ fibring by rational curves. From standard facts about twistor spaces [20], [14] one knows that $\mathrm{Pic}(\mathcal{F}_{g,g-1})$ is generated by a holomorphic line bundle L on $\mathcal{F}_{g,g-1}$ such that

i. the restriction of L to each fibre $\mathbb{CP}^1$ equals $\mathcal{O}(2)$,
ii. L^{2g-1} is isomorphic to the anticanonical bundle κ^{-1} of $\mathcal{F}_{g,g-1}$.

In our case, L also corresponds to $\det Q$, and it admits a square root over ${\mathcal{G}_g}' \subset \mathcal{G}_g$ on which U_1 and U_2 are defined; there is a C^∞-isomorphism

$$q^*U_1 \cong L^{1/2} \oplus L^{-1/2}\,.$$

Let $l = c_1(L)$ in $H^2(\mathcal{F}_{g,g-1}, \mathbb{Z})$. From the Leray-Hirsch theorem, there is an identity $\left(\frac{l}{2}\right)^2 + q^*c_2(U_1) = 0$ of real cohomology classes. In terms of integral classes and omitting q^*

$$l^2 = 4u.$$

The bundle Q is actually

$$Q \cong L^{-1/2} \otimes q^*U_2$$

where the right hand side is well defined on $\mathcal{F}_{g,g-1}$, even though the individual factors only make sense locally.

We can compute the Hilbert polynomial $\dim H^0(\mathcal{M}_{g,g-1}, \mathcal{O}(L^k))$ (which is the *Verlinde formula* of the moduli space) by using a Koszul complex (see [22], [8]). Let (k) denote the operation of tensoring with L^k. Since $\mathcal{M}_{g,g-1}$ is the zero set of a section of the bundle $\sigma^* = S^2Q = S^2V(1)$, we have

$$\begin{aligned}0 \longrightarrow \mathcal{O}_{\mathcal{F}}(\wedge^3 Q^*(k)) \longrightarrow \mathcal{O}_{\mathcal{F}}(\wedge^2 Q^*(k)) \longrightarrow \mathcal{O}_{\mathcal{F}}(Q^*(k)) \longrightarrow \\ \longrightarrow \mathcal{O}_{\mathcal{F}}(k) \longrightarrow \mathcal{O}_{\mathcal{M}}(k) \longrightarrow 0\,.\end{aligned}$$

which gives

$$h^0(\mathcal{M}_{g,g-1}, \mathcal{O}(L^k)) = a_k - b_{k-1} + b_{k-2} - a_{k-3}$$

where

$$a_k = \chi(\mathcal{F}_{g,g-1}, \mathcal{O}(k)), b_k = \chi(\mathcal{F}_{g,g-1}, \mathcal{O}(S^2V(k))).$$

Using the Borel-Weil-Bott theorem we can compute these numbers [8].

Proposition 4.2.

$$a_k = \frac{(2g+2k-1)(g+k)(g+k-1)}{g(g-1)(2g-1)(2g-2)} \binom{2g+k-2}{2g-3} \binom{2g+k-3}{2g-3}$$

$$b_k = \frac{3(2g+2k-1)(g+k+1)(g+k-2)}{g(g-1)(2g-1)(2g-2)} \binom{2g+k-1}{2g-3} \binom{2g+k-4}{2g-3}.$$

Since the coefficient of the top power of k in the Verlinde formula must be $\frac{l^{4g-6}}{(4g-6)!}$, we have the following table.

Table 1.

g	2	3	4	5	6	7	8
l^{4g-6}	32	224	2112	22880	268736	3328192	42791040

Recall that $l^2 = 4u$, so that this time we get

Proposition 4.3. *The quaternionic form $\Omega = 4u$ when pulled back to $\mathcal{M}_{g,g-1}$ via $\mathcal{M}_{g,g-1} \hookrightarrow \mathcal{F}_{g,g-1} \longrightarrow \mathcal{G}_{g,g-1}$ is non-degenerate.*

PROOF. This follows from the fact that l corresponds to the first Chern class of a positive line bundle on $\mathcal{M}_{g,g-1}$. ∎

REMARK 1. This actually corresponds to the fact that $q^*V_c = Q \oplus Q^*$, and it is in contrast to proposition 4.1. This is because the quaternionic structure in the case $n = 2$ is determined by the factor of the isotropy group of $\mathcal{G}_g$ which is fixed in the twistor fibration. On the other hand, in the case $n = g-1$ the quaternionic structure is determined by the factor of the isotropy group for which the twistor fibration is being considered.

REMARK 2. When $g = 3$ the two lines in fig. 1 meet, so that both propositions 4.1 and 4.3 actually hold. This is no inconsistency since as was said before, the quaternionic structures being considered, although denoted by the same symbols, are different because they come from the two different $SO(4)$ factors of the isotropy group.

Several interesting questions are still open, such as for example defining universal classes for $\mathcal{M}_{g,n}$ and determine their intersection pairings. For $\mathcal{M}_{g,2}$ this has been done by several authors [4], [25], [27], [9]. Also, it would be interesting to know more about the image of $\mathcal{M}_{g,n}$ in $\mathcal{G}_{g,n}$.

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