Rendiconti di Matematica, Serie VII
Volume 18, Roma (1998), 103-129

On the group of Poisson diffeomorphisms of the torus

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RIASSUNTO: *Si considera il gruppo dei diffeomorfismi hamiltoniani del toro con la struttura standard di Poisson. Si mostra che questo gruppo è perfetto e che lo sono anche i suoi ricoprimenti universali. Si estende questo risultato ad alcuni sottogruppi del gruppo dei diffeomorfismi hamiltoniani di* $\mathbb{R}^n$*. Il risultato generalizza alcuni noti teoremi di Banyaga sugli omomorfismi simplettici.*

ABSTRACT: *We consider the group of all Hamiltonian diffeomorphisms of the torus with the standard Poisson structure. We show that this group as well as its universal covering are perfect. Next we extend this result to some subgroup of the group of Hamiltonian diffeomorphisms of* $\mathbb{R}^n$ *with the standard Poisson structure. The results and their proofs generalize well known theorems of Banyaga for symplectomorphisms.*

1 – Introduction

Let (M, Ω) be a symplectic manifold. A diffeomorphism ϕ of M onto M is called a symplectomorphism if $\phi^*(\Omega) = \Omega$. By $G(M, \Omega)_0$ we denote the group of all symplectomorphisms which are isotopic to the identity through compactly supported symplectomorphisms. Next $G^*(M, \Omega)$ stands for the subgroup of all Hamiltonian diffeomorphisms.

For any topological group G the symbol $\widetilde{G}$ denotes the universal

Supported in part by KBN grant 2 P03A 024 10
KEY WORDS AND PHRASES: *Torus – Poisson manifold – Hamiltonian diffeomorphism – Perfectness of group – Calabi homomorphism.*
A.M.S. CLASSIFICATION: 57S05 – 58F05

covering group of G. Recall that the group is perfect if its abelianization $H_1(G) = G/[G,G]$ is trivial.

The starting point is the following result being a corollary of a well known paper [1] by A. Banyaga.

Theorem 1.1. *For any compact symplectic manifold* (M,Ω) *the groups* $G^*(M,\Omega)$ *and* $\widetilde{G^*(M,\Omega)}$ *are simple.*

Indeed, this is a consequence of a main result of [1] and some properties of the flux homomorphism (see Section 3). Notice that in view of [4] showing the simplicity amounts to showing the perfectness. Notice as well that if M is noncompact the group $G^*(M,\Omega)$ is neither simple nor perfect (see Theorem 3.3).

The first aim of this paper is to give a generalization of Theorem 1.1 to the torus T^n with the standard Poisson structure Λ_{2k}, $2k < n$.

Theorem 1.2. *Let* $G^*(T^n,\Lambda_{2k})$ *be the group of all Hamiltonian diffeomorphisms of* (T^n,Λ_{2k}). *Then* $G^*(T^n,\Lambda_{2k})$ *is a perfect group, and the same is true for its universal covering group.*

For the proof, see Section 6. Observe that in our case the group in question cannot be simple for obvious reasons.

By using the homology theory as in W. Thurston's paper [18] one can have the following result for the standard Poisson structure Λ_{2k} on $\mathbb{R}^n$.

Theorem 1.3. *Let* $G^{**}(\mathbb{R}^n,\Lambda_{2k})$ *be the group of all special Hamiltonian diffeomorphisms of* $(\mathbb{R}^n,\Lambda_{2k})$ (*for the definition of* G^{**} *see Section* 3 *and* 8). *Then* $G^{**}(\mathbb{R}^n,\Lambda_{2k})$ *and* $\widetilde{G^{**}(\mathbb{R}^n,\Lambda_{2k})}$ *are perfect.*

The proof will occupy Section 7. Certain consequences of this theorem are pointed out in the last section.

In the proofs of the above results on one hand we generalize a method of [7[, [18], [1] to the non-transitive case, and on the other we develop modifications of this method in [14]. The extension of [1] is possible as we restrict ourselves to Hamiltonian diffeomorphisms. It seems likely that some analogues of results of [1] still hold for the identity component of the group of all automorphisms of (M,Λ) but the proof would be essentially more complicated. Another difficulty is that the notion of the second Calabi homomorphism for Poisson manifolds is not well understood, or even not relevant (cf. [15]).

Other important candidates for the perfectness theorem are specified in [16], where the problem of n-transitivity along leaves is studied. Let us remark that only little is known about the automorphism groups of non-transitive geometric structures (e.g. Poisson) in spite of their numerous applications in Mechanics.

In the sequel all manifolds, tensors and diffeomorphisms are assumed to be of class C^∞. The main reason is that Implicit Function theorem in Section 5 is no longer true in the space of C^r-mappings.

2 – Diffeomorphism groups of a Poisson manifold

Let M be a second countable C^∞-smooth manifold. A Poisson structure can be introduced by a skew-symmetric $(2,0)$-tensor Λ on M such that $[\Lambda,\Lambda]=0$, where $[.,.]$ is the Schouten-Nijenhuis bracket (cf. [19]). Then the rank of Λ_p may vary but it is even everywhere. The ring of the real smooth functions on M, $C^\infty(M)$, can be given a Lie algebra structure by means of the bracket

$$\{f,g\} := \Lambda(df,dg) \quad \text{for any } f,g\in C^\infty(M),$$

and every adjoint homomorphism of this bracket is a derivation of $C^\infty(M)$. We have the "musical" bundle homomorphism associated with Λ

$$\sharp : T^*M \to TM, \quad \beta(\alpha^\sharp) = \Lambda(\alpha,\beta),$$

where $\alpha^\sharp = \sharp(\alpha)$, for any $\alpha,\beta\in T^*M$. In case Λ is nondegenerate (i.e. $\mathrm{rank}(\Lambda)=\dim(M)$), $\sharp$ is an isomorphism and we get a symplectic structure. We then denote by $\flat : TM\to T^*M$ the inverse isomorphism of $\sharp$.

In general, the distribution $\sharp(T^*_pM)$, $p\in M$, integrates to a generalized foliation such that Λ restricted to any leaf induces a symplectic structure. This foliation is called symplectic and will be denoted by $\mathcal{F}(\Lambda)$. The symplectic form living on $L\in\mathcal{F}(\Lambda)$ will be denoted by Ω_L.

If the dimension of leaves is constant, i.e. if $\mathcal{F}(\Lambda)$ is a regular foliation, the Poisson structure Λ is called regular. Since we shall appeal to the diffeomorphism group on the torus T^n with the standard Poisson structure Λ_{2k}, we shall be concerned with regular Poisson structures exclusively.

Suppose that (M,Λ) is a regular Poisson manifold such that $\dim(M) = n$, $\dim(\mathcal{F}(\Lambda)) = 2k$, $h = n-2k$. By extending the Darboux theorem for a regular Poisson manifold (cf.[10, 19]) one has the existence of canonical coordinates , namely for any $p \in M$ one has a local coordinate system at p $(x_i, y_j),\ i = 1,\dots,2k,\ j = 1,\dots h$, such that the following relations hold

$$\{x_i, x_{i+k}\} = 1, \quad \{x_i, x_j\} = 0 \quad \text{if} \quad |i-j| \neq k,$$
$$\{x_i, y_j\} = 0, \quad \{y_i, y_j\} = 0.$$

Let $G(M) \subset Diff^\infty(M)$ be any diffeomorphism group. By a smooth path in $G(M)$ we mean any family $\{f_t\}_{t\in\mathbb{R}}$ with $f_t \in G(M)$ such that the map $(t,x) \mapsto f_t(x)$ is smooth. Next, $G(M)_0$ denotes the subgroup of all $f \in G(M)$ such that there is a smooth path $\{f_t\}_{t\in\mathbb{R}}$ with $f_t = \mathrm{id}$ for $t \leq 0$ and $f_t = f$ for $t \geq 1$, and such that each f_t stabilizes outside a fixed compact set. Notice that $G(M)_0$ is the connected component of id if $G(M)$ is locally contractible and M is compact. Notice as well that the group of all automorphisms of a regular Poisson manifold is locally contractible (see [20, p. 339-40], the argument used there extends to Poisson manifolds, cf. [6], [15]).

To any smooth path f_t in $G(M)_0$ we attach a family of vector fields

$$\dot{f}_t = \frac{df_t}{dt}(f_t^{-1}).$$

Then the time-dependent family $\dot{f}_t$ is a unique smooth path in the Lie algebra corresponding to $G(M)_0$ which satisfies the equality

$$\frac{df_t}{dt} = X_t \circ f_t \quad \text{with} \quad f_0 = \mathrm{id}. \tag{1}$$

Conversely, given a smooth family X_t of compactly supported vector fields there exists a unique solution f_t of (1) (see e.g. [9], where such a one-to-one correspondence for infinite dimensional Lie groups is called regularity).

In particular, f_t is a flow if and only if the corresponding $X_t = X$ is time-independent, namely it is a unique compactly supported vector X generating this flow. It is well known that only few diffeomorphisms are elements of some flow (cf. [5], [13]).

A smooth mapping f of (M, Λ) into itself is called a Poisson morphism if

$$\{u \circ f, v \circ f\} = \{u, v\} \circ f \quad \text{for any } u, v \in C^\infty(M).$$

Let $G(M, \Lambda)$ stand for the group of all Poisson automorphisms of (M, Λ) which are tangent to the leaves of $\mathcal{F}(\Lambda)$.

Let us denote by $\mathcal{A} \subset C^\infty(M)$ the subspace of all functions u such that $[\Lambda, u] = 0$. Recall that a vector field X is an infinitesimal automorphism of (M, Λ) if $[\Lambda, X] = 0$, that is $\mathcal{L}_X \Lambda = 0$, where $\mathcal{L}$ is the Lie derivative. By $L(M, \Lambda)$ we denote the Lie algebra of all infinitesimal automorphisms with compact support which are tangent to $\mathcal{F}(\Lambda)$.

Next, let $L^*(M, \Lambda)$ be the subspace of $L(M, \Lambda)$ of all Hamiltonian vector fields, i.e. $X \in L^*(M, \Lambda)$ iff there exists compactly supported $u \in C^\infty(M)$ such that

$$X = [\Lambda, u] \quad \text{or, equivalently,} \quad X = (du)^\sharp.$$

Both $L(M, \Lambda)$ and $L^*(M, \Lambda)$ are $\mathcal{A}$-modules.

For $Y \in L(M, \Lambda)$, $X \in L^*(M, \Lambda)$ we get $[Y, X] = \mathcal{L}_Y[\Lambda, u] = [\Lambda, \mathcal{L}_Y u]$, so that $L^*(M, \Lambda)$ is an ideal of $L(M, \Lambda)$. Moreover we have

$$[L(M, \Lambda), L(M, \Lambda)] \subset L^*(M, \Lambda)$$

as a consequence of the equality $[X_1, X_2] = [\Lambda, u]$, where u is defined by $u(p) = \iota(X_1(p) \wedge X_2(p))\Omega_{L_p}$, L_p being the leaf passing through p (cf. [10]).

Proposition 2.1. *Suppose that f_t, X_t are related by the equation* (1). *Then $f_t \in G(M, \Lambda)_0$ for each t if and only if $X_t \in L(M, \Lambda)$ for each t.*

Proof. When restricting f_t to a leaf L we have

$$\frac{d}{dt}f_t^*\Omega_L = f_t^*(\iota(X_t)d\Omega_L + d(\iota(X_t)\Omega_L)) = f_t^*d(\iota(X_t)\Omega_L).$$

It follows that the claim is true on any leaf, and consequently so is on the whole M (cf. [19]). ∎

Definition 2.2. A smooth path satisfying Proposition 2.1 is called a *Poisson isotopy.* A Poisson isotopy f_t is said to be *Hamiltonian* if the corresponding $X_t \in L^*(M,\Lambda)$ for each t. A diffeomorphism f of (M,Λ) is called *Hamiltonian* if there exists a Hamiltonian isotopy f_t such that $f_0 = \mathrm{id}$ and $f_1 = f$. The totality of all Hamiltonian diffeomorphisms is denoted by $G^*(M,\Lambda)$. Clearly $G^*(M,\Lambda)_0 = G^*(M,\Lambda)$.

Proposition 2.3. *$G^*(M,\Lambda)$ is a normal subgroup of $G(M,\Lambda)$.*

Proof. First we check that $G^*(M,\Lambda)$ is a group. Let f_t, g_t be Hamiltonian isotopies, that is $\dot{f}_t = (du_t)^\sharp$, $\dot{g}_t = (dv_t)^\sharp$ for some smooth families of C^∞-functions u_t and v_t. Then $f_t \circ g_t$ is still a Hamiltonian isotopy as

$$\overset{\cdot}{\overbrace{(f_t \circ g_t)}} = (d(u_t + v_t \circ f_t^{-1}))^\sharp,$$

and f_t^{-1} is Hamiltonian since

$$\overset{\cdot}{\overbrace{f_t^{-1}}} = (d(-u_t \circ f_t))^\sharp.$$

It follows that $G^*(M,\Lambda)$ is a group.

Next, if f_t is a Hamiltonian isotopy as above and g is a Poisson diffeomorphism then

$$\overset{\cdot}{\overbrace{(g^{-1} \circ f_t \circ g)}} = (d(u_t \circ g))^\sharp.$$

This means that $G^*(M,\Lambda)$ is a normal subgroup of $G(M,\Lambda)$. ∎

Proposition 2.4. *Let (M,Ω) be a symplectic manifold. If $t \mapsto f_t$ is a smooth path in $G^*(M,\Omega)$ then X_t is Hamiltonian for any t and, consequently, f_t is a Hamiltonian isotopy.*

For the (nontrivial) proof, see [12, p. 319-20].

Lemma 2.5. *Let (M,Λ) be the product Poisson manifold, $M = L_{2k}\times Q$, where L_{2k} is a symplectic manifold and Q is a manifold. If $X\in L(M,\Lambda)$ then: $X\in L^*(M,\Lambda) \iff X|L\in L^*(L,\Omega_L)$ for any leaf L.*

Proof. ($\Rightarrow$) It follows by definition and by the assumption on (M,Λ).

($\Leftarrow$) Let θ_L be a unique smooth 1-form on a leaf L such that we have $\flat_L(\theta_L) = X|L$ where $\flat_L : TL\to T^*L$ is the isomorphism associated with (L,Ω_L). We have $\theta_L = du_L$ for any L with $u_L\in C^\infty(L)$. The functions u_L are defined uniquely up to a constant.

Let us fix $x^0\in L_{2k}$, and let $(x_1,\dots,x_{2k})$ be a canonical chart on L_{2k} at x^0. As $M = L_{2k}\times Q$ one can choose for any $y\in Q$ a canonical product chart $(x,y) = (x_1,\dots,x_{2k},y_1,\dots,y_h)$ at (x^0,y).

One can choose a 1-form θ on M such that $\theta|TL = \theta_L$ for any leaf L. If $\theta = \sum_{i=1}^{2k} a_i(x,y)dx_i$ in this chart then the equation

$$u_{L_y}(x) = \int_{x_1^0}^{x_1} a_1(x,y)dx_1 + C(y)$$

determines uniquely the constant $C(y)$, where L_y is the leaf through (x^0,y). Clearly $C(y)$ is a basic function. It follows that the function

$$u(x,y) = u_{L_y}(x) - C(y)$$

is smooth and satisfies $X = (du)^\sharp$. □

Proposition 2.6. *Let $M = L_{2k}\times Q$ be the product Poisson manifold. If f_t is a smooth path in $G^*(M,\Lambda)$ then f_t is a Hamiltonian isotopy.*

Proof. For each leaf L a smooth path $t\mapsto f_t|L$ is in $G^*(L,\Omega_L)$. By Proposition 2.4 $f_t|L$ is a Hamiltonian isotopy in (L,Ω_L). It follows that $X_t|L\in L^*(L,\Omega_L)$ for each t. In view of Lemma 2.5 $X_t\in L^*(M,\Lambda)$ for each t. This means that f_t is a Hamiltonian isotopy. □

Corollary 2.7. *Under the above assumption, let f_t be a Poisson isotopy. Then f_t is Hamiltonian if and only if $f_t|L$ is Hamiltonian of $(L,\Omega|L)$ for each L.*

Now we would like to put forward some questions. Denote by $\widehat{G}(M,\Lambda)$ (resp. $\widehat{G}^*(M,\Lambda)$) the subgroup of $G(M,\Lambda)$ generated by all $\exp(X)$ where $X\in L(M,\Lambda)$ (resp. $L^*(M,\Lambda)$). It is a trivial observation that $\widehat{G}(M,\Lambda)_0 = \widehat{G}(M,\Lambda)$ and $\widehat{G}^*(M,\Lambda)_0 = \widehat{G}^*(M,\Lambda)$.

QUESTIONS. Is it true that $G(M,\Lambda)_0 = \widehat{G}(M,\Lambda)$? Note that this is the case for symplectic manifolds (cf. [11]).

An analogue for Hamiltonian diffeomorphisms is whether $\widehat{G}^*(M,\Lambda) = G^*(M,\Lambda)$ holds true. Again, this is so for symplectic manifolds (see Corollary 3.2).

We end this section with the fragmentation properties for $G^*(M,\Lambda)$ and $G(M,\Lambda)$. From now on we adopt the following notation: $G_U(M)$ is the subgroup of all elements of $G(M)$ compactly supported in an open ball U. In the sequel open balls are always assumed to be relatively compact and *extendable* i.e. the closure of an open ball must be contained in another open ball.

DEFINITION 2.8 (Fragmentation property). If $\{U_i\}$ is any finite family of open balls and $h\in G(M)_0$ such that $\mathrm{supp}(h)\subset\bigcup U_i$, then there exists a decomposition $h = h^s\circ\ldots\circ h^1$ such that $h^j\in G_{U_{i(j)}}(M)_0$ for $i = 1,\ldots,s$.

LEMMA 2.9. *Let (M,Λ) be a regular Poisson manifold and let $\{U_i\}$ be a finite family of open balls of M. If f_t is a Hamiltonian isotopy of (M,Λ) such that $\overline{\bigcup_t \mathrm{supp}(f_t)}\subset\bigcup U_i$ then there are Hamiltonian isotopies f_t^j supported in $U_{i(j)}$ and such that $f_t = f_t^s\circ\cdots\circ f_t^1$.*

PROOF. First observe that by considering $f_{(p/m)t}f^{-1}_{(p-1/m)t}$, $p=1,\ldots,m$, instead of f_t we may assume that f_t is sufficiently near the identity.

We choose a new family of open balls, $\{V_j\}_{j=1}^s$, satisfying $\mathrm{supp}(f_t)\subset V_1\cup\ldots\cup V_s$ for each t and which is starwise finer that $\{U_i\}$: $\forall\, j\,\exists\, i\,\mathrm{star}(V_j)\subset U_{i(j)}$.

f_t being a Hamiltonian isotopy, for the corresponding family X_t we have the equality

$$X_t = (du_t)^\sharp$$

for some smooth path u_t in $C^\infty(M)$. Let $(\lambda_j)_{j=1}^s$ be a partition of unity subordinate to (V_j), and let $v_t^j = \lambda_j u_t$. One then has $X_t = (dv_t^1)^\sharp + \cdots +$

$(dv_t^s)^\sharp$. We define

$$X_t^j = (dv_t^1)^\sharp + \dots + (dv_t^j)^\sharp.$$

Each of the smooth families X_t^j integrates to a Hamiltonian isotopy g_t^j with support in $V_1 \cup \ldots \cup V_j$. We get

$$f_t = g_t^s = f_t^s \circ \dots \circ f_t^1$$

where $f_t^j = g_t^j \circ (g_t^{j-1})^{-1}$ ($g_t^0 = \mathrm{id}$). Finally, the inclusions

$$\mathrm{supp}(f_t^j) = \mathrm{supp}(g_t^j \circ (g_t^{j-1})^{-1}) \subset \mathrm{star}(V_j) \subset U_{i(j)},$$

which hold whenever f_t is close to id, give the required property. ∎

COROLLARY 2.10. *$G^*(M, \Lambda)$ verifies the fragmentation property.*

CONVENTION. In the sequel we shall omit for simplicity the composition sign $\circ$.

LEMMA 2.11. *Let $M = L_{2k} \times Q$ be the product Poisson manifold and let f_t be a Poisson isotopy joining $f_0 = \mathrm{id}$ with $f_1 = f$. If U, V are two open balls on M such that $V = V_1 \times V_2$, where V_1 is a ball on L_{2k}, and the closure of $\bigcup_t f_t(U)$ is a subset of V, then there is a Hamiltonian isotopy g_t such that $f_t = g_t$ on U and $\mathrm{supp}(g_t) \subset V$.*

PROOF. We have $X_t \in L(M, \Lambda)$ where X_t is related to f_t by (1). V_1 being a ball, $X_t|L \in L^*(L, \Omega_L)$ for any leaf L. By Lemma 2.5 $X_t \in L^*(M, \Lambda)$. It follows that $X_t = (du_t)^\sharp$ for a smooth family u_t for compactly supported C^∞-functions.

Let $\lambda : M \to \mathbb{R}$ be C^∞ such that $\lambda = 1$ on U and $\mathrm{supp}\,\lambda \subset V$. Then g_t corresponding to $Y_t = (dv_t)^\sharp$ satisfies the claim. ∎

Although it seems likely that Lemma 2.5, Proposition 2.6 and Lemma 2.11 hold in a wider class of Poisson manifolds (e.g. for the regular case by a method from [15]), the present form of them is sufficient for our purpose.

3 – The flux homomorphism

Let G be a topological group. Provided G is locally arcwise connected, its universal covering $\widetilde{G}$ is described as the set of pairs $(g, \{g_t\})$, where $g_t, t \in I$, is a path in G connecting g with e, and $\{g_t\}$ is the homotopy class of g_t rel. endpoints.

Let (M, Ω) be a symplectic manifold of dimension $n = 2k$. The group $\widetilde{G(M,\Omega)_0}$ can be endowed with multiplication in two equivalent ways: either by the juxtaposition of isotopies, or by the pointwise composition of isotopies. If f_t is an isotopy in $G(M,\Omega)_0$ one defines

$$\widetilde{S}(\{f_t\}) = \int_0^1 \flat(\dot{f}_t)dt.$$

This integral depends on the homotopy class rel. endpoints only (cf. [3], [12]), so that one has a continuous epimorphism $\widetilde{S}: \widetilde{G(M,\Omega)_0} \to H_0^1(M, \mathbb{R})$, where $H_0^1(M,\mathbb{R})$ is the first de Rham cohomology group with compact support. This epimorphism is called the flux homomorphism or the first Calabi homomorphism.

Let Γ be the image of the fundamental group $\pi_1(G(M,\Omega)_0, \mathrm{id})$ under $\widetilde{S}$. Then Γ is countable (cf. [1], [12]). Since

$$G(M,\Omega)_0 = \widetilde{G(M,\Omega)_0}/\pi_1(G(M,\Omega)_0)$$

we get a continuous epimorphism

$$S : G(M,\Omega)_0 \to H_0^1(M,\mathbb{R})/\Gamma.$$

The subgroup $\mathcal{K} = \mathrm{Ker}(S)$ is arcwise connected (cf. [1, p. 189]). Moreover, we have the following

Proposition 3.1. *Let f_t be an isotopy in $G(M,\Omega)_0$. Then f_t is an isotopy in $\mathcal{K}$ if and only if $\dot{f}_t = (du_t)^\sharp$ is a smooth curve in $L^*(M,\Omega)$, the Lie algebra of Hamiltonian vector fields of (M,Ω).*

For the proof, see [1, p. 190]. Observe that due to Theorem 10.12 in [12] $\widetilde{S}(\{f_t\}) = 0$ if and only if f_t is homotopic rel. endpoints to a Hamiltonian isotopy.

Corollary 3.2. *For any symplectic manifold* (M, Ω) *we have* $\mathcal{K} = G^*(M, \Omega) = \widehat{G}^*(M, \Omega)$ (*the notation as in Section* 2).

Proof. In [11] it is shown that for any symplectic manifold (not necessarily compact) we have $\mathcal{K} = \widehat{G}^*(M, \Omega)$. The equality $\mathcal{K} = G^*(M, \Omega)$ follows by Propositions 3.1 and 2.4. ∎

Observe that Theorem 1.1 is a consequence of this corollary and the fact that $\mathcal{K}$ is simple whenever M is compact (cf. Theorem II.6.1 in [1]).

Now if M is noncompact, the family u_t from Proposition 3.1 is uniquely determined. This enables us to define a new continuous epimorphism $\widetilde{R} : \widetilde{\mathcal{K}} \to \mathbb{R}$ (the second Calabi homomorphism) by

$$\widetilde{R}(\{\dot{f}_t\}) = \int_0^1 \Big(\int_M u_t \eta \Big) dt,$$

where η is a symplectic volume. Then $\widetilde{R}$ descends to a continuous epimorphism $R : \mathcal{K} \to \mathbb{R}/\Lambda$ (cf. [3], [12]).

Theorem 3.3 [1]. *For any open connected symplectic manifold* (M, Ω) *the group* $\mathrm{Ker}(R)$ *is simple.*

In particular, the group $G^*(M, \Omega)$ is not simple in this case. It follows easily by [4], or by an argument of Thurston [1, p. 225-6] that $G^*(M, \Omega)$ is neither perfect.

Definition 3.4. Let $2k < n$ and let $(\mathbb{R}^n, \Lambda_{2k})$ be the standard Poisson structure, i.e. $\mathbb{R}^n = \mathbb{R}^{2k} \times \mathbb{R}^h$ where $\mathbb{R}^{2k}$ is equipped with the standard symplectic form $\Omega_{2k} = \sum_{i=1}^k dx_i \wedge dx_{k+i}$. For any $L \in \mathcal{F}(\Lambda_{2k})$ we denote by $R_L : G^*(L, \Omega_L) \to \mathbb{R}/\Lambda$ the corresponding second Calabi homomorphism. Now $f \in G^{**}(\mathbb{R}^n, \Lambda_{2k})$ if, by definition, $R_L(f|L) = 0$ for any L.

Similarly we define $G^{**}(T^n, \Lambda_{2k})$ where Λ_{2k} is the standard Poisson structure on $T^n = T^{2k} \times T^h$.

Now we turn our attention to the case (T^{2k}, Ω_{2k}), where Ω_{2k} is the standard symplectic structure on the torus T^{2k}, i.e. $\Omega = \sum_{i=1}^{k} dx_i \wedge dx_{k+i}$ in the canonical chart $(x_i)_{i=1}^{2k}$.

For each $\alpha \in T^{2k}$ we define the rotation $R_\alpha \in G(T^{2k}, \Omega_{2k})_0$ by

$$R_\alpha(z) = (e^{2\pi i\alpha_1} z_1, \dots, e^{2\pi i\alpha_{2k}} z_{2k}).$$

We have the canonical inclusion

$$\alpha \in T^{2k} \hookrightarrow R_\alpha \in G(T^{2k}, \Omega_{2k})_0.$$

REMARK 3.5. It is straightforward that R_α is not a Hamiltonian unless $\alpha = 0$. Indeed, one has $\dot{R}_{t\alpha} = \sum t\alpha_i\partial_i$, where $\alpha = (\alpha_1, \dots, \alpha_{2k})$, $\partial_i = \partial/\partial x_i$. We then have

$$\int_0^1 \iota(\dot{R}_{t\alpha})\Omega_{2k} dt = (1/2)\sum_{i=1}^{k}(\alpha_i x_{k+i} - \alpha_{k+i} dx_i) \neq 0$$

so that in view of Theorem 10.12 in [12] R_α is not Hamiltonian.

For any $\alpha \in \mathbb{R}^{2k}$ we may take a smooth path $R_{t\alpha}$ in $G(T^{2k}, \Omega_{2k})$. Therefore, we obtain a map $j : \mathbb{R}^{2k} \to \widetilde{G(T^{2k}, \Omega_{2k})_0}$. It is visible that $j(\mathbb{Z}^{2k}) \subset \pi_1(G(T^{2k}, \Omega_{2k})_0)$. Consequently, j induces the map

$$j : T^{2k} = \mathbb{R}^{2k}/\mathbb{Z}^{2k} \to G(T^{2k}, \Omega_{2k})_0.$$

Further, since $\widetilde{S}(\pi_1(G(T^{2k}, \Omega_{2k})_0)) \subset H_0^1(T^{2k}, \mathbb{Z}) = \mathbb{Z}^{2k}$ we have the map

$$S : G(T^{2k}, \Omega_{2k})_0 \to H_0^1(T^{2k}, \mathbb{R}^{2k})/H_0^1(T^{2k}, \mathbb{Z}^{2k}) = \mathbb{R}^{2k}/\mathbb{Z}^{2k} = T^{2k}.$$

Thus we have defined the map

$$J := S \circ j : T^{2k} \to T^{2k}.$$

In other words, J is the restriction of S to $T^{2k} \subset G(T^{2k}, \Omega_{2k})_0$.

Proposition 3.6. *The map $J : T^{2k} \to T^{2k}$ is actually the isomorphism defined by*

$$J(x_1, x_2, \dots, x_{2k-1}, x_{2k}) = (-x_2, x_1, \dots, -x_{2k}, x_{2k-1}).$$

For the (rather obvious) proof, see [1, p. 222].

4 – Topological preliminaries

Let G be a topological group. We shall be concerned with $H_1(G)$, the first homology group of G in the sense of Eilenberg-Mclane. This group is identified with the abelianization $G/[G,G]$ (see e.g. [2]).

With any topological group G we can associate some simplicial set as follows (cf. [2]). The symbol G^{Δ^n} will stand for the set of all continuous mappings of Δ^n into G, Δ^n being the standard n-dimensional simplex. The group G acts on G^{Δ^n} by the pointwise multiplication. Let $B_n\overline{G} = G^{\Delta^n}/G$, that is $B_n\overline{G}$ is the set of orbits of the action of G. It is visible that $B_n\overline{G}$ can be identified with the set $(G,e)^{(\Delta^n,0)}$ of continuous mappings of Δ^n into G sending 0 to e. Then $B\overline{G} = \bigcup B_n\overline{G}$ is a simplicial set with some face operators $\partial_i : B_n\overline{G} \to B_{n-1}\overline{G}$ and degeneracy operators $s_i : B_n\overline{G} \to B_{n+1}\overline{G}$ (for the definition see [2] or [1]). These operators verify relevant compatibility conditions.

It is important that $B\overline{G}$ is a Kan complex and it is possible to give a purely combinatorial definition of homotopy groups (cf. [8]). Namely the following equivalence relation is given on $B\overline{G}$: for any 1-simplices σ, $\tau \in B_1\overline{G}$

$$\sigma \sim \tau \quad \text{iff} \quad \exists\, c \in B_2\overline{G} : \partial_0 c = \sigma, \partial_1 c = \tau, \partial_2 c = e$$

where e is the constant map. Then the first homotopy group of $B\overline{G}$ is defined by $\pi_1(B\overline{G}) = B_1\overline{G}/\sim$.

It follows by definition that for any $\sigma \in B_1\overline{G}$ the classes of σ with respect to the relation $\sim$ and with respect to the homotopy rel. endpoints are the same, that is

$$\pi_1(B\overline{G}) \;=\; B_1\overline{G}/\sim = \widetilde{G}.$$

The following will be useful in the next section.

LEMMA 4.1. *Let σ, τ be 1-simplices in G. Then σ, $\mathrm{Ad}_{\tau(1)}\sigma$ are homological.*

PROOF (See e.g. [1]). The paths $\mu(t) = [\tau(1), \sigma(t)]$ and $\nu(t) = [\tau(t), \sigma(t)]$ are homotopic rel. the endpoints $\mu(0) = \nu(0) = e$ and $\mu(1) = \nu(1) = [\tau(1), \sigma(1)]$. In fact, the homotopy can be given by

$$H(s,t) \;=\; [\tau(s+t-st), \sigma(t)].$$

Since the class of ν in $\widetilde{G}$ belongs to $[\widetilde{G}, \widetilde{G}]$, $\{\nu\} \in H_1(B\overline{G}, \mathbb{Z}) = \widetilde{G}/[\widetilde{G}, \widetilde{G}]$ is equal to 0. Hence $\{\mu\} = 0$. Thus the paths $t \mapsto \sigma(t)$ and $t \mapsto \tau(1)\sigma(t)\tau(1)^{-1}$ are homological. ∎

Further, it is well known that

$$H_1(B\overline{G}, \mathbb{Z}) \;=\; \frac{\pi_1(B\overline{G})}{[\pi_1(B\overline{G}), \pi_1(B\overline{G})]},$$

and, consequently,

$$H_1(B\overline{G}, \mathbb{Z}) = H_1(\pi_1(B\overline{G})) \;=\; H_1(\widetilde{G}). \tag{2}$$

Thus to prove that $f_t \in \widetilde{G}$ is in the commutator subgroup it suffices to have that $\{f_t\} = 0$ in $H_1(B\overline{G}, \mathbb{Z})$. This will be useful in the proof of Theorem 1.3.

5 – Implicit function theorem

The concept of $\mathcal{L}$-category was introduced in [17]. Roughly speaking, an object in this category is a quadruple $(E, B, \mathcal{N}, \mathcal{S})$, where E is a Fréchet space, $\mathcal{N} = (|\ \ |_i)$ is an increasing sequence of norms defining the topology of E, $\mathcal{S} = (S_t)$, $t > 0$, is a one-parameter family of "approximation" operators on E, and B is an open subset with respect to some norm from $\mathcal{N}$. Let E_i denote the completion of E with respect to the norm $|\ \ |_i$, and let $\rho_{ji} : E_j \to E_i$ be an extension if id_E, $j \geq i$. Then topologically $E = \lim_{\leftarrow}(E_i, \rho_{ji})$. An interpretation of the operators S_t is the following. Each S_t extends to an $S_t : E_0 \to E$ and S_t approximates an element from E_0 by an element from E. The greater is t the better is an approximation.

The concept of C^r (weak) morphism in the $\mathcal{L}$-category is even more complicated (see [17]). Of course, all morphisms are continuous mappings.

By means of the $\mathcal{L}$-category one can introduce the notion of $\mathcal{L}$-manifold of class C^r, $1 \le r \le \infty$. This is a topological space endowed with an $\mathcal{L}$-atlas, i.e. an atlas modeled on an $\mathcal{L}$-object in the usual way. In particular, the concept of tangent space of $\mathcal{L}$-manifold at a point is well defined. The spaces of C^r mappings are clue examples of $\mathcal{L}$-manifolds, and the need of a generalized smooth structure on them motivated the definition of $\mathcal{L}$-category.

The object of our interest will be $\mathcal{L}$-groups, that is topological groups such that their group products and inverse mappings are $\mathcal{L}$-morphisms. Of course, the diffeomorphism groups are here the main example. In obvious way one can define also a notion of $\mathcal{L}$-action of an $\mathcal{L}$-group on an $\mathcal{L}$-manifold.

We begin with an Implicit function theorem in the case of $\mathcal{L}$-actions (cf. [17]). Let G, H be $\mathcal{L}$-groups of class C^r $(r \ge 2)$ and $\mathcal{M}$ be an $\mathcal{L}$-manifold. Denote by $\alpha : G \times G \to G, \beta : H \times H \to H$ the group products and let $\Phi : G \times \mathcal{M} \to \mathcal{M}, \Psi : H \times \mathcal{M} \to \mathcal{M}$ be $\mathcal{L}$-actions of class C^r. Next, let $\Delta : G \times H \times \mathcal{M} \to \mathcal{M}$, be an "action" of $G \times H$, defined by

$$\Delta(g, h, x) = \Phi(g, \Psi(h, x))$$

for $g \in G, h \in H, x \in \mathcal{M}$. By $d\Delta$ we denote the differential of Δ with respect to two first variables. By the chain rule one has

$$d\Delta(g, h, x, \hat{g}, \hat{h}) = d_1\Phi(g, \Psi(h, x), \hat{g}) + d_2\Phi(g, \Psi(h, x), d_1\Psi(h, x, \hat{h})).$$

(Here we adopt the notation $\hat{g} \in T_g(G)$, $\hat{x} \in T_x(\mathcal{M})$ and so on.) Let us fix $x_0 \in \mathcal{M}$. By making use of the local triviality of the tangent bundle $T\mathcal{M}$ one can identify $T_x(\mathcal{M})$ with $T = T_{x_0}(\mathcal{M})$ for x being near x_0. Likewise, $T_g(G)$ is identified with $T_1 = T_e(G)$, whenever $g \in G$ is near e, and $T_h(H)$ is identified with $T_2 = T_e(H)$, whenever $h \in H$ is near e. Then by applying Implicit function theorem one has the following

Theorem 5.1 [17, 4.2.5]. *Suppose that there exists an* $\mathcal{L}$*-morphism of class* C^∞ , $L : \mathcal{U}\times T \to T_1\times T_2$, *where* $\mathcal{U}$ *is a neighborhood of* e *in* H, *such that if* $L(h,\hat{x}) = (\hat{g},\hat{h})$, *then*

$$d\Delta(e,e,\Psi(h,x),\hat{g},\hat{h}) = \hat{x}.$$

Then there exists a neighborhood $\mathcal{V}$ *of* x_0 *in* $\mathcal{M}$ *and a weak* $\mathcal{L}$*-morphism of class* C^∞ $s : \mathcal{V}\to G\times H$ *such that* $\Delta(g,h,x_0) = x$ *if* $s(x) = (g,h)$.

Now let $k \geq 1$ and let $\mathcal{F}_k$ denote the trivial k-dimensional foliation on the torus T^n, i.e. $\mathcal{F}_k = \{T^k\times\{pt\}\}$. The symbol $G(T^n,L_k)_0$ stands for the group of all leaf preserving diffeomorphisms on $(T^n,\mathcal{F}_k)$ which are isotopic to the identity through leaf preserving diffeomorphisms.

Notice that we have the canonical inclusion $\alpha\in T^k\hookrightarrow R_\alpha\in G(T^n,L_k)_0$, where

$$R_\alpha(z_1,\dots,z_n) = (e^{2\pi i\alpha_1}z_1,\dots,e^{2\pi i\alpha_k}z_k,z_{k+1},\dots,z_n).$$

The following result is a version of Theorem 5.2.1 in [17], or fundamental lemma in [7].

Theorem 5.2. *Let* $\alpha\in T^k$ *verify Diophantine condition. There exist a neighborhood* $\mathcal{V}$ *of* R_α *in* $G(T^n,L_k)_0$ *and a weak* $\mathcal{L}$*-morphism of class* C^∞

$$s : \mathcal{V}\to G(T^n,L_k)_0\times T^k$$

such that $h = R_\lambda g^{-1}R_\alpha g$ *whenever* $h\in\mathcal{V}$, *and* $s(h) = (g,\lambda)$. *Moreover, if* h_t, $t\in I$, *is a smooth isotopy in* $\mathcal{V}$ *and* $s(h_t) = (g_t,\lambda_t)$ *then* g_t, λ_t *depend smoothly on* t.

Recall that $\alpha = (\alpha_1,\dots,\alpha_n)\in\mathbb{R}^n$ satisfies the *Diophantine condition* if there are small $c > 0$ and large N such that for any $(n+1)$-tuple of integers $(q_0,q_1,\dots,q_n)$ with $(q_1,\dots,q_n)\neq 0$ one has

$$|q_0+q_1\alpha_1+\dots+q_n\alpha_n| \;>\; c(|q_1|+\dots+|q_n|)^{-N}.$$

We can extend this definition to elements of T^n by saying that $\alpha\in T^n$ is Diophantine if so is its representant in $\mathbb{R}^n$; this definition is independent of the choice of representant. It is well known that the set of all

Diophantine elements of T^n is dense. Moreover, it has the Haar measure equal to 1.

It is obvious that if $\alpha \in T^k$ is Diophantine then so is $\alpha' = (\alpha, 0) \in T^n$.

Proof. The proof follows closely [17], or [7], and it is given here for the completeness sake.

Let $G = T^k, H = G(T^n, L_k)_0$ endowed with the structure opposite to the usual. Define actions of G and H, respectively, on H by

$$\Phi(\lambda, h) = R_\lambda h,$$
$$\Psi(g, h) = g^{-1}hg.$$

Let $\Delta : G \times H \times H \to H$ be the composition of these actions

$$\Delta(\lambda, g, h) = \Phi(\lambda, \Psi(g, h)) = R_\lambda g^{-1}hg.$$

We make use of Theorem 5.1. We have

$$d\Delta(e, e, h, \hat{\lambda}, \hat{g}) = \hat{\lambda} + dh \cdot \hat{g} - \hat{g} \cdot \Delta,$$

where $\hat{\lambda} \in \mathbb{R}^k = T_e(T^k)$, $\hat{g} \in C^\infty(T^n, \mathbb{R}^k) = T_{\mathrm{id}}(G(T^n, L^k)_0)$. Consider the equation

$$\hat{\lambda} + d(g^{-1}R_\alpha g) \cdot \hat{g} - \hat{g} \cdot (g^{-1}R_\alpha g) = \hat{h}. \tag{3}$$

In view of Theorem 5.1 we have to solve (3) for given $g, h \in G(T^n, L_k)$, and with respect to the unknowns $\hat{\lambda}$, $\hat{g}$. Set $\hat{f} = dg \cdot \hat{g} \cdot g^{-1} \in C^\infty(T^n, \mathbb{R}^k)$. Since $dR_\alpha = \mathrm{id}$, we get

$$\hat{f} - \hat{f} \cdot R_\alpha = dg \cdot (\hat{h} - \hat{\lambda}) \cdot g^{-1} \tag{4}$$

If m be the normalized Haar measure on T^n, we have

$$\int_{T^n} dg \cdot (\hat{h} - \hat{\lambda}) \cdot g^{-1} dm = 0. \tag{5}$$

The equality (5) determines uniquely $\hat{\lambda} \in \mathbb{R}^k$, provided g is sufficiently near id in $G(T^n, L_k)_0$.

The rest consists in showing the existence of $\hat{f} \in C^\infty(T^n, \mathbb{R}^k)$ satisfying (4). This will follow by using the condition on α. Suppose that we have the following expansion in Fourier series

$$\hat{f}(x) = \sum_{p \in \mathbb{Z}^n - \{0\}} a_p e^{2\pi i(p,x)}, \quad a_p \in \mathbb{R}^k.$$

Further, suppose that the right hand side of (4) has the Fourier expansion

$$\sum_{p \in \mathbb{Z}^n} b_p e^{2\pi i(p,x)}, \quad b_p \in \mathbb{R}^k.$$

Then, in view of (4), we get

$$a_p = \frac{b_p}{1 - e^{2\pi i(p,\alpha)}},$$

for $p > 0$, and $a_0 = 0$. The Diophantine condition now gives

$$|a_p| \leq c|b_p||p|^N,$$

where $|p| = \sum_i |p_i|$. This implies that $\hat{f}$ is of class C^∞ (when C^r, r finite, would be considered, one could not avoid the "loss of smoothness", i.e. $\hat{f}$ is of class $C^{r-\beta}$, β depending on α).

The second assertion follows from the fact that s is a C^∞ $\mathcal{L}$-morphism (Theorem 5.1), and it sends smooth curves to smooth curves. □

6 – The case of $G^*(T^n, \Lambda_{2k})$

Let Λ_{2k} be the standard Poisson structure on the torus T^n, i.e. $T^n = T^{2k} \times T^h$ is the product of T^{2k} with the standard symplectic structure by T^h. In particular, for any leaf L we have $\Omega_L = \sum_{i=1}^k dx_i \wedge dx_{k+i}$ in the canonical chart $(x_1, \ldots, x_{2k}, y_1, \ldots, y_h)$. Theorem 5.2 enables us the study of the group of Hamiltonian diffeomorphisms of this structure. The following argument is a modification of [1, Theorem III.6.2].

Proof of Theorem 1.2. Clearly it suffices to prove the second assertion. Let f_t be an isotopy in $G^*(T^n, \Lambda_{2k})$ connecting $f = f_1$ with

$\mathrm{id} = f_0$. Our purpose is to show that f_t is in the commutator subgroup. In view of Sect.4 we have to show that $\{f_t\} = 0$ in $H_1(B\overline{G^*(T^n, \Lambda_{2k})}, \mathbb{Z})$.

We may assume that $f_t \in \mathcal{V}'$, $\mathcal{V}'$ being a neighborhood of id such that $R_\alpha \mathcal{V}' \subset \mathcal{V}$, where $\mathcal{V}$, α are as in Theorem 5.2, and α is so small that R_α is in a contractible neighborhood of id. This assumption can be accomplished by replacing f_t by the product of $f_{(p/m)t} f^{-1}_{(p-1/m)t}$, $p = 1, \dots, m$, for a sufficiently large m. Then, in view of Theorem 5.2, we have $R_\alpha f_t = R_{\lambda_t} g_t^{-1} R_\alpha g_t$ or

$$f_t = R_{\lambda_t} R_\alpha^{-1} g_t^{-1} R_\alpha g_t \tag{6}$$

where $g_t \in G(T^n, L_{2k})_0$. Since $R_\alpha \in G(T^n, L_{2k})$ we get

$$\Lambda_{2k} = f_{t*} \Lambda_{2k} = (g_t^{-1})_* R_{\alpha *} g_{t*} \Lambda_{2k},$$

that is

$$g_{t*} \Lambda_{2k} = R_{\alpha *} g_{t*} \Lambda_{2k}. \tag{7}$$

Let $\bar{\alpha} \in \mathbb{R}^{2k}$ be a representant of $\alpha \in T^{2k}$. The Diophantine condition on α ensures us that the components of $\bar{\alpha}$ are linearly independent over $\mathbb{Q}$. Consequently α generates a dense subgroup of T^{2k}. By (7) the tensor $g_{t*}\Lambda_{2k}$ is invariant by $R_{\alpha *}$. This implies that $g_{t*}\Lambda_{2k}$ is T^{2k}-invariant, and $g_{t*}\Lambda_{2k}$ has constant coefficients in the canonical chart. Let $L \in \mathcal{F}(\Lambda_{2k})$. The forms $(g_t|L)_*\Omega_L$ and Ω_L are cohomologous on L and they must have the same periods (cf. [12, p. 319]). It follows that all the coefficients of $(g_t|L)_*\Omega_L$, and of $(g_t|L)_*\Lambda_{2k}$, are equal to 1. Therefore we get $g_t \in G(T^n, \Lambda_{2k})_0$.

Note that another method to obtain the last claim is to use Theorem 2.3 in [6].

Now we return to the equality (6). Let L be any leaf of $\mathcal{F}(\Lambda_{2k})$. In view of Corollary 3.2 we get

$$\begin{aligned} 0 = S_L(f_t|L) = S_L(R_{\lambda_t}|L) - S_L(R_\alpha|L) - S_L(g_t) + S_L(R_\alpha|L) + S_L(g_t|L) = \\ = S_L(R_{\lambda_t}|L) = J(\lambda_t), \end{aligned}$$

where S_L is the flux homomorphism for (L, Ω_L). It follows from Proposition 3.6 that $\lambda_t = 0$. Therefore

$$f_t = R_\alpha^{-1} g_t^{-1} R_\alpha g_t. \tag{8}$$

Now it suffices to make a modification of (8) in order to have all factors of it in $G^*(T^n, \Lambda_{2k})$.

Let $L \in \mathcal{F}(\Lambda_{2k})$. We set

$$\beta_t = J^{-1} S_L(g_t|L), \quad \gamma_t = \alpha + \beta_t.$$

Note that β_t and γ_t depend on leaves, i.e. they are basic functions with respect to $\mathcal{F}(\Lambda_{2k})$. For any basic function λ by $\widehat{R}_\lambda$ we denote a diffeomorphism such that $\widehat{R}_\lambda|L_y = R_{\lambda(y)}$, i.e. $\widehat{R}_\lambda$ is a rotation when restricted to a leaf.

Next we set

$$h_t = \widehat{R}_{\beta_t}^{-1} g_t \quad \text{and} \quad k_t = \widehat{R}_{\beta_t} g_t^{-1}.$$

Then we have $h_t, k_t \in G^*(T^n, \Lambda_{2k})$ by corollaries 3.2 and 2.7 as e.g. for h_t

$$S_L(h_t|L) = S_L(R_{\beta_t}^{-1}) + S_L(g_t|L) = -J(\beta_t) + J(\beta_t) = 0.$$

We obtain from (8) the equality

$$f_t = \widehat{R}_{\gamma_t}^{-1} k_t \widehat{R}_{\gamma_t} h_t. \tag{9}$$

Let $T^n = \bigcup_{i=1}^r U_i$ be a covering by open balls. Due to Lemma 2.9 we have a decomposition $k_t = k_t^r \ldots k_t^1$ with $\operatorname{supp}(k_t^i) \subset U_i$. Notice that by definition

$$R_\alpha^{-1} h_t R_\alpha = \widehat{R}_{\gamma_t}^{-1} k_t^{-1} \widehat{R}_{\gamma_t}.$$

Hence we have the decomposition

$$R_\alpha^{-1} h_t R_\alpha = \bar{h}_t^1 \ldots \bar{h}_t^r, \quad \text{where} \quad \bar{h}_t^i = \widehat{R}_{\gamma_t}^{-1} (k_t^i)^{-1} \widehat{R}_{\gamma_t}.$$

By making use of Lemma 2.11 we can find $\rho_t^i \in G_{U_i}^*(M, \Lambda)$ satisfying $\rho_t^i = \widehat{R}_{\gamma_t}$ on $\operatorname{supp}(k_t^i)$. In particular, $(k_t^i)^{-1} = \rho_t^i \bar{h}_t^i (\rho_t^i)^{-1}$. Likewise, there are $\sigma_t^i \in G^*_{(\rho_1^i)^{-1}(U_i)}(M, \Lambda)$ such that $\sigma_t^i = R_{t\alpha}$ on $(\rho_1^i)^{-1}(U_i)$.

Thus the equality (9) is transformed into

$$\begin{aligned} f_t &= (\rho_t^r)^{-1} k_t^r \rho_t^r \cdots (\rho_t^1)^{-1} k_t^1 \rho_t^1 R_\alpha \bar{h}_t^1 \cdots \bar{h}_t^r R_\alpha^{-1} \\ &= (\rho_t^r)^{-1} k_t^r \rho_t^r \cdots (\rho_t^1)^{-1} k_t^1 \rho_t^1 \sigma_1^1 \bar{h}_t^1 (\sigma_1^1)^{-1} \cdots \sigma_1^r \bar{h}_t^r (\sigma_1^r)^{-1}. \end{aligned} \tag{10}$$

By the abelianity of $H_1(G^*\widetilde{(T^n, \Lambda_{2k})})$ we get that the r.h.s. of (10) on the homology level is trivial. This completes the proof. ∎

Actually we shall need in Sect.7 a more specified version of Theorem 1.2.

THEOREM 6.1. *Let V', W' be open balls in T^h such that $\overline{V}' \subset W'$, and let $V = T^{2k} \times V'$, $W = T^{2k} \times W'$. If f_t is an isotopy in $G^*_V(T^n, \Lambda_{2k})$ then $\{f_t\} = 0$ in $H_1(\widetilde{G^*_W(T^n, \Lambda_{2k})})$.*

PROOF. Let $f_t \in G^*_V(T^n, \Lambda_{2k})$. By Theorem 1.2 we have

$$f_t \sim [h^1_t, k^1_t] \dots [h^r_t, k^r_t]$$

where $\sim$ stands for the homotopy rel. endpoints, and $h^i_t, k^i_t \in G^*(T^n, \Lambda_{2k})$. Choose a smooth bump function $\mu : T^h \to [0, 1]$ such that $\operatorname{supp} \mu \subset W'$ and $\mu = 1$ on V'. Then we set

$$\bar{h}^i_t(x, y) = h^i_{\mu(y)t}(x, y), \quad \bar{k}^i_t(x, y) = k^i_{\mu(y)t}(x, y)$$

where $(x, y) = (x_1, \dots, x_{2k}, y_1, \dots, y_h)$ is the canonical chart. First note that as h^i_t, k^i_t are leaf preserving diffeomorphisms then so are $\bar{h}^i_t$, $\bar{k}^i_t$. Note as well that in view of Corollary 2.7 $\bar{h}^i_t, \bar{k}^i_t \in G^*_W(T^n, \Lambda_{2k})$. Finally, observe that we have

$$f_t \sim [\bar{h}^1_t, \bar{k}^1_t] \dots [\bar{h}^r_t, \bar{k}^r_t].$$

Indeed, the initial homotopy is leafwise so that the required modification of it is obvious. □

By interpreting the above theorems in terms of homology we get

COROLLARY 6.2. *The group $H_1(B\overline{G^*(T^n, \Lambda_{2k})}, \mathbb{Z})$ is trivial. Furthermore, if $\iota : G^*_V(T^n, \Lambda_{2k}) \hookrightarrow G^*_W(T^n, \Lambda_{2k})$ is the canonical inclusion then the image of*

$$\iota_* : H_1(B\overline{G^*_V(T^n, \Lambda_{2k})}, \mathbb{Z}) \to H_1(B\overline{G^*_W(T^n, \Lambda_{2k})}, \mathbb{Z})$$

is trivial.

7 – Proof of Theorem 1.3

Let Λ_{2k} be the standard Poisson structure on T^n. For simplicity we denote $G^*_U = G^*_U(T^n, \Lambda_{2k})$ and $G^{**}_U = G^{**}_U(T^n, \Lambda_{2k})$ for any open $U \subset T^n$.

First we define some open subsets of $T^n = T^{2k} \times T^h$. We set

$$\begin{aligned} U &= U_1 \times U_2 \\ B &= U_1 \times W_2 \\ V &= T^{2k} \times V_2 \\ W &= T^{2k} \times W_2 \\ W' &= T^{2k} \times W_2'. \end{aligned}$$

Here U_1 is an open ball in T^{2k}, and U_2, V_2, W_2, W_2' are open balls in T^h satisfying $\overline{U}_2 \subset V_2 \subset \overline{V}_2 \subset W_2 \subset \overline{W}_2 \subset W_2'$. Then we have the following commutative diagram

$$\begin{array}{ccc} H_1(B\overline{G^{**}_U}, \mathbb{Z}) & \xrightarrow{\iota_{1*}} & H_1(B\overline{G^{**}_B}, \mathbb{Z}) \\ \iota_{4*}\big\downarrow & & \iota_{2*}\big\downarrow \\ H_1(B\overline{G^{*}_U}, \mathbb{Z}) & & H_1(B\overline{G^{*}_B}, \mathbb{Z}) \\ \iota_{5*}\big\downarrow & & \iota_{3*}\big\downarrow \\ H_1(B\overline{G^{*}_V}, \mathbb{Z}) & \xrightarrow{\iota_{6*}} & H_1(B\overline{G^{*}_W}, \mathbb{Z}) \end{array}$$

where all the arrows come from the canonical inclusions, i.e. $\iota_1 : G^{**}_U \hookrightarrow G^{**}_B$, etc. The commutativity follows from the identification (2), the functoriality of H_*, and the fact that such a diagram holds for $\tilde{\iota}_1 : \widetilde{G^{**}_U} \to \widetilde{G^{**}_B}$ etc. The latter is due to the definition of the universal covering. Observe that thanks to Corollary 6.2 we know that the images of ι_{5*} and ι_{6*} are trivial.

The following two lemmas will be of use.

Lemma 7.1. *With the above notation, there exist a finite family of open balls $\{B^i\}_{i=1}^s$ such that $W = \bigcup B^i$ and a related family of isotopies $\{\phi^i_t\}_{i=1}^s$ in $G^*_{W'}$ such that $\phi^i_1(B^i) \subset B$ and*

$$\phi^i_1 | B^i \cap B^j \;=\; \phi^{ij}_1 \circ \phi^j_1 | B^i \cap B^j \;,$$

where ϕ^{ij}_t is an isotopy in G^*_B, for each (i,j) such that $B^i \cap B^j \neq \emptyset$. Moreover, we may have that $B^i \cap B^j$, whenever nonempty, is a ball.

Proof. In view of Lemma III.5.2 [1] there exists a covering $\{U_1^i\}_{i=1}^s$ of T^{2k} by open balls such that $U_1^i \cap U_1^j$ is a ball whenever nonempty. Furthermore, there exist Hamiltonian isotopies ψ_t^i in $G^*(T^{2k}, \Omega_{2k})$, and ψ_t^{ij} in $G^*_{U_1}(T^{2k}, \Omega_{2k})$ such that

$$\psi_t^i | U_1^i \cap U_1^j = \psi_1^{ij} \circ \psi_1^j | U_1^i \cap U_1^j.$$

Then we let $B^i = U_1^i \times W_2$ and $\phi_t^i = \psi^i_{\mu(y)t} \times \mathrm{id}$, $\phi_t^{ij} = \psi^{ij}_{\mu(y)t} \times \mathrm{id}$, where $\mu : T^l \to [0,1]$ is a bump function such that $\operatorname{supp} \mu \subset W_2'$ and $\mu = 1$ on W_2. ∎

Let us recall that $c \in B_n\overline{G}$ has its support in U if and only if $\forall\, x$, $y \in \Delta^n$ the diffeomorphism $c(x)c(y)^{-1}$ is supported in U.

The following is a version of Lemma III.5.3 in [1] specified to our case.

Lemma 7.2. *Let U, W, $\{B^i = U_1^i \times W_2\}_{i=1}^s$ be as above. If a 1-chain $\sigma \in B\overline{G_U^{**}}$ is a boundary of a 2-chain $c = \sum c_\alpha \in B\overline{G_V^*}$ then σ is a boundary of a 2-chain $C = \sum C_\alpha$ such that the supports of C_α are subordinate to $\{B^i\}$.*

The original proof in [18], [1] is very long and difficult. It still works in our case as the groups in question are still locally contractible and the fragmentation property holds. We shall not reproduce a reasoning from [1, p. 213-19] only indicating why it applies to our case.

$\mathcal{B}_1 = \{U_1^i\}_{i=1}^s$ is a covering of T^{2k} by open balls. It is well known that there exists a triangulation of T^{2k}, $T(\mathcal{B}_1) = \{\Delta_i^j\}, i \in I_j, j = 0, 1, \dots, 2k$, which is starwise finer than $\mathcal{B}_1$. Next by induction on j one can choose $\mathcal{V}_1 = \{V_1^{j,i}\}, i \in I_j$, a covering by open balls of T^{2k} finer than $\mathcal{B}_1$, such that $\{V_1^{j,i}\}_{i \in I_l, j=1,\dots,m}$, is a covering of the m-skeleton of $T(\mathcal{B}_1)$ and $\overline{V}_1^{j,i} \cap \overline{V}_1^{j,i'} = \emptyset$ for any $i \neq i'$. We set $\mathcal{V} = \{V^{j,i}\}$, where $V^{j,i} = V_1^{j,i} \times W_2$.

The proof consists in applications of Lemma 2.9 many times. This is done by means of complicated induction reasonings with respect to the covering $\mathcal{V}$. The "product" form of it enables us to choose the functions λ_i in the proof of Lemma 2.9 defined on T^{2k} rather than on T^n, i.e. they are common for each leaf. Therefore the whole construction from [1] can be carried over to our case leaf by leaf. The assumption that $\sigma \in B\overline{G_U^{**}}$ is necessary in order to accomplish a construction of edges of new simplices exactly for the same reasons as in [1].

PROPOSITION 7.3. *If* $\sigma \in B_1\overline{G_U^{**}}$ *satisfies* $\iota_{3*}\iota_{2*}\iota_{1*}\{\sigma\} = 0$ *then* $\iota_{2*}\iota_{1*}\{\sigma\} = 0$.

PROOF. Let $\sigma \in B_1\overline{G_U^{**}}$. Thanks to Corollary 6.2 $\sigma = \partial c$ where $c = \sum c_\alpha \in B_2\overline{G_V^{*}}$. In view of Lemma 7.2 we assume that support of each c_α is contained in some B^i.

Under the notation of Lemma 7.1 the following convention will be useful:

(i) $\mathrm{supp}(\partial_j c_\alpha) \subset B^{i(j,\alpha)}$ and by $\phi_t^{i(j,\alpha)}$ we denote the corresponding isotopy;
(ii) we assume that $B^{i(j,\alpha)} = B$ and $\phi_t^{i(j,\alpha)} = \mathrm{id}$, if $\mathrm{supp}(\partial_j c_\alpha) \subset B$;
(iii) if $\partial_j c_\alpha = \pm\partial_l c_\beta$ then $B^{i(j,\alpha)} = B^{i(l,\beta)}$ and $\phi_t^{i(j,\alpha)} = \phi_t^{i(l,\beta)}$;
(iv) $\mathrm{supp}(c_\alpha) \subset B^{i(\alpha)}$ and $\phi_t^{i(\alpha)}$ denotes the corresponding isotopy;
(v) we denote $\chi_{j\alpha\beta} = \phi^{i(j,\alpha)i(\beta)}$.

The following equality holds

$$\sigma = \sum_\alpha \sum_{j=0}^{2} (-1)^j \partial_j c_\alpha \ = \ \sum_\alpha \sum_{j=0}^{2} (-1)^j \mathrm{Ad}_{\phi_1^{i(j,\alpha)}}(\partial_j c_\alpha). \tag{11}$$

If fact, if support of the edge $\partial_j c_\alpha$ is in B then nothing will change. Otherwise, this edge must be reduced in the sum on the left hand side, and by (iii) so must be on the right hand side.

By making use of Lemmas 7.1 and 4.1 we have

$$\mathrm{Ad}_{\phi_1^{i(j,\alpha)}}(\partial_j c_\alpha) = \mathrm{Ad}_{\chi_{j\alpha\alpha}}\mathrm{Ad}_{\phi_1^{i(\alpha)}}(\partial_j c_\alpha) \sim \mathrm{Ad}_{\phi_1^{i(\alpha)}}(\partial_j c_\alpha) = \partial_j \mathrm{Ad}_{\phi_1^{i(\alpha)}}(c_\alpha).$$

By substituting this to (11)

$$\sigma \sim \sum_\alpha \sum_{j=0}^{2} (-1)^j \partial_j \mathrm{Ad}_{\phi_1^{i(\alpha)}}(c_\alpha) = \partial \sum_\alpha \mathrm{Ad}_{\phi_1^{i(\alpha)}}(c_\alpha)\ ,$$

and, by (iv), $\mathrm{supp}(\sum_\alpha \mathrm{Ad}_{\phi_1^{i(\alpha)}}(c_\alpha)) \subset B$. Thus σ is a coboundary in $B\overline{G_B^{*}}$, as required. ∎

Proposition 7.4. *The map* $\iota_{2*} : H_1(B\overline{G_B^{**}}, \mathbb{Z}) \to H_1(B\overline{G_B^{*}}, \mathbb{Z})$ *is a monomorphism.*

The proof is a Poisson version of that of Lemma III.5.4 [1]. As $B = U_1 \times W_2$ with a canonical chart the required modifications are obvious.

Proof of Theorem 1.3. If f_t is a Hamiltonian isotopy of $(\mathbb{R}^n, \Lambda_{2k})$ we may assume that $\mathrm{supp}(f_t) \subset U$, where U is in a domain of a canonical chart. Moreover, we may and do assume that $\overline{U} \subset B$ such that B is still in this domain, and U and B are identified with those fixed on T^n at the beginning of this section.

Let $\sigma = \{f_t\} \in B_1\overline{G_U^{**}}$. First we have that in view of Corollary 6.2 $\iota_{3*}\iota_{2*}\iota_{1*}\{\sigma\} = 0$. By Proposition 7.3 we get $\iota_{2*}\iota_{1*}\{\sigma\} = 0$. Next, by Proposition 7.4 $\iota_{1*}\{\sigma\} = 0$. Therefore $\{f_t\} = 0$ in $H_1(B\overline{G_B^{**}}, \mathbb{Z})$. This means that f_t is homologous (or homotopic rel. endpoints) to a product of commutators of special Hamiltonian isotopies supported in B. Finally if $\widetilde{G}$ is perfect then so is G. ∎

8 – The case of regular Poisson manifolds

First we give the definition of $G^{**}(M, \Lambda)$, where (M, Λ) is an arbitrary regular Poisson manifold. A Hamiltonian diffeomorphism $f \in G^{**}(M, \Lambda)$ if there is a finite covering of $\mathrm{supp}(f)$ by canonical chart domains, $\{U_i\}$, such that $f = f^s \cdots f^1$ and $f^j \in G^{**}_{U_{i(j)}}$. (The definition of G_U^{**} is given in Section 3.)

The following is a Poisson counterpart of Theorem 3.3.

Theorem 8.1. $G^{**}(M, \Lambda)$ *is a perfect group.*

Indeed, this follows immediately from Theorem 1.3.

It is still an open problem whether $G^*(M, \Lambda)$ is perfect for any compact regular Poisson manifold (M, Λ). It seems likely that it is so at least if all leaves of $\mathcal{F}(\Lambda)$ are compact.

Another question is whether $G^{**}(M, \Lambda) = G^*(M, \Lambda)$ for any compact regular Poisson manifold (M, Λ). Notice that due to a difficult Lemma III.3.2 in [1] this is the case of any compact symplectic manifold. Notice as well that, in general, $G^{**}(M, \Lambda) \neq G^*(M, \Lambda)$ for M noncompact (Theorem 3.3).

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Lavoro pervenuto alla redazione il 4 marzo 1997
ed accettato per la pubblicazione il 3 dicembre 1997.
Bozze licenziate il 6 febbraio 1998

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