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Einstein manifolds and obstructions to the existence of Einstein metrics

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Riassunto: *Questo articolo costituisce una panoramica sulle varietà di Einstein. Si è scelto di menzionare alcuni fatti classici che sembrano avvalorare lo studio di tale nozione, e si discutono alcuni esempi tipici di varietà di Einstein che entrano naturalmente in gioco in geometria. Quindi, vengono trattati i problemi concernenti lo spazio dei moduli, l'unicità e l'esistenza di metriche di Einstein su una varietà fissata, presentando i nuovi sviluppi in materia.*

Abstract: *This article is a panorama about Einstein manifolds* (*which has not to be intended as a complete report on the subject*). *We have chosen to mention some classical facts which make the notion of Einstein metric worth of investigation, and we discuss how Einstein manifolds naturally arise in geometry by means of typical examples. Then, we survey the problems of moduli space, uniqueness and existence of Einstein metrics on a given manifold, introducing recent developments on the subject.*

1 – Curvature and Einstein metrics

1.1 A first, classical, motivation for introducing the notion of Einstein metric is given by trying to answer to the following "naive" question:

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given any differentiable manifold X, do there exist any "distinguished" or "nicest" Riemannian metrics on X?

In dimension 2, it comes natural to think about the constant curvature metrics: every compact surface admits at least one Riemannian metric of constant curvature, whose sign is equal to that of its Euler characteristic [4]. On the other hand, even if a constant curvature metric is distinguished among all metrics that one can consider on a given compact surface, such a metric is not unique, exceptly in the cases of the sphere and of the real projective plane: one knows that, generally, the metrics of constant curvature on a fixed surface X, modulo the isometries and a multiplicative factor, form a (singular) finite dimensional manifold, i.e. the *Moduli Space* (see [2], [21]).

In dimension n greater than 2, the situation changes completely. In fact, there are at least three significative and different notions of curvature. If R is the curvature tensor of a metric g on X, one can talk about

- the *sectional curvature* of tangent planes P of X: $\sigma(P) = \sigma(e_1, e_2) = R(e_1, e_2, e_1, e_2)$ (where $\{e_1, e_2\}$ is any orthonormal basis of P), whose complete knowledge permits to recover the whole curvature tensor R;

- the *Ricci tensor*, which is a symmetric bilinear form defined on each tangent space T_xX by $\mathrm{Ric}(u, v) = \sum_{i=1}^{n} R(u, e_i, v, e_i)$ (where $\{e_i\}_{i=1,\dots,n}$ is any orthonormal basis of T_xX). If u is a unit tangent vector, one let $\mathrm{Ric}(u) = \mathrm{Ric}(u, u)$ denote the *Ricci curvature* in the direction of u;

- the *scalar curvature* function on X, defined as the trace of the Ricci tensor; namely, at a point x, $\mathrm{scal}(x) = \sum_{i \neq j} R(e_i, e_j, e_i, e_j)$, for some orthonormal basis $\{e_i\}_{i=1,\dots,n}$ of T_xX.

It is known (see for instance [10]) that a manifold which admits a metric with constant sectional curvature has universal covering which is diffeomorphic either to the standard n-sphere S^n or to $\mathbb{R}^n$. Moreover, after renormalization, such a metric is locally isometric to only one of the three classical models: S^n with its canonical metric, the Euclidian space or the real hyperbolic space. The manifolds which admit a metric of constant sectional curvature, therefore, seem to constitute a very restricted class.

On the other hand, the condition of constancy of the scalar curvature is too weak to characterize any metric: actually, by the solution of the Yamabe problem (due to Yamabe, Trudinger, Aubin, Schoen and Yau), it has been showed that for every compact manifold X and every metric g on X, there exists a function f on X such that $e^f \cdot g$ is a metric with constant scalar curvature. Since the conformal classes of metrics, on a given manifold, form an infinite dimensional family, we see that the constant scalar curvature metrics are too many to be considered “distinguished”.

Then, one is naturally brought to consider the notion of *constant Ricci curvature.*

1.2 A metric which satisfies the condition $\mathrm{Ric}(g) = \lambda \cdot g$ for some constant λ (equivalently, such that the Ricci curvature in every direction and in every point is constant) is said to be an *Einstein metric.* A manifold which admits an Einstein metric is called an *Einstein manifold.* The constant λ is unessential, since, by rescaling the metric, one can always assume to have $\lambda = -1, 0$ or 1, so that only the sign of λ is important. It is called the *sign* of the Einstein metric.

In dimension $n = 2, 3$ this notion coincides with that of metric of constant sectional curvature (see [4]) but, in higher dimensions, as we shall see in section 2, Einstein metrics constitute a quite larger class, which seems already to lead to a promising notion.

1.3 From the algebraic viewpoint, the Einstein condition in 1.2 is translated into a simpler expression for the curvature tensor R.

A metric g on a manifold X gives rise to an action of the orthogonal group on each tangent space T_xX and, consequently, on all associated vector spaces (tensor, symmetric and exterior powers of T_xX). In particular, one deduces an action on the vector spaces $\mathcal{C}_xX$ of the *algebraic curvature tensors*, that is the subspace of tensors of $S^2\bigwedge^2 T_xX$ which satisfy the formal Bianchi identity (i.e. the tensors of the same type and which satisfy the same algebraic properties of the curvature tensor). As a result, one has a decomposition of $\mathcal{C}_xX = \mathcal{U}_xX \oplus \mathcal{Z}_xX \oplus \mathcal{W}_xX$ into irreducible representations of the orthogonal group. With respect to this decomposition, R splits into a sum $R = U + Z + W$, where

- $U = \frac{\mathrm{scal}(g)}{2n(n-1)} \cdot g \odot g$ is a tensor involving the scalar curvature;
- $Z = \frac{1}{n-2}(\mathrm{Ric}(g) - \frac{\mathrm{scal}(g)}{n} g) \odot g$ involves the trace-free part of the Ricci tensor;
- W is the *Weyl tensor* of g.

(Here, $\odot$ stands for the Kulkarni-Nomizu product of two symmetric 2-tensors, see [4]).

Consequently, Einstein metrics are characterized by the vanishing of the tensor $Z(g)$. This fact plays an important role in dimension 4, because of the expression of the Gauss-Bonnet formula in this dimension (see 4.2).

1.4 Let $\mathcal{S}^2X$ denote the set of all (differentiable) symmetric 2-tensor fields on a compact manifold X and let $\mathcal{M}_1X \subset \mathcal{S}^2X$ be the (infinite dimensional) differentiable manifold made up of all metrics g on X with fixed volume $\mathrm{Vol}(g) = 1$.

Consider the functional *total scalar curvature* $S : g \mapsto \int_X \mathrm{scal(g)}dv_g$, defined on $\mathcal{M}_1X$. It can be shown [4] that S is differentiable and that

$$S'_g(h) = \Big\langle \frac{\mathrm{scal}(g)}{2} g - \mathrm{Ric}(g), h \Big\rangle_g$$

(where $\langle h, k\rangle_g = \int_X \mathrm{Tr}_g(H \circ K)dv_g$, if H and K are the symmetric endomorphisms associated to h, k via g). Since $T_g\mathcal{M}_1 = \{h \in \mathcal{S}^2X,\ \langle h, g\rangle_g = 0\}$ one sees that g is a critical point for S if and only if $\mathrm{Ric}(g)$ is proportional in every point to g. This last condition is easily seen to be equivalent to the Einstein condition, in dimension greater than 2 (see [4]).

So, Einstein metrics correspond exactly to the critical points of the total scalar curvature functional.

2 – Examples

2.1 The first examples of Einstein metrics are given by the so called *two point homogeneous spaces*. These are the Riemannian manifolds whose isometry group acts transitively on the unit tangent bundle: consequently, the Ricci curvature in all directions is constant. This class is known [25] to consist of all the symmetric spaces of rank one: namely, beyond the constant curvature spaces $\mathbb{E}^n, S^n$ and $\mathbb{R}H^n$, there are the hyperbolic and projective spaces over the field of the complex numbers and of quaternions $\mathbb{C}H^n, \mathbb{C}P^n, \mathbb{H}H^n, \mathbb{H}P^n$ and the Cayley hyperbolic and projective plane $\mathbb{C}aH^2, \mathbb{C}aP^2$.

Clearly, since the Einstein condition is local, any quotient X/G of these spaces, by a subgroup G of isometries acting properly discontinuosly and without fixed points, will become an Einstein manifold, endowed with the metric induced by that of X. When $X = \mathbb{R}H^n, \mathbb{C}H^n$ we call such a quotient, respectively, a *real* or *complex hyperbolic manifold*. Since a complex hyperbolic manifold turns out to be a *complex* manifold, we shall call a complex hyperbolic manifold of real dimension 4 a *complex hyperbolic surface* (complex dimension equal to 2).

2.2 More generally, any homogeneous Riemannian manifold $X = G/H$ (i.e. a Riemannian manifold X whose isometry group acts transitively on X), such that the isotropy representation of H into T_oX is irreducible (where o is the base point of $X = G/H$), is Einstein.

In fact, since H acts by isometries on X, its action on S^2T_oX leaves fixed both the metric g and the Ricci tensor $\mathrm{Ric}(g)$ in o. Since g is positive definite, from the Schur lemma it follows that the space of invariant bilinear forms on T_oX has dimension 1, so g and $\mathrm{Ric}(g)$ are proportional (in o, hence everywhere).

In particular, all irreducible Riemannian symmetric spaces are Einstein, since their adjoint representation is irreducible (see [4], pp. 201-202 for a complete list). Some examples are the Grassmann manifolds.

2.3 (G. Jensen, [14]) Consider the *Hopf fibration* on the quaternion projective space $h : S^{4m+3} \to \mathbb{H}P^m$, defined by sending $u \in S^{4m+3} \subset \mathbb{H}^{m+1}$ into the quaternionic line passing through u. Each fiber is a copy of the group of the units of quaternions, that is diffeomorphic to S^3. With respect to the canonical constant curvature metric of S^{4m+3} and to the symmetric metric of $\mathbb{H}P^m$, the map h is a *Riemannian submersion with totally geodesic fibers.*

One can modify the metric $g = g_{\mathrm{hor}} \oplus g_{\mathrm{vert}}$ along the fibers, obtaining the family of metrics $g_t = g_{\mathrm{hor}} \oplus t \cdot g_{\mathrm{vert}}$ on the sphere S^{4m+3}.

Now, motivated by 1.4, one looks for the critical points of

$$S(t) = \frac{1}{\mathrm{Vol}(g_t)^{(4m+1)/(4m+3)}} \int_{S^{4m+3}} \mathrm{scal}(g_t) dv_{g_t} .$$

For $t_0 = 1/(2m+3)$ one finds that g_{t_0} is Einstein.

The metric g_{t_0} is still a homogeneous metric but it is not isometric to the canonical metric of S^{4m+3}. In 1982 W. Ziller [32] showed that, for $m \neq 3$, every homogeneous Einstein metric on S^{4m+3} is homothetic to g_{t_0} or to the canonical one, while S^{15} has 3 different homogeneous Einstein metrics.

2.4 Non-homogeneous Einstein metrics on compact manifolds are more difficult to find. A well known example was found in 1979 by

D. Page [5], who constructed an Einstein metric on the connected sum $\mathbb{C}P^2\sharp\overline{\mathbb{C}P^2}$ (where $\overline{\mathbb{C}P^2}$ denotes $\mathbb{C}P^2$ endowed with the orientation opposite to that given by the complex structure). Recall that this manifold is diffeomorphic to the complex surface obtained by blowing up $\mathbb{C}P^2$ at a point. On the contrary, we shall see in 4.2 that blowing up $\mathbb{C}P^2$ at more than 8 points gives a complex surface which does not admit any Einstein metric.

2.5 We discuss below some relevant non-homogeneous examples of Einstein metrics which concern kähler geometry: they come from the Yau and Aubin solution (see [7]) of the Calabi conjecture (which exploits hard analysis machineries). For the sequel, every manifold is meant to have (real) dimension greater than two.

A *kähler manifold* can be described as a Riemannian manifold (X, g) such that

1) X has a complex structure J;
2) the tensor J is an isometry, i.e. $g(Ju, Jv) = g(u, v)$;
3) the 2-form defined by $\omega_g(u, v) = g(Ju, v)$ is parallel (hence closed).

A Hermitian metric, corresponding to g, will be given by $G = g + i\omega_g$; G or, equivalently, g are called *kähler metrics*. One says that a manifold X is of kähler type if it admits a kähler metric. A *Kähler-Einstein manifold* is a kähler manifold (X, g) such that g is an Einstein metric.

Every kähler manifold (X, g) has the property that its first Chern class $c_1(X)$ is represented, in De Rham cohomology, by $\frac{1}{2\pi}\widetilde{\mathrm{Ric}}(g)$, where $\widetilde{\mathrm{Ric}}(g)$ is the *Ricci form* of g (that is, the closed, real 2-form of type $(1,1)$ associated to the Ricci tensor: $\widetilde{\mathrm{Ric}}(g)(u, v) = \mathrm{Ric}(g)(Ju, v)$). Yau proved (Calabi conjecture, [7]) that, given a compact complex manifold X of kähler type, every closed real 2-form of type $(1,1)$ in the De Rham cohomology class of $c_1(X)$ is equal to $\frac{1}{2\pi}\widetilde{\mathrm{Ric}}(g)$, for some kähler metric g on X; moreover, for each fixed kähler metric g_0 on X, every $(1,1)$-form in $c_1(X)$ is equal to $\frac{1}{2\pi}\widetilde{\mathrm{Ric}}(g)$ for a *unique* kähler metric g such that ω_g is cohomologous to ω_{g_0}. As a consequence we have

Theorem 2.5.1 (S.T. Yau). *A compact complex manifold X of kähler type has a Ricci flat Kähler-Einstein metric if and only if $c_1(X)=0$.*

For instance, hypersurfaces of degree $n+1$ of $\mathbb{C}P^n$ have zero first Chern class (see [5]), so they admit a Ricci flat Kähler-Einstein metric (which never is the metric induced by the Fubini-Study kähler metric of $\mathbb{C}P^n$, whose restriction is not Einstein [15]).

The first Chern class of a complex manifold X is said to be *negative* (resp. *positive*) if it can be represented by a real form γ of type (1,1) that is negative definite (resp. positive definite). (This means that the corresponding symmetric bilinear form $\gamma(u, Jv)$ is negative or positive definite). We see that *a necessary condition for a compact kähler manifold to have a Kähler-Einstein metric is that its first Chern class has a sign or is zero*: this follows simply from the fact that, if g is a Kähler-Einstein metric on X, then $\frac{1}{2\pi}\widetilde{\mathrm{Ric}}(g) = \frac{1}{2\pi}\lambda\tilde{g}$ represents $c_1(X)$.

The same techniques used by Yau to prove the Calabi conjecture also give

Theorem 2.5.2 [7]. *A compact complex manifold X of kähler type has a Kähler-Einstein metric of negative sign if and only if $c_1(X)$ is negative definite.*

The analogue of 2.5.3 in the case where $c_1(X)$ is positive is false. Actually, there are other obstructions to the existence of Kähler-Einstein metrics on compact complex manifolds, which are not always trivial when $c_1(X) > 0$ (see [4], [9]).

For instance, the identity component $\mathrm{Aut}_0(X)$ of the group of biholomorphic transformations of a compact Kähler-Einstein manifold X must be reductive (i.e. the Lie algebra of $\mathrm{Aut}_0(X)$ is the direct sum of its center and its commutator Lie subalgebra). This shows that $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$, which is a compact kähler surface with positive first Chern class (see [31]), admits no Kähler-Einstein metric because $\mathrm{Aut}(\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2})$ is not reductive (p. 331, [4]). (Watch out that the Einstein metric mentioned in 2.4, constructed by D. Page on $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$, is not kähler!).

Some sufficient conditions for the existence of Kähler-Einstein metrics on compact complex manifolds with positive first Chern class are discussed in [1], [29] and, recently, in [22].

From now on, we shall be concerned only about *compact* manifolds.

3 – Moduli space of Einstein structures

3.1 We have seen in 1.1 that, in dimension greater than 2, the set of metrics of constant sectional curvature on a given manifold is "generally" empty, while the set of metrics of constant scalar curvature is too big, actually infinite dimensional. Another cause of considerable interest in studying Einstein metrics is the fact that the metrics of constant Ricci curvature on a given manifold give rise to a reasonable space.

Namely, the *Moduli Space* $\mathcal{E}(X)$ *of Einstein structures on a compact manifold* X is defined by the set of equivalence classes of Einstein metrics on X, where two metrics g, g' are considered equivalent (i.e. they define the same *Einstein structure*) if g is isometric to $\lambda \cdot g'$ for some positive constant λ (such an isometry is called a *homothety* between g and g').

A theorem due to N. Koiso [16] says that, in a neighbourhood of an Einstein structure $[g]$, $\mathcal{E}(X)$ is a real analytic subset of a smooth real analytic manifold of finite dimension.

3.2 The only examples where the moduli space $\mathcal{E}(X)$ is known are in dimension 4 (apart from real surfaces).

We have for instance the case of the torus T^4 (and his quotients). Here, every Einstein metric is easily seen to be flat, from the Gauss-Bonnet formula, since the Euler characteristic of T^4 is zero (compare with 4.2). Then, one can show that $\mathcal{E}(T^4)$ is the quotient, by a discrete group, of a convex, open subset of a vector space of dimension 9.

3.3 Another relevant case is that of the $K3$-surfaces. These are simply connected complex manifolds of kähler type, such that the first Chern class vanishes. So, by 2.5.1, they are Ricci flat Kähler-Einstein manifolds. The $K3$-surfaces are known to be each diffeomorphic to the other (but they are different as *complex* manifolds), so let X denote the unique underlying differentiable manifold. N. Hitchin [13], (compare with 4.2.1) proved that any Einstein metric on X is kähler with respect to some complex structure of X. Starting from this result, it has been shown ([30], [4]) that $\mathcal{E}(X)$ is the quotient, by a discrete group, of an open set of the symmetric space $SO(3,19)/SO(3)\times SO(19)$ (which has dimension 57).

3.4 There are two very interesting cases, which we shall be concerned about later on, of *rigidity* of Einstein metrics, where $\mathcal{E}(X)$ is reduced to a point. In 1994, G. Besson, G. Courtois, S. Gallot [6] proved:

THEOREM 3.4.1. *The only Einstein metric on a compact real hyperbolic* 4*-manifold* (X, g_0) *is the real hyperbolic metric* g_0 (*up to homotheties*).

In the same year C. LEBRUN [18] obtained also the complex analogue:

THEOREM 3.4.2. *The only Einstein metric on a compact complex hyperbolic surface* (*complex dimension* 2) (X, g_0) *is the complex hyperbolic metric* g_0 (*up to homotheties*).

These are the only known cases of uniqueness of Einstein structures on a manifold, and both of them are connected with the rigidity properties of the locally symmetric manifolds (i.e. smooth quotients of symmetric manifolds).

Actually, G. BESSON, G. COURTOIS, S. GALLOT [6] proved the following stronger rigidity property of Einstein metrics: if (Y, g) is a compact Einstein 4-manifold homotopy equivalent to a compact real hyperbolic 4-manifold (X, g_0), then there exists an isometry between (Y, g) and $(X, \lambda \cdot g_0)$, for some $\lambda > 0$.

In contrast with this, the result of uniqueness 3.4.2 only holds for manifolds *diffeomorphic* to a complex hyperbolic surface, since LeBrun's argument involves differential invariants, namely the Seiberg Witten invariants. The question whether there exist, on some complex hyperbolic surface, smooth structures non-diffeomorphic to the canonical one, and Einstein metrics compatible with such structures, remains unsettled.

In the next section we are going to take care of the particular case where $\mathcal{E}(X) = \emptyset$.

4 – Obstructions

4.1 We have seen in 2.5 that there exist some obstructions to the existence of Kähler-Einstein metrics on kähler manifolds. There are also some well known obstructions to the existence of Einstein-metrics of *positive* sign on a given manifold: for instance, one knows that a compact manifold with positive Ricci curvature has compact universal covering, hence finite fundamental group. What we are going to discuss now are the *obstructions to the existence of any Einstein metric* on a given manifold (that is, not only Kähler-Einstein metrics or Einstein metrics of a specified sign). Actually, it will be about *topological* obstructions, that is conditions in terms of the topological invariants.

Somewhat surprisingly, no obstruction is known in dimension greater than 4. Even worst, we cannot answer to the following natural

Question: Does every compact manifold of dimension greater than 4 have an Einstein metric?

The answer to the above question is probably negative, but, so far, no example is known of a compact manifold of dimension greater than 4 which admits no Einstein metric.

We remarked, in 1.2, that in dimensions 2 and 3 the notion of Einstein metric is nothing more than that of metric of constant sectional curvature. So, if on the one hand, in dimension 2, every manifold has an “Einstein” metric, on the other hand we definitely know some examples of 3-manifolds which admits no “Einstein” metric, such as $S^2 \times S^1$ (whose fundamental covering is not diffeomorphic either to $\mathbb{R}^3$ or to S^3). The problem to find which 3-manifolds admit a metric of constant sectional curvature is a deep problem (see [27], [28]) but we shall not be concerned about.

Therefore, in all this section and the following one, we shall restrict our attention to (compact) 4-manifolds.

4.2 In dimension 4, the Gauss-Bonnet formula for the Euler characteristic $\chi(X)$ of a Riemannian manifold (X, g) takes the following expression:

$$\chi(X) = \frac{1}{8\pi^2} \int_X (\| U_g \|^2 - \| Z_g \|^2 + \| W_g \|^2) d\nu_g$$

where U_g, Z_g and W_g are the irreducible components of the curvature tensor, discussed in 1.3. Since $Z_g = 0$ for an Einstein metric g, we see that the Euler characteristic of an Einstein 4-manifold is always non-negative, and it is equal to zero if and only if g is a flat metric. This fact was pointed out in 1961 by M. Berger [3].

If X is an oriented manifold, the Weyl tensor W splits again in a sum of two terms W^+ and W^- corresponding to the further decomposition in irreducible subspaces of the vector space of the algebraic curvature tensors, under the action of $SO(4)$. The signature $\tau(X)$ of X can be then expressed, by the Hirzebruch formula, as

$$\tau(X) = \frac{1}{12\pi^2} \int_X (\| W_g^+ \|^2 - \| W_g^- \|^2) d\nu_g \,.$$

So, more generally, combining the Gauss-Bonnet formula with the Hirzebruch formula one obtains:

Theorem 4.2.1 (J. Thorpe [26] - N. Hitchin [13]).
Let X be a compact oriented manifold of dimension 4.
If $\chi(X) < \frac{3}{2}\,|\tau(X)|$ then X doesn't admit any Einstein metric.
Moreover, if $\chi(X) = \frac{3}{2}\,|\tau(X)|$ then X admits no Einstein metric unless it is either flat or a K3 surface or an Enriques surface or the quotient of an Enriques surface by a free antiholomorphic involution.

The equality case was studied by N. Hitchin. An Enriques surface $\mathcal{E}$ is a holomorphic quotient of a $K3$ surface such that $\pi_1(\mathcal{E}) = \mathbb{Z}_2$, and, as a $K3$ surface, it actually admits Ricci flat Kähler-Einstein metrics, from the Aubin and Yau solution of the Calabi conjecture, since its real first Chern class vanishes (Theorem 2.5.1).

The above theorem enabled to exhibit some examples of 4-manifolds (even simply connected) which don't admit any Einstein metric, with positive, arbitrarily high, Euler characteristic . For instance, it shows that the connected sum $n\mathbb{C}P^2$ of $n \geq 4$ copies of the complex projective plane or the blow-up $\mathbb{C}P^2\sharp n\overline{\mathbb{C}P^2}$ of the complex projective plane at $n > 8$ points admit no Einstein metric.

Let us use the following suitable *convention: for a non-orientable manifold X, $\tau(X) = 0$ by definition.* Then, for a non-orientable manifold X, it is easily seen that Thorpe's theorem remains valid, but it is reduced to the result of Berger.

4.3 In 1982, M. Gromov [11] showed that the Thorpe obstruction condition was not the only obstruction to the existence of Einstein metrics in dimension 4. In order to study the minimal volume problem on a manifold X, he introduced a homotopy invariant, the *simplicial volume* $\| X \|$, which, for closed manifolds, can be defined by

$$\| X \| = \inf_{\sigma=\Sigma_i r_i \sigma_i} \sum_i |\, r_i \,|,$$

where σ runs over all real singular cycles representing the fundamental class of X.

He proved that, if a compact Riemannian manifold (X, g) satisfies $\mathrm{Ric}(g) \geq \frac{1}{n-1} g$, then $\mathrm{Vol}(X, g) \geq c_n \cdot \| X \|$, where c_n is a constant only depending on the dimension n of X. A consequence of this result is:

THEOREM 4.3.1 (M. GROMOV [11]). *Let X be a compact manifold of dimension* 4.

If $\chi(Y) < \frac{1}{2592\pi^2} \| Y \|$ then Y doesn't admit any Einstein metric.

The simplicial volume of manifolds with amenable fundamental group (for instance abelian groups, or groups of subexponential growth) vanishes, so Gromov's theorem says something new only in the case of manifolds with "large" fundamental group. (However, remark that, not every manifold with large fundamental group has non-trivial simplicial volume: for instance, $\Sigma_k \times S^2$, where Σ_k is a surface of genus $k \geq 2$, has fundamental group of exponential growth but trivial simplicial volume (see [11]) so that Gromov's obstruction theorem cannot be invoked; nevertheless, $\Sigma_k \times S^2$ admits no Einstein metric by Thorpe's theorem.)

4.4 In the sequel, we shall consider compact 4-manifolds Y which admit a map of non-zero degree onto some compact real or complex hyperbolic 4-manifold. To treat also non-orientable manifolds, it is worth to consider the *absolute degree* of a map, rather than the usual degree (see [8]).

Given any map $f : Y \to X$, the absolute degree can be defined by $\mathrm{Adeg}(f) = \inf\{G(g)$, for all g homotopic to $f\}$, where $G(g)$ is the geometric degree of g, i.e. the smallest number of connected components of $g^{-1}(D)$, when D runs over all disks D in X such that $g^{-1}(D) \longrightarrow D$ is a (generally non-connected) topological covering (if such disks don't exist, $G(g)$ is set equal to ∞). The absolute degree of a map between oriented manifolds coincides with the absolute value of the usual degree.

With the convention fixed in 4.2 for the signature of a non-orientable manifold, the following results have been recently proved in [23]:

THEOREM 4.4.1. *Let Y be a compact 4-manifold which has a map f of absolute degree* $\mathrm{Adeg}(f) = d > 0$ *onto a real hyperbolic compact*

4-manifold (X, g_0). *If*

$$\chi(Y) - \frac{3}{2} \mid \tau(Y) \mid < d \cdot \Big(\chi(X) - \frac{3}{2} \mid \tau(X) \mid \Big)$$

then Y *doesn't admit any Einstein metric.*

In addition, if the equality $\chi(Y) - \frac{3}{2} \mid \tau(Y) \mid = d \cdot \big(\chi(X) - \frac{3}{2} \mid \tau(X) \mid \big)$ occurs, then Y admits no Einstein metric unless f is homotopic to a $\mid d \mid$-sheeted smooth covering of X (in this case, Y has a real hyperbolic metric, which is the unique Einstein metric on Y up to homotheties, by the result 3.4.1).

Theorem 4.4.2. *Let* Y *be a compact 4-manifold which admits a map* f *of absolute degree* $\operatorname{Adeg}(f) = d > 0$ *onto a compact complex hyperbolic surface* (X, g_0). *If*

$$\chi(Y) - \frac{3}{2} \mid \tau(Y) \mid \leq \frac{4^3}{3^4} \cdot d \cdot \Big(\chi(X) - \frac{3}{2} \mid \tau(X) \mid \Big)$$

then Y *doesn't admit any Einstein metric.*

Both results come from the Gauss-Bonnet and the Hirzebruch formulas, combined with an estimate of the minimal volume of a compact Riemannian n-manifold (Y, g) which admits a degree $d > 0$ map f onto a compact real or complex hyperbolic n-manifold (X, g_0). The minimal volume estimate is due to G. Besson, G. Courtois, S. Gallot [6], and it is extended to non-orientable manifolds in [24].

The above Theorems 4.4.1 and 4.4.2 improve the Thorpe-Hitchin theorem, under the additional *topological* hypothesis of the existence of a map f above described.

Notice that, when Y is a covering of a real hyperbolic manifold X, the inequality of Theorem 4.4.1 is an equality and Y actually admits an (Einstein) real hyperbolic metric, which shows that the result 4.4.1 is optimal. (Remark also that the signature of a hyperbolic 4-manifold is always zero, by the Hirzebruch formula)

On the contrary, Theorem 4.4.2 seems to miss the sharpness because of the constant $4^3/3^4$, in the right hand side, which is smaller than 1.

We shall see in 5.4 that the Theorem 4.4.1 also gives in some cases an improvement of Gromov's inequality (Theorem 4.3.1).

5 – Examples of 4-manifolds without Einstein metrics

5.1 We shall now see how the Theorems 4.4.1 and 4.4.2 provide *new* examples of 4-manifolds without Einstein metrics.

Recall that the *connected sum* $X\sharp M$ of two n-dimensional manifolds X and M is obtained by removing from X and M two n-cells B_X and B_M and then pasting together the resulting boundaries ∂B_X and ∂B_M (see, for instance, [17] for more details).

If $Y = X\sharp M$, there exists a map $Y \to X$, of absolute degree 1, which maps $X \setminus \overline{B_X}$ and $M \setminus B_M$ (considered as subsets of Y) respectively onto $X \setminus \{x_0\}$ and $\{x_0\}$, for some $x_0 \in B_X$. When $Y = kX$ (seen as the gluing of k copies of $X \setminus B_X$ on $S^4 \setminus \bigcup_{i=1}^k B_i$ by identifying each ∂B_X with one of the ∂B_i's), there exists a map $Y \to X$, of absolute degree k, which maps each copy of $X \setminus \overline{B_X}$ and $S^4 \setminus \bigcup_{i=1}^k B_i$, considered as subsets of Y, respectively onto $X \setminus \{x_0\}$ and x_0.

The invariants $\chi(Y), \tau(Y), \| Y \|$ are easily computed from those of X and M (the signature and the simplicial volume are additive with respect to connected sums, while $\chi(X\sharp M) = \chi(X)+\chi(M)-2$ when the dimension is even, see [5], [11]).

Then, as a consequence of the Theorem 4.4.1, one finds:

Corollary 5.1.1 [23]. *Let X and M be 4-dimensional, compact manifolds and suppose X real hyperbolic:*

(i) *if $\chi(M) < 2$, or if $\chi(M) = 2$ and M is not homeomorphic to S^4, then $X\sharp M$ admits no Einstein metric;*

(ii) *if X and M are oriented (M not homeomorphic to S^4) and $\chi(M) - \frac{3}{2} \, |\tau(M)| \leq 2$, then $X\sharp M$ admits no Einstein metric;*

(iii) *the connected sum kX of k copies of X admits no Einstein metric, if $k > 1$.*

5.2 For every real hyperbolic manifold X, the manifold $Y = X\sharp\mathbb{R}P^4$ admits no Einstein metrics, by 5.1.1 (i), since $\chi(\mathbb{R}P^4) = 1$. Nevertheless, if $\chi(X) \geq 2$ (for instance, if X is oriented), then $\chi(Y) \geq 1$, so this result cannot be obtained from Thorpe's obstruction 4.2.1. On the other hand, $\|Y\|=\|X\|$ since $\mathbb{R}P^4$ has trivial simplicial volume (being finitely covered by S^4); then, one verifies that Gromov's obstruction condition is not satisfied by Y if $\chi(X) \geq 2$.

5.3 Let $M_1 = S^1 \times N$ and $M_2 = S^2 \times \Sigma_g$, where N is any compact 3-manifold and Σ_g is a compact oriented surface of genus $g \geq 2$. We have $\chi(M_i) < 2$ so, by Corollary 5.1.1(i), for every real hyperbolic manifold X, $X\sharp M_i$ admits no Einstein metric.

For $X\sharp M_1$, it is straightforward to check, with the formulas quoted in 5.1, that this result cannot be obtained neither from Thorpe's nor from Gromov's obstruction, if $\chi(X) \geq 3$. Analogously, one finds that neither Thorpe's obstruction condition nor Gromov's one hold for $X\sharp M_2$, as soon as $\chi(X) \geq \frac{2}{1-(1/1944\cdot T_4)}(2g-1)$.

Similar examples can be obtained by taking $X\sharp M$, with $M = S^2 \times U_h$, where U_h is the non-orientable compact surface with h crosscaps, or $M = \mathbb{R}P^2 \times \Sigma$ or $M = U_2 \times \Sigma$, where Σ is any compact surface.

5.4 Let us check on a particular case the non-optimality of the constant $\frac{1}{2592\pi^2}$ in Gromov's Theorem 4.3.1.

Take a compact real hyperbolic 4-manifold X and a compact 4-manifold M of trivial simplicial volume (for instance, simply connected), and consider $Y = X\sharp M$. Since $\| M \| = 0$, Gromov's theorem then says that:

$$\text{if } \chi(Y) < \frac{1}{2592\pi^2} \| X \|, \text{ then } Y \text{ admits no Einstein metric.}$$

Now, let T_4 be the volume of the regular 4-dimensional ideal geodesic simplex in the real hyperbolic 4-dimensional space: it is explicitly computable and its value is (see [12]) $T_4 = \frac{10\pi}{3} \arcsin \frac{1}{3} - \frac{\pi^2}{3} \approx 0.26889$. In [11] M. Gromov computed the simplicial volume of any real hyperbolic manifold, e.g. $\| X \| = \frac{1}{T_4} \cdot \mathrm{Vol}(X) = \frac{1}{T_4} \cdot \frac{4\pi^2}{3} \cdot \chi(X)$.

So, the obstruction Theorem 4.4.1 implies that:

$$\text{if } \chi(Y) < \frac{3T_4}{4\pi^2} \| X \|, \text{ then } Y \text{ admits no Einstein metric}$$

which is an optimal result since, for $M = S^4$, $Y = X$ and the above inequality is an equality.

Remark that $\frac{3T_4}{4\pi^2} \ll \frac{1}{2592\pi^2}$.

Some simple examples can be given by taking the connected sum Y of a real hyperbolic 4-manifold X with $\mathbb{C}P^2$ or with a $K3$ surface K (recall that $\chi(\mathbb{CP}^2) = 3, \tau(\mathbb{CP}^2) = 1, \chi(K) = 24, \tau(K) = -16$). In both cases, Y admits no Einstein metric, but Gromov's obstruction condition is not satisfied.

5.5 Let us recall that the complex surface obtained by blowing up h points from a complex surface Y is diffeomorphic to the connected sum of Y with h copies of $\overline{\mathbb{C}P^2}$ (i.e., $\mathbb{C}P^2$ endowed with the opposite orientation to that given by the complex structure).

Corollary 5.5.1 [23]. *Let X be a compact complex hyperbolic surface and let X_k be a k-sheeted covering of X. The complex surface $X_k \sharp h\overline{\mathbb{C}P^2}$, obtained by blowing up h points from the surface X_k, admits no Einstein metric if $h \geq \frac{179}{243} k \cdot c_1^2(X)$.*

One can verify that Gromov's obstruction (Theorem 4.3.1) is not conclusive in this case. Thorpe's obstruction says that there exist no Einstein metrics, on $X_k \sharp h\overline{\mathbb{C}P^2}$, if $\frac{h}{k} \geq c_1^2(X)$; so, every k-sheeted covering X_k of X and integer h such that $\frac{179}{243} c_1^2(X) \leq \frac{h}{k} < c_1^2(X)$ give rise to a complex surface without Einstein metrics, that we cannot deduce neither from Gromov's nor from Thorpe's obstruction theorems. Notice that this gives an infinity of new examples of *complex surfaces without Einstein metrics*: in fact, compact complex hyperbolic surfaces have coverings of arbitrarily high degree [23], so that we can choose k and h arbitrarily high.

This result should be compared to another one, obtained by C. LeBrun [19] for blow-ups of *complex surfaces of general type* (more generally, for blow-ups of symplectic 4-manifolds of general type), by means of the Seiberg-Witten invariants. He finds the sharper constant $\frac{2}{3}$ ($< \frac{179}{243}$), and he exhibits non-minimal (with respect to the blow-up operation), *simply connected* examples of complex surfaces without Einstein metrics, which don't come from Thorpe's obstruction.

Notice that, similarly to the remarks which follow the results 3.4.1 and 3.4.2 in 3.4, the Corollary 5.5.1 excludes the existence of Einstein metrics on any 4-manifold *homotopy equivalent* to one of the complex surfaces X_k above described, while the result of LeBrun only works for manifolds *diffeomorphic* to blow-ups of complex surfaces of general type.

5.6 As an application of Theorems 4.4.1 or 4.4.2, one finds the following result about "genericity" of 4-manifolds which don't admit any Einstein metric:

THEOREM 5.6.1 [23]. *For every compact 4-manifold X, there exists a compact 4-manifold Y which has the same Euler characteristic and the same signature as X, and which doesn't admit any Einstein metric.*

Therefore, one can fill also the region D of the $\mathbb{Z}$-plane defined by $D = \{k > \frac{3}{2}|t|\}$ (in which Thorpe's obstruction condition is not satisfied) with 4-manifolds without Einstein metrics.

Actually, by taking suitable connected sums with hyperbolic manifolds, one can prove that, for each (k, t) such that $k - t \in \mathbb{Z}$, there exist *infinitely many non-homeomorphic* manifolds which have Euler characteristic k and signature t, and which don't admit any Einstein metric (see [23]).

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