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Almost Kähler manifolds whose antiholomorphic sectional curvature is pointwise constant

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RIASSUNTO: *Si dimostra che ogni varietà almost Kähler* (M, g, J) *connessa, con* $\dim M \geqq 8$ *e curvatura sezionale antiolomorfa puntualmente costante è una varietà di Kähler di curvatura sezionale olomorfa costante.*

ABSTRACT: *We prove that an almost Kähler manifold* (M, g, J) *with* $\dim M \geqq 8$ *and pointwise constant antiholomorphic sectional curvature is a complex space-form.*

1 – Introduction and preliminaries

Let $(M,\ g,\ J)$ be a $2n$-dimensional almost Hermitian manifold. A 2-plane α in the tangent space T_xM at a point x of M is antiholomorphic if it is orthogonal to $J\alpha$.

The manifold $(M,\ g,\ J)$ has pointwise constant antiholomorphic sectional curvature (p.c.a.s.c.) ν if, at any point x, the Riemannian sectional curvature $\nu(x) = K_x(\alpha)$ is independent on the choice of the antiholomorphic 2-plane α in T_xM.

If (g, J) is a Kähler structure, the previous condition means that (M, g, J) is a complex space-form, i.e. a Kähler manifold with constant

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holomorphic sectional curvature $\mu = 4\nu$ ([2]). Moreover, the Riemannian curvature tensor R satisfies:

$$R = \nu(\pi_1 + \pi_2)\,, \tag{1.1}$$

ν being a constant function and π_1, π_2 the tensor fields such that:

$$\begin{aligned} \pi_1(X,Y,Z,W) &= g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\,; \\ \pi_2(X,Y,Z,W) &= 2g(JX,Y)g(JZ,W) + g(JX,Z)g(JY,W) + \\ &\quad - g(JY,Z)g(JX,W)\,. \end{aligned} \tag{1.2}$$

According to [16], for any $(0,2)$-tensor field S, we consider the $(0,4)$-tensor fields $\phi(S)$, $\psi(S)$ defined by:

$$\begin{aligned} \phi(S)(X,Y,Z,W) &= g(X,Z)S(Y,W) + g(Y,W)S(X,Z) + \\ &\quad - g(X,W)S(Y,Z) - g(Y,Z)S(X,W)\,, \\ \psi(S)(X,Y,Z,W) &= 2g(X,JY)S(Z,JW) + 2g(Z,JW)S(X,JY) + \\ &\quad + g(X,JZ)S(Y,JW) + g(Y,JW)S(X,JZ) + \\ &\quad - g(X,JW)S(Y,JZ) - g(Y,JZ)S(X,JW)\,. \end{aligned} \tag{1.3}$$

A generalization of (1.1) is obtained by G. Ganchev ([5]). In fact, he proves that the almost Hermitian manifold (M,g,J) has p.c.a.s.c. ν iff

$$R = \frac{1}{2(n+1)}\psi(\rho^*(R)) + \nu\pi_1 - \frac{2(n+1)\nu + \tau^*(R)}{2(n+1)(2n+1)}\pi_2 \tag{1.4}$$

$\rho^*(R)$, $\tau^*(R)$ respectively denoting the $*$-Ricci tensor and the $*$-scalar curvature.

The previous formula allows to relate the symmetric part of $\rho^*(R)$ to the Ricci tensor $\rho(R)$ and thus $\tau^*(R)$ to the scalar curvature $\tau(R)$.

Indeed, putting

$$L_3R(X,Y,Z,W) = R(JX,JY,JZ,JW)\,, \tag{1.5}$$

one has: $\rho^*(R + L_3R)(X,Y) = \rho^*(R)(X,Y) + \rho^*(R)(Y,X) = \rho^*(R + L_3R)(JX,JY)$, and (1.4) implies:

$$\rho^*(R + L_3R) = \frac{2}{3}(n+1)\rho(R) - \frac{(n+1)\tau(R) - 3\tau^*(R)}{3n}g\,, \tag{1.6}$$

$$8n(n^2-1)\nu = (2n+1)\tau(R) - 3\tau^*(R)\,. \tag{1.7}$$

Another characterization of the p.c.a.s.c. condition can be obtained regarding the Riemannian curvature tensor as a section of the vector bundle $\mathcal{R}(M)$ of the algebraic curvature tensor fields on M. According to the splitting $\mathcal{R}(M) = \bigoplus_{1\leqq i\leqq 10} \mathcal{W}_i(M)$ considered in [16], the formula (1.4) can be interpreted in terms of the vanishing of suitable $\mathcal{W}_i$-projections $p_i(R)$ of R.

More precisely, an application of the Theorem 8.1 in [16] yields to the following result.

Proposition 1.1. *Let (M, g, J) be an almost Hermitian manifold. If* $\dim M = 4$*, (M, g, J) has p.c.a.s.c. iff* $p_3(R) = p_7(R) = p_8(R) = 0$*. If* $\dim M \geqq 6$*, then (M, g, J) has p.c.a.s.c. iff* $p_3(R) = p_6(R) = p_7(R) = p_8(R) = p_{10}(R) = 0$ *and* (1.6) *holds.*

Combining with the Theorem 8.1 in [4], one has:

Proposition 1.2. *Let (M, g, J) be an almost Hermitian manifold with p.c.a.s.c. Then g is an Einstein metric iff (M, g, J) has pointwise constant holomorphic sectional curvature.*

The classification of the almost Hermitian manifolds with p.c.a.s.c. is still an open problem, even if nowadays several partial results are known.

In [1] V. Apostolov, G. Ganchev and S. Ivanov classify the compact Hermitian surfaces with constant antiholomorphic sectional curvature. Moreover, they, construct an example of conformal Kähler surface with p.c.a.s.c. ν, the function ν being non-constant. Thus, the Schür's lemma of antiholomorphic type is not valid in the 4-dimensional case.

Furthermore, the third author of the present paper has already solved the above-mentioned problem for $2n$-dimensional, $n \geqq 3$, connected, $\mathcal{R}_3$-manifolds, i.e. almost Hermitian manifolds such that $R = L_3 R$ (equivalently, $p_8(R) = p_9(R) = p_{10}(R) = 0$).

In fact, any connected $\mathcal{R}_3$-manifold M with p.c.a.s.c. and $\dim M \geqq 6$ has constant antiholomorphic sectional curvature ([9]) and turns out to be a real space-form or a complex space-form ([11]).

This results allows the classification of nearly Kähler as well as locally conformal Kähler manifolds with p.c.a.s.c. In fact, any nearly Kähler manifold is a $\mathcal{R}_3$-manifold ([7]). Since for a locally conformal Kähler manifold the projections $p_9(R)$ vanishes, the locally conformal Kähler manifolds with p.c.a.s.c. turn out to be $\mathcal{R}_3$-manifolds ([3]).

Moreover, combining the results stated in [10] and [13], any connected $\mathcal{R}_3$-almost Kähler manifold with p.c.a.s.c. and $\dim M \geqq 6$ turns out to be a complex space-form.

Since the projection $p_9(R)$, a priori, does not vanish in the almost Kähler case the classification of the almost Kähler manifolds with p.c.a.s.c. is meaningful.

We recall the almost Kähler condition, i.e.:

$$\underset{(V,X,Y)}{\sigma}(\nabla_V \omega)(X,Y) = 0\,, \tag{1.8}$$

σ denoting the cyclic sum and $\nabla\omega$ the covariant derivative of the fundamental 2-form ω ($\omega(X,Y) = g(JX,Y)$) with respect to the Levi-Civita connection ∇.

Moreover, (1.8) implies:

$$(\nabla_X J)Y + (\nabla_{JX} J)JY = 0\,; \tag{1.9}$$

$$\sum_i (\nabla_{e_i} J)e_i = 0\,, \tag{1.10}$$

for any local orthonormal frame $\{e_i\}_{1\leqq i\leqq 2n}$.

In this paper, we state the following theorem, whose proof is divided into several steps.

THEOREM 1. *Let (M,g,J) be a $2n$-dimensional, $n \geqq 4$, connected, almost-Kähler manifold. If (M,g,J) has pointwise constant antiholomorphic sectional curvature, then (M,g,J) is a complex space-form.*

2 – Some auxiliary lemmas

Given a $2n$-dimensional almost Hermitian manifold (M,g,J), the tensor field

$$Q = \frac{1}{6}\rho(R) + \frac{1}{4(n+1)}\rho^*(R - L_3 R) \tag{2.1}$$

is,in general, neither symmetric nor skew-symmetric, since $\rho(R)$, $\rho^*(R-L_3R)$ respectively determine its symmetric, skew-symmetric components. Moreover, we assume that (M,g,J) has p.c.a.s.c.; then the formula (1.6) implies:

$$Q(JX,JY)=Q(Y,X)\,, \tag{2.2}$$

and thus one has:

$$Q((\nabla_V J)X,JY)=Q(Y,(\nabla_V J)JX))\,; \tag{2.3}$$

$$(\nabla_V Q)(JX,JY)=(\nabla_V Q)(Y,X)-Q((\nabla_V J)X,JY)+ \\ -Q(JX,(\nabla_V J)Y)\,; \tag{2.4}$$

$$\sum_i Q((\nabla_V J)e_i,Je_i)=-\sum_i Q(Je_i,(\nabla_V J)e_i)\,, \tag{2.5}$$

for any local orthonormal frame $\{e_i\}_{1\leqq i\leqq 2n}$.

If (g,J) is an almost Kähler structure, (1.8) and (1.9) imply also:

$$\sum_i Q(V,e_i)(\nabla_{e_i}\omega)(Y,X)=Q(V,(\nabla_X J)Y-(\nabla_Y J)X)\,; \tag{2.6}$$

$$2\sum_i Q(Je_i,(\nabla_{e_i}J)V)=\sum_i Q(Je_i,(\nabla_V J)e_i)\,. \tag{2.7}$$

Now, we observe that (2.1), (1.6) and (1.7) allow to rewrite (1.4) as follows:

$$R=\psi(Q)+\nu\pi_1-\frac{2n-1}{3}\nu\pi_2\,. \tag{2.8}$$

By means of (2.8) and the second Bianchi identity, we'll state some properties of Q and ∇Q useful for the proof of the Theorem 1.

First of all, from (2.8), one has:

$$\begin{aligned}(\nabla_V R)(X,Y,Z,W)&=2g(X,JY)\{(\nabla_V Q)(Z,JW)+\\ &+Q(Z,(\nabla_V J)W)\}+2g(Z,JW)\{(\nabla_V Q)(X,JY)+\\ &+Q(X,(\nabla_V J)Y)\}+g(X,JZ)\{(\nabla_V Q)(Y,JW)+\\ &+Q(Y,(\nabla_V J)W)\}+g(Y,JW)\{(\nabla_V Q)(X,JZ)+\\ &+Q(X,(\nabla_V J)Z\}-g(Y,JZ)\{(\nabla_V Q)(X,JW)+\\ &+Q(X,(\nabla_V J)W)\}-g(X,JW)\{(\nabla_V Q)(Y,JZ)+\end{aligned} \tag{2.9}$$

$$
\begin{aligned}
&+ Q(Y, (\nabla_V J)Z)\} + 2(\nabla_V \omega)(Y, X)Q(Z, JW)+\\
&+ 2(\nabla_V \omega)(W, Z)Q(X, JY) + (\nabla_V \omega)(Z, X)Q(Y, JW)+\\
&+ (\nabla_V \omega)(W, Y)Q(X, JZ) - (\nabla_V \omega)(Z, Y)Q(X, JW)+\\
&- (\nabla_V \omega)(W, X)Q(Y, JZ)+\\
&+ V(\nu)\Big(\pi_1 - \frac{2n-1}{3}\pi_2\Big)(X, Y, Z, W)+\\
&- \frac{2n-1}{3}\nu\{2g(X, JY)(\nabla_V \omega)(W, Z)+\\
&+ 2g(Z, JW)(\nabla_V \omega)(Y, X) + g(X, JZ)(\nabla_V \omega)(W, Y)+\\
&+ g(Y, JW)(\nabla_V \omega)(Z, X) - g(X, JW)(\nabla_V \omega)(Z, Y)+\\
&- g(Y, JZ)(\nabla_V \omega)(W, X)\}\,.
\end{aligned}
$$

LEMMA 2.1. *Let (M, g, J) be a $2n$-dimensional $(n \geqq 2)$ almost-Kähler manifold with p.c.a.s.c. ν. The covariant derivative ∇Q is given by:*

$$
\begin{aligned}
2(n+1)(2n-1)(\nabla_V Q)(X, JY) = {}& (2n+3)(Q(Y, (\nabla_X J)V)+\\
&- Q(X, (\nabla_Y J)V)) + (4n+3)Q(V, (\nabla_X J)Y - (\nabla_Y J)X)+\\
&- Q(Y, (\nabla_V J)X) - (4n^2+2n-3)Q(X, (\nabla_V J)Y)+\\
&+ g(X, JY)\Big\{2n\sum_i Q(Je_i, (\nabla_V J)e_i)+\\
&+ \frac{4}{3}(n+1)(n-2)V(\nu) + \frac{2n-1}{6}V(\tau(R))\Big\}+\\
&+ g(X, JV)\Big\{\frac{4n-1}{2}\sum_i Q(Je_i, (\nabla_Y J)e_i)+\\
&- \frac{2}{3}(n+1)(2n^2-4n+3)Y(\nu) + \frac{2n-1}{6}Y(\tau(R))\Big\}+\\
&- g(Y, JV)\Big\{\frac{4n-1}{2}\sum_i Q(Je_i, (\nabla_X J)e_i)+\\
&- \frac{2}{3}(n+1)(2n^2-4n+3)X(\nu) + \frac{2n-1}{6}X(\tau(R))\Big\}+\\
&- 2(n+1)\{JX(\nu)g(Y, V) - JY(\nu)g(X, V)\}+\\
&+ \frac{1}{3}(n+1)(\tau(R) - 2(2n-1)^2\nu)(\nabla_V \omega)(X, Y)\,,
\end{aligned}
\tag{2.10}
$$

where $\{e_i\}_{1 \leqq i \leqq 2n}$ is a local orthonormal frame.

Proof. In fact, by the second Bianchi identity, we have:

$$\underset{(V,X,Y)}{\sigma}\sum_i(\nabla_V R)(X,Y,e_i,Je_i)=0\,,$$

which, combined with (2.9), (2.4) and (1.8), yields to:

$$\begin{aligned}2(n+1)\underset{(V,X,Y)}{\sigma}(\nabla_V Q)(X,JY)&=\underset{(V,X,Y)}{\sigma}\Big\{Q(Y,(\nabla_V J)X)+\\&-(2n+3)Q(X,(\nabla_V J)Y)+g(JX,Y)\Big[\frac{1}{6}V(\tau(R))+\\&-\frac{4}{3}(n^2-1)V(\nu)+\sum_i Q(Je_i,(\nabla_V J)e_i)\Big]\Big\}\,.\end{aligned}\tag{2.11}$$

Moreover, by the second Bianchi identity, we obtain:

$$\sum_{i,q}\{2(\nabla_{e_i}R)(V,e_q,Je_i,Je_q)-(\nabla_V R)(e_i,e_q,Je_i,Je_q)\}=0\,,$$

which, combined with (2.9), (2.5) and (2.7) implies:

$$\begin{aligned}\sum_i(\nabla_{e_i}Q)(V,e_i)&=\frac{4n+1}{4(n+1)}\sum_i Q(Je_i,(\nabla_V J)e_i)+\\&+\frac{n}{6(n+1)}V(\tau(R))-\frac{2}{3}(n-1)^2V(\nu)\,.\end{aligned}\tag{2.12}$$

This formula, with (2.4), (2.6), (1.10) and the condition

$$\sum_i\{(\nabla_V R)(e_i,Je_i,X,Y)+2(\nabla_{e_i}R)(Je_i,V,X,Y)\}=0$$

yields to:

$$\begin{aligned}&2n(\nabla_V Q)(X,JY)+(\nabla_X Q)(Y,JV)+(\nabla_Y Q)(V,JX)=2Q(V,(\nabla_X J)Y)+\\&\quad-3Q(V,(\nabla_Y J)X)-2nQ(X,(\nabla_V J)Y)-Q(X,(\nabla_Y J)V)+\\&\quad+g(X,JY)\Big\{\frac{2n-1}{2(n+1)}\sum_i Q(Je_i,(\nabla_V J)e_i)+\frac{2}{3}(2n-3)V(\nu)+\end{aligned}$$

$$
\begin{aligned}
&+\frac{n-1}{6(n+1)}V(\tau(R))\Big\}+g(V,JY)\Big\{\frac{4n+1}{4(n+1)}\sum_i Q(Je_i,(\nabla_X J)e_i)+\\
&-\frac{1}{3}(2n^2-2n+1)X(\nu)+\frac{n}{6(n+1)}X(\tau(R))\Big\}+\\
&-g(V,JX)\Big\{\frac{4n+1}{4(n+1)}\sum_i Q(Je_i,(\nabla_Y J)e_i)+\\
&-\frac{1}{3}(2n^2-2n+1)Y(\nu)+\frac{n}{6(n+1)}Y(\tau(R))\Big\}+\\
&-JX(\nu)g(Y,V)+JY(\nu)g(X,V)+\\
&+\frac{1}{6}(\tau(R)-2(2n-1)^2\nu)(\nabla_V\omega)(X,Y)\,.
\end{aligned}
$$

Thus, combining with (2.11), one proves the statement. □

LEMMA 2.2. *In the hypothesis of the Lemma 2.1, when $n \geqq 3$, one has:*

$$
\sum_i Q(Je_i,(\nabla_V J)e_i)=\frac{4}{3}(n^2-1)V(\nu)\,. \tag{2.13}
$$

PROOF. In fact, the second Bianchi identity and (2.1) give:

$$
V(\tau(R))=2\sum_i(\nabla_{e_i}\rho(R))(V,e_i)=6\sum_i\{(\nabla_{e_i}Q)(V,e_i)+(\nabla_{e_i}Q)(e_i,V)\}\,.
$$

Moreover, the formulas (2.10), (2.12), (2.7), (1.8), (1.9), (1.10) imply:

$$
\begin{aligned}
\sum_i\{(\nabla_{e_i}Q)(V,e_i)+(\nabla_{e_i}Q)(e_i,V)\}&=\frac{1}{6}V(\tau(R))+\\
&+\frac{n-2}{2n-1}\Big\{\sum_i Q(Je_i,(\nabla_V J)e_i)-\frac{4}{3}(n^2-1)V(\nu)\Big\}\,,
\end{aligned}
$$

and then the statement. □

Proposition 2.1. *In he hypothesis of the Lemma* 2.1, *if* $n \geqq 3$, *one has:*

$$
\begin{aligned}
&4(2n-3)(Q(X,(\nabla_Y J)W) - Q(Y,(\nabla_X J)W))+\\
&\quad + 4n(Q(X,(\nabla_W J)Y) - Q(Y,(\nabla_W J)X))+\\
&\quad - 4(n-3)Q(W,(\nabla_X J)Y - (\nabla_Y J)X) + \tau(R)(\nabla_W \omega)(X,Y)+\\
&\quad - \frac{8}{3}(2n^2-4n+3)(X(\nu)g(JY,W)+\\
&\quad - Y(\nu)g(JX,W) + 2W(\nu)g(X,JY))+\\
&\quad + 8n(n-2)(JX(\nu)g(Y,W)+\\
&\quad - JY(\nu)g(X,W)) = 0\,.
\end{aligned}
\tag{2.14}
$$

Proof. The Lemmas 2.1 and 2.2, the formula (2.4) and the condition:

$$\sum_i \{(\nabla_{e_i} R)(X,Y,e_i,W) + (\nabla_Y R)(e_i,X,e_i,W) - (\nabla_X R)(e_i,Y,e_i,W)\} = 0$$

imply the vanishing of the tensor field S defined by:

$$
\begin{aligned}
S(W,X,Y) &= 4(2n^2-3)\{Q(X,(\nabla_Y J)W) - Q(Y,(\nabla_X J)W)\}+\\
&\quad - 4n\{Q((\nabla_Y J)W,X) - Q((\nabla_X J)W,Y)\}+\\
&\quad + 2(2n^2+3n+3)\{Q(X,(\nabla_W J)Y) - Q(Y,(\nabla_W J)X)\}+\\
&\quad - 2(n+3)\{Q((\nabla_W J)Y,X) - Q((\nabla_W J)X,Y)\}+\\
&\quad + 2(n-3)\{Q(W,(\nabla_X J)Y - (\nabla_Y J)X) - (2n+3)Q((\nabla_X J)Y+\\
&\quad - (\nabla_Y J)X,W)\} + (n+1)\tau(R)(\nabla_W \omega)(X,Y)+\\
&\quad - 4(n+1)(2n^2-2n-3)\{X(\nu)g(JY,W) - Y(\nu)g(JX,W)\}+\\
&\quad + \frac{4}{3}(n+1)(4n^2-4n+3)\{JX(\nu)g(Y,W)+\\
&\quad - JY(\nu)g(X,W) - 2W(\nu)g(X,JY)\}\,.
\end{aligned}
$$

In particular, by means of (1.9) and (2.3), the conditions:

$$
\begin{aligned}
&S(W,X,Y) - S(W,JX,JY) - S(JW,JX,Y) - S(JW,X,JY) = 0\,;\\
&S(W,X,Y) - S(W,JX,JY) + S(JW,JX,Y) + S(JW,X,JY) = 0\,,
\end{aligned}
$$

turn out to be equivalent to:

$$
\begin{aligned}
&2(n-3)\{Q(W,(\nabla_X J)Y-(\nabla_Y J)X)+Q((\nabla_X J)Y+\\
&\quad-(\nabla_Y J)X,W)\}-2(2n-3)\{Q(X,(\nabla_Y J)W)+\\
&\quad+Q((\nabla_Y J)W,X)-Q(Y,(\nabla_X J)W)-Q((\nabla_X J)W,Y)\}+\\
&\quad-2n\{Q(X,(\nabla_W J)Y)+Q((\nabla_W J)Y,X)+\\
&\quad-Q(Y,(\nabla_W J)X)-Q((\nabla_W J)X,Y)\}+\\
&\quad-\tau(R)(\nabla_W\omega)(X,Y)=0\,;
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
&(n-3)\Big\{Q(W,(\nabla_Y J)X-(\nabla_X J)Y)-Q((\nabla_Y J)X+\\
&\quad-(\nabla_X J)Y,W)+\frac{2}{3}(n+1)[X(\nu)g(JY,W)+\\
&\quad-Y(\nu)g(JX,W)+JX(\nu)g(Y,W)-JY(\nu)g(X,W)]\Big\}=0\,.
\end{aligned}
\tag{2.16}
$$

Thus, if $n=3$, the statement follows from (2.15) combined with the condition $S=0$. If $n>3$, (2.16) implies also, with suitable changes of the involved variables, the relation:

$$
\begin{aligned}
&Q((\nabla_W J)X,Y)-Q((\nabla_W J)Y,X)=Q((\nabla_X J)W,Y)+\\
&\quad-Q((\nabla_Y J)W,X)+Q(Y,(\nabla_W J)X-(\nabla_X J)W)+\\
&\quad-Q(X,(\nabla_W J)Y-(\nabla_Y J)W)+\frac{2}{3}(n+1)\Big\{X(\nu)g(Y,JW)+\\
&\quad-Y(\nu)g(X,JW)+2W(\nu)g(X,JY)+\\
&\quad+JX(\nu)g(Y,W)-JY(\nu)g(X,W)\Big\}\,.
\end{aligned}
\tag{2.17}
$$

Thus, applying (2.17) and (2.16), the relation (2.15) yields to:

$$
\begin{aligned}
&6(n-1)\{Q((\nabla_X J)W,Y)-Q((\nabla_Y J)W,X)\}=\\
&\quad=2(n-3)\{Q(X,(\nabla_Y J)W)-Q(Y,(\nabla_X J)W)\}+\\
&\qquad+4n\{Q(X,(\nabla_W J)Y)-Q(Y,(\nabla_W J)X)\}+\\
&\qquad-4(n-3)Q(W,(\nabla_X J)Y-(\nabla_Y J)X)+\\
&\qquad+\tau(R)(\nabla_W\omega)(X,Y)+\frac{4}{3}(n+1)\{(2n-3)(X(\nu)g(JY,W)+\\
&\qquad-Y(\nu)g(JX,W))-2nW(\nu)g(X,JY)\}+\\
&\qquad-4(n+1)\{JX(\nu)g(Y,W)-JY(\nu)g(X,W)\}\,.
\end{aligned}
\tag{2.18}
$$

Moreover, via (2.17), (2.18) and (2.16), with a direct computation, one has:

$$\begin{aligned} S(W,X,Y) = \frac{n(n+1)}{n-1}\Big\{&4(2n-3)(Q(X,(\nabla_Y J)W)+\\ &-Q(Y,(\nabla_X J)W)) + 4n(Q(X,(\nabla_W J)Y)+\\ &-Q(Y,(\nabla_W J)X)) - 4(n-3)Q(W,(\nabla_X J)Y - (\nabla_Y J)X)+\\ &+\tau(R)(\nabla_W \omega)(X,Y) + 8n(n-2)(JX(\nu)g(Y,W)+\\ &-JY(\nu)g(X,W)) + \frac{8}{3}(2n^2-4n+3)(X(\nu)g(Y,JW)+\\ &-Y(\nu)g(X,JW) - 2W(\nu)g(X,JY))\Big\}\,. \end{aligned}$$

Therefore, the vanishing of S implies the statement. □

PROPOSITION 2.2. *In the hypothesis of the Lemma* 2.1, *if* $n \geqq 4$, *one has:*

$$\begin{aligned} Q(X,(\nabla_Y J)V) - Q((\nabla_Y J)V,X) = \frac{2}{3}\{&(2n-1)(Y(\nu)g(JV,X)+\\ &+JY(\nu)g(V,X)) + (n-2)(V(\nu)g(JY,X)+\\ &+JV(\nu)g(Y,X))\}\,. \end{aligned} \tag{2.19}$$

PROOF. We consider the $(0,3)$-tensor field T such that:

$$T(V,X,Y) = Q(V,(\nabla_X J)Y - (\nabla_Y J)X)\,.$$

Since T satisfies:

$$T(V,X,Y) = -T(V,Y,X) = -T(V,JX,JY)\,,$$

T can be regarded as a section of the vector bundle $\mathcal{W}(M)$ whose fibre, at any point x of M, is the linear space $\mathcal{W}_x$ considered in [8].

According to the splitting $\mathcal{W}(M) = \bigoplus_{1\leqq i\leqq 4} \mathcal{W}_i(M)$ defined in [8], we denote by $q_1(T)$ the $\mathcal{W}_1$-projection of T; it is the skew-symmetric tensor field such that:

$$6q_1(T)(V,X,Y) = \underset{(V,X,Y)}{\sigma}(T(V,X,Y) - T(JV,JX,Y))\,.$$

Since $n \geqq 4$, applying (2.16) and then (2.14), one obtains:

$$\begin{aligned}
3q_1(T)(V,X,Y) = & \underset{(V,X,Y)}{\sigma}(Q(V,(\nabla_X J)Y - (\nabla_Y J)X) + \\
& + \frac{2}{3}(n+1)V(\nu)g(X,JY)) = \\
= & \frac{1}{n}\Big\{3Q(V,(\nabla_X J)Y - (\nabla_Y J)X) + \\
& + 3(n-1)(Q(X,(\nabla_Y J)V + \\
& - Q(Y,(\nabla_X J)V)) + \frac{1}{4}\tau(R)(\nabla_V \omega)(X,Y) + \\
& + 2n(n-2)(JX(\nu)g(V,Y) - JY(\nu)g(V,X)) + \\
& + 2(n^2-n+1)(X(\nu)g(JV,Y) - Y(\nu)g(JV,X)) + \\
& - 2(n-1)(n-2)V(\nu)g(X,JY)\Big\}\,.
\end{aligned}$$

Then, the condition: $q_1(T)(V,X,Y) + q_1(T)(JV,JX,Y) = 0$ combined with (1.9), (2.3), (2.16) proves the statement. ∎

3 – The proof of the Theorem 1

To the Riemannian curvature of a manifold satisfying the hypothesis of the Theorem 1, we apply the second Bianchi identity in the form:

$$\begin{aligned}
& \underset{(V,X,Y)}{\sigma}\{(\nabla_V R)(X,Y,Z,W) + (\nabla_V R)(X,Y,JZ,JW)\} + \\
& + \underset{(V,JX,JY)}{\sigma}\{(\nabla_V R)(JX,JY,Z,W) + \\
& + (\nabla_V R)(JX,JY,JZ,JW)\} = 0\,.
\end{aligned} \tag{3.1}$$

The complete expression of the first member in (3.1), evaluated by means of (2.9), is a tensor field which contains four blocks of terms, respectively depending on $g\otimes(\nabla Q + Q(\cdot,\nabla J))$, $\nabla\omega\otimes Q$, $d\nu\otimes(\pi_1 - \frac{2n-1}{3}\pi_2)$, $g\otimes\nabla\omega$.

Since (g,J) is an almost Kähler structure, the whole term in $g\otimes\nabla\omega$ vanishes, while only the skew-symmetric component of Q, i.e. $\rho^*(R - L_3 R)$, is involved in the block depending on $\nabla\omega\otimes Q$.

After a quite long computation, applying the Lemmas 2.1, 2.2 and then the Proposition 2.2, the whole expression in $g\otimes(\nabla Q + Q(\cdot,\nabla J))$

turns out to depend only on $d\nu \otimes g \otimes g$. Thus, the condition (3.1) is equivalent to:

$$
\begin{aligned}
&\frac{3}{4(n+1)}\{\rho^*(R-L_3R)(X,Z)(\nabla_W\omega)(JY,V)+\\
&-\rho^*(R-L_3R)(Y,Z)(\nabla_W\omega)(JX,V)+\\
&-\rho^*(R-L_3R)(JX,Z)(\nabla_W\omega)(Y,V)+\\
&+\rho^*(R-L_3R)(JY,Z)(\nabla_W\omega)(X,V)-\rho^*(R-L_3R)(X,W)\times\\
&\times(\nabla_Z\omega)(JY,V)+\rho^*(R-L_3R)(Y,W)(\nabla_Z\omega)JX,V)+\\
&+\rho^*(R-L_3R)(JX,W)(\nabla_Z\omega)(Y,V)-\rho^*(R-L_3R)(JY,W)\times\\
&\times(\nabla_Z\omega)(X,V)\}-X(\nu)\{\pi_1(V,Y,Z,W)+\pi_1(V,Y,JZ,JW)+\\
&+2g(Y,JV)g(Z,JW)\}+Y(\nu)\{\pi_1(V,X,Z,W)+\\
&+\pi_1(V,X,JZ,JW)+2g(X,JV)g(Z,JW)\}+\\
&-JX(\nu)\{\pi_1(V,JY,Z,W)+\\
&-\pi_1(JV,Y,Z,W)+2g(Y,V)g(Z,JW)\}+\\
&+JY(\nu)\{\pi_1(V,JX,Z,W)+\\
&-\pi_1(JV,X,Z,W)+2g(X,V)g(Z,JW)\}+\\
&+2V(\nu)\{\pi_1(X,Y,Z,W)+\\
&+\pi_1(X,Y,JZ,JW)-2g(X,JY)g(Z,JW)\}+\\
&+2W(\nu)\{\pi_1(X,Y,Z,V)+\\
&+\pi_1(X,Y,JZ,JV)-2g(X,JY)g(Z,JV)\}+\\
&-2Z(\nu)\{\pi_1(X,Y,W,V)+\\
&+\pi_1(X,Y,JW,JV)-2g(X,JY)g(W,JV)\}+\\
&-2JW(\nu)\{\pi_1(X,Y,Z,JV)+\\
&-\pi_1(X,Y,JZ,V)+2g(X,JY)g(Z,V)\}+\\
&+2JZ(\nu)\{\pi_1(X,Y,W,JV)+\\
&-\pi_1(X,Y,JW,V)+2g(X,JY)g(W,V)\}=0\,.
\end{aligned}
\tag{3.2}
$$

First of all, this formula implies that ν is a constant function.

Indeed, given a vector field V, let Y be a vector field such that $g(V,Y)=g(JV,Y)=0$ in an open set. Putting in (3.2) $X=Z=JV$, $W=Y$, one has:

$$\frac{8}{3}(n+1)V(\nu)g(Y,Y)g(V,V)=\rho^*(R-L_3R)(JV,Y)(\nabla_V\omega)(V,Y)+ \\ -\rho^*(R-L_3R)(V,Y)(\nabla_V\omega)(V,JY). \tag{3.3}$$

Therefore, $V(\nu)=0$, if $(\nabla_V J)V=0$.

Assuming that $(\nabla_V J)V$ does not vanish at some point, we consider an open set where $(\nabla_V J)V$ never vanishes and apply (3.3) to a local vector field Y orthogonal to V, JV, $(\nabla_V J)V$, $J((\nabla_V J)V)$. Then, we obtain again: $V(\nu)=0$.

Therefore, one has: $d\nu=0$; hence, since M is connected, ν is a constant function.

Now, the condition (3.2) turns out to be equivalent to the vanishing of the tensor field H defined by:

$$\begin{aligned} H(V,X,Y,Z,W)=\ &\rho^*(R-L_3R)(X,Z)(\nabla_W\omega)(JY,V)+\\ &-\rho^*(R-L_3R)(Y,Z)(\nabla_W\omega)(JX,V)+\\ &-\rho^*(R-L_3R)(JX,Z)(\nabla_W\omega)(Y,V)+\\ &+\rho^*(R-L_3R)(JY,Z)(\nabla_W\omega)(X,V)+\\ &-\rho^*(R-L_3R)(X,W)(\nabla_Z\omega)(JY,V)+\\ &+\rho^*(R-L_3R)(Y,W)(\nabla_Z\omega)(JX,V)+\\ &+\rho^*(R-L_3R)(JX,W)(\nabla_Z\omega)(Y,V)+\\ &-\rho^*(R-L_3R)(JY,W)(\nabla_Z\omega)(X,V)\,. \end{aligned}$$

This implies also the vanishing of the tensor field H' defined by:

$$\begin{aligned} H'(V,X,Y,Z,W)=\ &2(H(V,X,Y,Z,W)+H(V,Z,W,X,Y))+\\ &-H(V,Y,Z,X,W)-H(V,X,W,Y,Z)+\\ &-H(V,Z,X,Y,W)-H(V,Y,W,Z,X)\,. \end{aligned}$$

Then, combining the conditions:

$$H'(V,X,Y,Z,W)=0,\quad H'(V,JX,JY,Z,W)=0,$$
$$H'(JV,X,Y,Z,JW)=0$$

and using (1.9), one has:

$$
\begin{aligned}
&\rho^*(R - L_3R)(X,Z)(\nabla_V\omega)(JY,W)+\\
&\quad -\rho^*(R - L_3R)(Y,Z)(\nabla_V\omega)(JX,W)+\\
&\quad -\rho^*(R - L_3R)(JX,Z)(\nabla_V\omega)(Y,W)+\\
&\quad +\rho^*(R - L_3R)(JY,Z)(\nabla_V\omega)(X,W)=0\,.
\end{aligned}
\tag{3.4}
$$

This implies the Kähler condition, i.e. $\nabla J = 0$. Indeed, if $\nabla J \neq 0$, we consider vector fields Y, V such that $(\nabla_V J)Y$ never vanishes in an open set.
Putting in (3.4) $W = Y$, $X = (\nabla_V J)Y$, one obtains, for any Z, $\rho^*(R - L_3R)(JY,Z)=0$ and also $\rho^*(R-L_3R)(Y,Z)=-\rho^*(R-L_3R)(JY,JZ)=0$. Thus, (3.4) reduces to:

$$\rho^*(R-L_3R)(X,Z)(\nabla_V\omega)(JY,W)-\rho^*(R-L_3R)(JX,Z)(\nabla_V\omega)(Y,W)=0\,,$$

or, equivalently, to:

$$\rho^*(R - L_3R)(X,Z)J((\nabla_V J)Y) + \rho^*(R - L_3R)(JX,Z)(\nabla_V J)Y = 0\,.$$

Therefore, $\rho^*(R - L_3R)$ vanishes. According to [16], this means the vanishing of the projection $p_9(R)$. Since also $p_8(R) = p_{10}(R) = 0$ (see the Proposition 1.1), (M,g,J) turns out to be a $\mathcal{R}_3$-almost Kähler manifold with p.c.a.s.c. Since $\dim M \geqq 8$, a direct application of the classification theorem in [10] implies that (M,g,J) is a Kähler manifold with constant holomorphic sectional curvature. This contradicts the condition $\nabla J\neq 0$. □

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