Rendiconti di Matematica, Serie VII
Volume 18, Roma (1998), 167-179

Curvature properties of solvable extensions of *H*-type groups

I. DOTTI – J. LAURET

RIASSUNTO: *Si studiano alcune proprietà geometriche di estensioni solubili unidimensionali di gruppi di tipo H e di tipo H modificato.*

ABSTRACT: *We study various geometric properties of one dimensional solvable extensions of H-type and modified H-type groups.*

Let M be a rank one symmetric space of non-compact type. If K denotes its sectional curvature, R its curvature tensor and ∇ the riemannian connection, it is well known that $K < 0$ and $\nabla R = 0$. We can represent M as $M = G/H$ where G is the identity component of the isometry group of M and H is the isotropy subgroup at some fixed point $p \in M$. The Iwasawa decomposition $G = HAN$ gives a diffeomorphism between M and $S = AN$ so that M may be viewed as a solvable Lie group with a left invariant metric such that $K < 0$ and $\nabla R = 0$.

Partially supported by grants from CONICOR, SECYTUNC (Argentina) The second author holds a fellowship from CONICET (Argentina).

Il contenuto di questo lavoro è stato oggetto di una conferenza tenuta dal primo Autore, I. Dotti al Convegno "Recenti sviluppi in Geometria Differenziale", Università "La Sapienza", Roma, 11-14 Giugno 1996.

KEY WORDS AND PHRASES: *Curvature – Solvable Lie groups – H-type groups*
A.M.S. CLASSIFICATION: 22E25 – 53C30

In [12] E. Heintze characterized the corresponding solvable Lie algebras as follows.

Let $\mathfrak{s}$ denote the Lie algebra of S and let $\langle\ ,\ \rangle$ denote the inner product on $\mathfrak{s}$ induced by the symmetric metric on M. Then,

(1) $\mathfrak{s}$ admits an orthogonal decomposition $\mathfrak{s}=\mathbf{R}A\oplus\mathfrak{n}_1\oplus\mathfrak{n}_2$ with $\langle A,A\rangle=1$, $\mathfrak{s}'=\mathfrak{n}_1\oplus\mathfrak{n}_2$, $[\mathfrak{s}',\mathfrak{s}']=\mathfrak{n}_2$, $[\mathfrak{s}',\mathfrak{n}_2]=0$ and $\mathrm{ad}_A|_{\mathfrak{n}_i}=\frac{i}{2}I$, $i=1,2$;

(2) the endomorphisms J_Z of $\mathfrak{n}_1$ defined by

$$\langle J_Z U,V\rangle=\langle Z,[U,V]\rangle \quad \forall\, U,V\in\mathfrak{n}_1$$

satisfy $J_Z^2=-\langle Z,Z\rangle I$ for every $Z\in\mathfrak{n}_2$;

(3) for every $V\in\mathfrak{n}_1$, the subspace of $\mathfrak{n}_1$ spanned by $\{V, J_{Z'}V : Z'\in\mathfrak{n}_2\}$ is stable by J_Z, for every $Z\in\mathfrak{n}_2$.

By imitating the above construction of rank one symmetric spaces of non compact type, as solvable Lie groups, one can obtain a larger class of homogeneous manifolds. It is the purpose of this note to consider some possible generalizations of the previous construction and analyze various geometric properties.

In the first section we survey some results on two step nilpotent metric Lie algebras satisfying condition (2) above. This class of nilpotent algebras, known as H-type algebras, was introduced by Kaplan in [13] in connection with the study of hypoelliptic differential equations. The geometry of the nilpotent associated groups is well understood and had provided an interesting source of examples. Among all the possible solvable extensions there is one of particular interest. It is obtained by requiring properties (1) and (2) above, at the Lie algebra level. In the second section we review some known results on this special solvable extensions of H-type groups, known as Damek-Ricci spaces.

Weakening condition (2) above, a larger class of homogeneous nilmanifolds appear. They correspond to new metrics on the same underlying nilpotent groups, but with some remarkable differences. We consider in the third section this generalization, consisting on modifying the H-type metric on the center and we study some curvature properties of their corresponding solvable extension. In particular we discuss the Einstein condition and the moduli space $\mathrm{QP}^*(\mathfrak{s})$ of isometry classes of left-invariant metrics that satisfy $-4\leq K\leq -1$.

1 – H-type algebras

Let $\mathfrak{n}$ be a two-step real nilpotent Lie algebra endowed with an inner product $\langle\,,\,\rangle$. Assume $\mathfrak{n}$ has an orthogonal decomposition $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$, where $\mathfrak{z}$ is a subspace of the center of $\mathfrak{n}$ and $[\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}$. Define a linear mapping $J : \mathfrak{z} \to \mathrm{End}(\mathfrak{v})$ by

$$\langle J_Z X, Y\rangle = \langle Z, [X, Y]\rangle \tag{1}$$

(note that J_Z is skew-symmetric). Now $\mathfrak{n}$ is said to be an *H-type algebra* if for any $Z \in \mathfrak{z}$

$$J_Z^2 = -\langle\, Z, Z\,\rangle I\,. \tag{2}$$

The corresponding H-type group is the simply connected Lie group N with Lie algebra $\mathfrak{n}$ endowed with the left invariant metric induced by the inner product $\langle\,,\,\rangle$ in $\mathfrak{n}$.

It is easily seen that if $\mathfrak{n}$ is H-type and $\mathfrak{z} \neq 0$ then $\mathfrak{z}$ is the center of $\mathfrak{n}$. If $\mathfrak{z} = 0$ then $\mathfrak{n} = \mathfrak{v}$ is abelian.

Main examples. The H-type algebras with $\dim \mathfrak{z} = 0, 1, 3, 7$ are constructed as follows (see [13] or [16]). This family contains the Lie algebras of the Iwasawa N groups associated to real rank one simple Lie groups.

Let $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ or $\mathfrak{o}$, the Cayley numbers. Take $\mathfrak{z} = \mathrm{Im}\,\mathbf{F}$ ($\mathfrak{z} = 0$ if $\mathbf{F} = \mathbf{R}$), $\mathfrak{v} = \mathbf{F}^p \times \mathbf{F}^q$.

Define

$$[X, Y] = \sum_{l=1}^{p} \mathrm{Im}\,\bar{x}_l y_l + \sum_{l=p+1}^{q} \mathrm{Im}\,y_l \bar{x}_l$$

where $X, Y \in \mathfrak{v}$, $X = \sum_{l=1}^{n} x_l E_l$, $Y = \sum_{l=1}^{n} y_l E_l$, x_l, $y_l \in \mathbf{F}$, $n = p + q$ and E_l denotes the element of $\mathbf{F}^n$ with 1 in the l-th position and zero elsewhere.

The inner product on $\mathfrak{z} \oplus \mathfrak{v}$ is given by

$$\langle z + X, u + Y\rangle = \mathrm{Re}\,\bar{z}u + \sum_{l=1}^{n} \mathrm{Re}\,\bar{x}_l y_l$$

for z, $u \in \mathfrak{z}$ and $X, Y \in \mathfrak{v}$.

Finally, it follows from the above definitions, that if $z \in \mathfrak{z}$, J_z is given by

$$J_z \sum_{l=1}^{n} x_l E_l = \sum_{l=1}^{p} x_l\, z\, E_l + \sum_{l=p+1}^{n} z\, x_l\, E_l$$

and the resulting algebra, $\mathfrak{n}(\mathbf{F}, p, q)$, is an H-type algebra.

In [4], Cowling *et al.*, defined and studied the J^2-condition on an H-type algebra (see condition (3) in the Introduction). We now recall its definition.

Given $X \in \mathfrak{v}$, let $J_{\mathfrak{z}}X = \{J_Z X : Z \in \mathfrak{z}\}$. Clearly (1) implies $(J_{\mathfrak{z}}X)^{\perp} = \ker(\mathrm{ad}_X)|\mathfrak{v})$, thus we may consider, for every $X \in \mathfrak{v}$, the orthogonal decomposition

$$\mathfrak{v} = J_{\mathfrak{z}}X \oplus \mathbf{R}X \oplus \mathfrak{w}_X \tag{3}$$

where $\mathfrak{w}_X$ is the orthogonal complement of $\mathbf{R}X$ in $\ker(\mathrm{ad}_X|\mathfrak{v})$.

An H-type algebra $\mathfrak{n}$ *satisfies the* J^2 *condition* if for every $X \in \mathfrak{v}$ the subspace $\mathbf{R}X \oplus J_{\mathfrak{z}}X$ is J_Z-invariant, for all $Z \in \mathfrak{z}$. In particular, if $X \in \mathfrak{v}$ and $Z_1, Z_2 \in \mathfrak{z}$ with $\langle Z_1, Z_2 \rangle = 0$, then there exists $Z_3 \in \mathfrak{z}$ such that $J_{Z_1} J_{Z_2} X = J_{Z_3} X$.

The above property characterizes, among the H-type algebras, the two step nilpotent algebras which are the nilpotent part of the Iwasawa decomposition of a real rank one simple Lie group.

Theorem 1.1 ([4], [16]). *The H-type algebras satisfying the J^2-condition are* $\mathfrak{n}(\mathbf{F}, p, 0)$ *if* $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ *and* $p \in \mathbf{N}$ *or* $\mathfrak{n}(\mathfrak{o}, 1, 0)$.

We now recall the NC-condition (which was introduced in [7]) motivated by the geometry of a solvable extension (see section 2). We will give a characterization of H-type Lie algebras which satisfy it.

An H-type algebra $\mathfrak{n}$ *satisfies the NC-condition* if $[X, J_{Z_1} J_{Z_2} X] \neq 0$ for every non zero $X \in \mathfrak{v}$ and any linearly independent Z_1, Z_2 in $\mathfrak{z}$. Or, equivalently, if the projection $P_{J_{\mathfrak{z}}X}(J_{Z_1} J_{Z_2} X)$ onto $J_{\mathfrak{z}}X$, with respect to the decomposition given by (3), is non zero.

It is clear that H-type algebras which verify the J^2-condition also satisfy the NC-condition. The fact that these conditions are actually equivalent is proved in [7].

Theorem 1.2. *The H-type algebras satisfying the NC-condition are* $\mathfrak{n}(\mathbf{F}, p, 0)$ *if* $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ *and* $p \in \mathbf{N}$ *or* $\mathfrak{n}(\mathfrak{o}, 1, 0)$.

In particular

Corollary 1.3. *In an H-type algebra* $\mathfrak{n}$ *the following conditions are equivalent*

(i) $\mathfrak{n}$ *satisfies the NC-condition*
(ii) $\mathfrak{n}$ *satisfies the J^2-condition.*
(iii) $\mathfrak{n}$ *is the nilpotent part* (*in the Iwasawa decomposition*) *of a real rank one simple Lie algebra.*

2 – Damek-Ricci extensions

The class of solvable extensions of H-type groups which we will consider in this section are constructed as follows. They are modelled on generalizing (1) and (2) in the Introduction.

Let $\mathfrak{n}$ be an H-type algebra with corresponding simply connected Lie group N. If $A = \mathbf{R}^+$ acts on N by the dilations $(z, x) \to (tz, t^{\frac{1}{2}} x)$, we let S be the semidirect product AN. Let $\mathfrak{s}$ be the Lie algebra of S. If D is the derivation of $\mathfrak{n}$ given by $D|_{\mathfrak{v}} = \frac{1}{2} I$ and $D|_{\mathfrak{z}} = I$ and $\mathfrak{a} = \mathbf{R}A$, then $\mathfrak{s}$ is the semi-direct product $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ where $\mathfrak{a}$ acts on $\mathfrak{n}$ via $\mathrm{ad}_A|_{\mathfrak{n}} = D$. We endow $\mathfrak{s}$ with the only inner product extending the given one in $\mathfrak{n}$ and such that $|A| = 1$, $\langle A, \mathfrak{n} \rangle = 0$. Finally, we give to S the riemannian structure obtained by left translating the inner product on $\mathfrak{s}$. The riemannian manifold obtained will be called a *Damek-Ricci space*.

Some geometric features of these spaces are given in

Theorem 2.1. (i) *S is an Einstein manifold with non positive sectional curvature* ([1], [5]).

(ii) *S has negative curvature if and only if* $\mathfrak{n}$ *satisfy the NC-condition* ([6]).

(iii) *S has negative sectional curvature if and only if S is symmetric* ([7], [14]).

(iv) *S is a harmonic manifold* ([6]).

The NC-condition can also be formulated in terms of the operator K_Y, $Y \in \mathfrak{n}$ studied by SZABO in [17]. For every $Y \in \mathfrak{n}$, $Y = Z+X$ and $\langle Z', Z\rangle = 0$ we set $K_Y(Z') = [\overline{X}, J_{\overline{Z}} J_{Z'} X]$ where $\overline{X} = X/|X|$, $\overline{Z} = Z/|Z|$. It is clear that K_Y is skew symmetric and that $\mathfrak{n}$ satisfies the NC-condition if and only if K_Y is an isomorphism, for every $Y = Z + X$, $Z \neq 0$, $X \neq 0$. Moreover, in Theorem 1.11 of [17] and also in section 4.2 of [3], the eigenvalues and the corresponding eigenspaces of the curvature operator in S are computed. It is shown that they depend on the eigenvalues of ${K_Y}^2$.

As a consequence one can deduce (see [3], end of section 4.2) that S has negative sectional curvature if and only if 0 is not an eigenvalue of ${K_Y}^2$, for every $Y = Z + X$, $Z \neq 0, X \neq 0$. Furthermore, as observed in [3], this characterization implies that when $\mathfrak{z}$ is even dimensional there exist planes of zero curvature, thus obtaining (iii) above for an H-type group N with even dimensional center.

3 – Solvable extensions endowed with modified H-type metrics

In the study of the geometry of a 2-step nilpotent Lie group endowed with a left-invariant metric $(N, \langle\,,\rangle)$, the maps $\{J_Z\}_{Z\in\mathfrak{z}}$ defined in (1) play a very important role. The Levi-Civita conexion, curvature and Ricci tensor and geodesics, for example, are described in terms of the maps $\{J_Z\}_{Z\in\mathfrak{z}}$ (see [8]) for any 2-step nilpotent Lie group $(N, \langle\,,\rangle)$. Furthermore, the expression of the geodesic γ of $(N, \langle\,,\rangle)$ with $\gamma(0) = e$ and $\gamma'(0) = X + Z$ ($X \in \mathfrak{v}$ and $Z \in \mathfrak{z}$) is given essentially in terms of the distinct nonzero eigenvalues of J_Z^2. Thus, it will be simpler depending on the number of distinct eigenvalues of J_Z^2.

We are thus led to study the following class of metrics, obtained by weakening the H-type condition on 2-step nilpotent Lie groups.

DEFINITION. A 2-step nilpotent Lie group $(N, \langle\,,\rangle)$ is said to be a *modified H-type group* if for any nonzero $Z \in \mathfrak{z}$, $J_Z^2 = \lambda(Z)I$ for some $\lambda(Z) < 0$.

Note that if $\lambda(Z) = -\langle Z, Z\rangle$ for all $Z \in \mathfrak{z}$ then $(N, \langle\,,\rangle)$ is an H-type group, and if for some $c > 0$, $\lambda(Z) = -c\langle Z, Z\rangle$ for all $Z \in \mathfrak{z}$ then $(N, \langle\,,\rangle)$ is an H-type group in the sense of [9].

THEOREM 3.1 [15]. (1) *If $(N, (\,,))$ is a modified H-type group, then there exists an H-type metric $\langle\,,\rangle$ on $\mathfrak{n}$ and a symmetric positive definite transformation P on $(\mathfrak{z}, \langle\,,\rangle)$ such that $(X+Z, Y+Z') = \langle X, Y\rangle + \langle PZ, Z'\rangle$ for all $X, Y \in \mathfrak{v}$, $Z, Z' \in \mathfrak{z}$ (we denote this inner product by $\langle\,,\rangle_P$).*

Let $(N, \langle\,,\rangle)$ be an H-type group.

(2) *If P and P' are symmetric positive transformations on $(\mathfrak{z}, \langle\,,\rangle)$ then $(N, \langle\,,\rangle_P)$ is isometric to $(N, \langle\,,\rangle_{P'})$ if and only if P and P' are conjugate on $\mathfrak{z}$.*

(3) *If P is a symmetric positive definite transformation of $(\mathfrak{z}, \langle\,,\rangle)$ then the isotropy group H_P of $(N, \langle\,,\rangle_P)$ is*

$$H_P = \{\varphi \in H : \varphi|_{\mathfrak{z}} P = P\varphi|_{\mathfrak{z}}\},$$

where H is the isotropy group of $(N, \langle\,,\rangle)$.

Thus, the modified H-type groups do not provide new Lie algebras other than the H-type algebras. Moreover, the modified H-type groups are nothing but pairs $(N, \langle\,,\rangle_P)$ where $(N, \langle\,,\rangle)$ is an H-type group and P a symmetric positive definite transformation on $(\mathfrak{z}, \langle\,,\rangle)$. However, geometrically, these Riemannian manifolds have some different properties. As an example, according to [2], an H-type group is a commutative space if and only if it is a weakly symmetric space. On the other hand, modified H-type groups $(N, \langle\,,\rangle_P)$ with $\dim\mathfrak{z} = 3$ and such that the eigenvalues of P are pairwise distinct yield examples of commutative spaces which are not weakly symmetric (see [15]).

We now consider the same class of solvable extensions of H-type groups $S = AN$ as in section 2, but we will endow S with extensions of modified H-type metrics $\langle\,,\rangle_P$.

We fix in $\mathfrak{s}$ the Damek-Ricci metric $\langle\,,\rangle$, i.e. $\langle\,,\rangle|_{\mathfrak{n}\times\mathfrak{n}}$ is an H-type metric, $\langle A, \mathfrak{n}\rangle = 0$, and $|A| = 1$, with $\mathrm{ad}A|_{\mathfrak{v}} = \frac{1}{2}I$, $\mathrm{ad}A|_{\mathfrak{z}} = I$. For each positive definite symmetric transformation P on $(\mathfrak{z}, \langle\,,\rangle)$, we take the inner product $\langle\,,\rangle_P$ on $\mathfrak{s}$ defined by $|A|_P = \frac{1}{2}$, $\langle A, \mathfrak{n}\rangle_P = 0$ and $\langle X+Z, Y+Z'\rangle_P = \langle X, Y\rangle + \langle PZ, Z'\rangle$ for all $X, Y \in \mathfrak{v}$, $Z, Z' \in \mathfrak{z}$. Finally, we give to S the Riemannian structure obtained by left translating the inner product $\langle\,,\rangle_P$. The Riemannian manifold obtained is denoted by $(S, \langle\,,\rangle_P)$. Note that $(N, \langle\,,\rangle_P|_{\mathfrak{n}\times\mathfrak{n}})$ is a modified H-type group.

It is easy to prove that $(S, 4\langle\,,\rangle_{4I})$ is isometric to the Damek-Ricci space, i.e. the space $(S, \langle\,,\rangle_{4I})$ is conformally equivalent to the Damek-Ricci space.

The reason for considering the above extensions with $|A|_P = \frac{1}{2}$ rests on the fact that, we will make use, in what follows, of results obtained in [10] where different normalizing constants are stablished.

Definition [10]. Let S be a simply connected, solvable Lie group with Lie algebra $\mathfrak{s}$, endowed with a left-invariant metric $\langle\,,\rangle$. We say that $(S, \langle\,,\rangle)$ is a *3-step Carnot solvmanifold* if $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ is 2-step nilpotent with codimension one and if $\mathfrak{s} = \mathbb{R}A \oplus \mathfrak{v} \oplus \mathfrak{z}$ is an orthogonal decomposition with $|A| = 1$ then $\mathrm{ad}A|_{\mathfrak{v}} = I$, $\mathrm{ad}A|_{\mathfrak{z}} = 2I$.

The spaces $(S, \langle\,,\rangle_P)$ are 3-step Carnot solvmanifolds. Actually, $\mathfrak{s} = \mathbb{R}A' \oplus \mathfrak{v} \oplus \mathfrak{z}$ with $A' = 2A$ and thus $|A'|_P = 1$, $\mathrm{ad}A'|_{\mathfrak{v}} = I$, $\mathrm{ad}A'|_{\mathfrak{z}} = 2I$.

We now compute the Ricci transformation Ric of the spaces $(S, \langle\,,\rangle_P)$ given by $\mathrm{Ric}(X) = \sum R(X, A')A' + \sum R(X, X_i)X_i + \sum R(X, Z_i)Z_i$, where $X \in \mathfrak{s} = \mathrm{T}_e S$ and $\{X_i\}$, $\{Z_i\}$ are orthonormal bases of $\mathfrak{v}$ and $\mathfrak{z}$ respectively. Furthermore choose $\{Z_i\}$ to be eigenvectors of P with eigenvalues $\{\lambda_i\}$.

Proposition 3.2. *The matrix of the Ricci transformation* Ric *of* $(S, \langle\,,\rangle_P)$ *in terms of the basis* $\{A', X_1, ..., X_n, Z_1, ..., Z_m\}$ *is*

$$\mathrm{Ric} = \begin{bmatrix} -(n+4m) & 0 & 0 \\ 0 & \left(-(n+2m) - \frac{1}{2}\sum \lambda_i\right) I_{n\times n} & 0 \\ 0 & 0 & -(2n+4m)I_{m\times m} + \frac{n}{4}P \end{bmatrix}.$$

Therefore $(S, \langle\,,\rangle_P)$ *is an Einstein manifold if and only if* $P = 4I$.

Proof. We can use lemma (3.17) in [10] or lemma 1.4 in [18]. Let $\{J_Z^P\}_{Z\in\mathfrak{z}}$ denote the transformation defined in (1) for $(\mathfrak{n}, \langle\,,\rangle_P)$. It is easy to see that $J_Z^P = J_{PZ}$, for any $Z \in \mathfrak{z}$, where $\{J_Z\}_{Z\in\mathfrak{z}}$ are the maps corresponding to the H-type metric $\langle\,,\rangle$. We have

$$\mathrm{Ric}(A') = -\mathrm{tr}(\mathrm{ad}\,A')^2\, A' = -(n+4m)A'.$$

If $X \in \mathfrak{v}$ then

$$\begin{aligned}\mathrm{Ric}(X) &= \frac{1}{2}\sum (J^P_{Z_i})^2 X - \mathrm{tr}(\mathrm{ad}\,A')\mathrm{ad}\,A'(X) = \\ &= \frac{1}{2}\sum (-\langle PZ_i, PZ_i\rangle)X - (n+2m)X = \\ &= -\frac{1}{2}\sum \langle PZ_i, Z_i\rangle_P X - (n+2m)X = \\ &= \Big(-\frac{1}{2}\sum \lambda_i - (n+2m)\Big)X.\end{aligned}$$

For all $1 \le i, j \le m$ we have

$$\begin{aligned}\langle \mathrm{Ric}(Z_i), Z_j\rangle_P &= -\frac{1}{4}\mathrm{tr}(J^P_{Z_i}J^P_{Z_j}) - \mathrm{tr}(\mathrm{ad}\,A')\langle \mathrm{ad}\,A'(Z_i), Z_j\rangle_P = \\ &= -\frac{1}{4}\lambda_i\lambda_j \mathrm{tr}(J_{Z_i}J_{Z_j}) - (n+2m)2\langle Z_i, Z_j\rangle_P = \\ &= -\frac{1}{4}\lambda_i\lambda_j(-n\langle Z_i, Z_j\rangle\delta_{ij}) - (2n+4m)\delta_{ij} = \\ &= \frac{n}{4}\lambda_i\langle Z_i, Z_i\rangle_P\delta_{ij} - (2n+4m)\delta_{ij} = \\ &= \Big(\frac{n}{4}\lambda_i - (2n+4m)\Big)\delta_{ij} = \\ &= \Big\langle\Big(\frac{n}{4}P - (2+4m)I\Big)Z_i, Z_j\Big\rangle_P.\end{aligned}$$

This conclude the computation of Ric. Suppose that $(S, \langle\,,\rangle_P)$ is Einstein, thus $P = \lambda I$ and $\frac{n}{4}\lambda - (2n+4m) = -(n+4m)$, this implies $\lambda = 4$. Conversely, if $P = 4I$ then $\mathrm{Ric} = -(n+4m)I$, concluding the proof.

In what follows we will give some curvature properties of the solvable extensions $(S, \langle\,,\rangle_P)$ considered previously.

Definition [10]. A solvable Lie algebra s is a *3-step Carnot algebra* if $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ is 2-step nilpotent with codimension one and s admits a decomposition $\mathfrak{s} = \mathbb{R}A \oplus \mathfrak{v} \oplus \mathfrak{z}$ with $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, $\mathfrak{z}$ the center of $\mathfrak{n}$ and $\mathrm{ad}A|_{\mathfrak{v}} = I$, $\mathrm{ad}A|_{\mathfrak{z}} = 2I$. An inner product $\langle\,,\rangle$ on $\mathfrak{s}$ is *admissible* if $|A| = 1$ and $\mathbb{R}A$, $\mathfrak{v}$ and $\mathfrak{z}$ are mutually orthogonal.

Note that a solvmanifold $(S, \langle\,,\rangle)$ is a 3-step Carnot solvmanifold if and only if the Lie algebra $\mathfrak{s}$ of S is a 3-step Carnot algebra and $\langle\,,\rangle$ is an admissible inner product on $\mathfrak{s}$.

Let $\mathrm{AM}(\mathfrak{s})$ denote the set of admissible inner products on a 3-step Carnot algebra $\mathfrak{s}$ with respect to a fixed decomposition $\mathfrak{s} = \mathbb{R}A \oplus \mathfrak{v} \oplus \mathfrak{z}$. Let $\mathrm{QP}(\mathfrak{s})$ denote the set of all inner products $\langle\,,\rangle$ on $\mathfrak{s}$, not necessarily admissible, that satisfy the quarter pinched condition $-4 \leq K \leq -1$, where K is the sectional curvature of the corresponding Riemannian manifold $(S, \langle\,,\rangle)$.

THEOREM 3.3 [10]. *Let $\mathfrak{s} = \mathbb{R}A \oplus \mathfrak{v} \oplus \mathfrak{z}$ be a 3-step Carnot algebra.*

(1) *For any $\langle\,,\rangle \in \mathrm{QP}(\mathfrak{s})$ there exists $\langle\,,\rangle' \in \mathrm{QP}(\mathfrak{s}) \cap \mathrm{AM}(\mathfrak{s})$ such that $(S, \langle\,,\rangle)$ is isometric to $(S, \langle\,,\rangle')$.*

(2) *If $\langle\,,\rangle, \langle\,,\rangle' \in \mathrm{AM}(\mathfrak{s})$ then $(S, \langle\,,\rangle)$ is isometric to $(S, \langle\,,\rangle')$ if and only if there exists $\varphi \in \mathrm{Aut}(\mathfrak{s})_A = \{\psi \in \mathrm{Aut}(\mathfrak{s}) : \psi A = A\} \simeq \mathrm{Aut}(\mathfrak{n})$ such that $\langle\,,\rangle' = \varphi^*\langle\,,\rangle$ (i.e. if and only if $(N, \langle\,,\rangle|_{\mathfrak{n}\times\mathfrak{n}})$ is isometric to $(N, \langle\,,\rangle'|_{\mathfrak{n}\times\mathfrak{n}})$).*

(3) *Let $\mathrm{AM}^*(\mathfrak{s})$ denote the space of isometry classes of solvmanifolds $(S, \langle\,,\rangle)$, where $\langle\,,\rangle$ is admissible. Then $\mathrm{AM}^*(\mathfrak{s})$ can be identified with the quotient space $\mathrm{Aut}(\mathfrak{s})_A\backslash\mathrm{AM}(\mathfrak{s})$ and the double coset space*

$$\mathrm{Aut}^*(\mathfrak{n})\backslash\mathrm{GL}(\mathfrak{v}) \times \mathrm{GL}(\mathfrak{z})/\mathrm{O}(\mathfrak{v}, \langle\,,\rangle) \times \mathrm{O}(\mathfrak{z}, \langle\,,\rangle),$$

where $\langle\,,\rangle$ is a fixed inner product $\langle\,,\rangle \in \mathrm{AM}(\mathfrak{s})$ and $\mathrm{Aut}^*(\mathfrak{n}) = \{\varphi \in \mathrm{Aut}(\mathfrak{n}) : \varphi(\mathfrak{v}) \subset \mathfrak{v}\}$.

(4) *Let $\mathrm{QP}^*(\mathfrak{s})$ denote the space of isometry classes of solvmanifolds $(S, \langle\,,\rangle)$, where $\langle\,,\rangle$ satisfies $-4 \leq K \leq -1$. Thus $\mathrm{QP}^*(\mathfrak{s})$ can be identified with a path connected subset of $\mathrm{AM}^*(\mathfrak{s})$ (also denoted by $\mathrm{QP}^*(\mathfrak{s})$) whose interior in $\mathrm{AM}^*(\mathfrak{s})$ is nonempty.*

REMARK. In all further discussions we give $\mathrm{AM}^*(\mathfrak{s})$ the double coset space topology from (3) and $\mathrm{QP}^*(\mathfrak{s})$ the topology induced from $\mathrm{AM}^*(\mathfrak{s})$.

We now describe a criteria, following [10], for the inequalities $K \leq -1$ and $K \geq -4$ to hold in terms of the norm of the map $J : \mathfrak{z} \longrightarrow \mathrm{End}(\mathfrak{v})$

(see (1)), which is defined by

$$\|J_Z\| = \max\{|J_Z X| : |X| = 1\},$$
$$\|J\| = \max\{\|J_Z\| : |Z| = 1\}.$$

Proposition 3.4 [10]. *Let $(S, \langle\,,\rangle)$ be a 3-step Carnot solvmanifold and let $J : \mathfrak{z} \longrightarrow \mathrm{End}(\mathfrak{v})$ be the map determined by $(\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}] = \mathfrak{v} \oplus \mathfrak{z}, \langle\,,\rangle)$.*
(1) If $K \leq -1$ or $K \geq -4$ in $(S, \langle\,,\rangle)$ then $\|J\| \leq 2$.
(2) If $\|J\| \leq 1$ then $K \leq -1$.
(3) If $\|J\| \leq \sqrt{2}$ then $K \geq -4$.

The next computations will allow to construct, using the modified H-type metrics $\langle\,,\rangle_P$, an $\mathbb{R}^m$ inside $\mathrm{QP}^*(\mathfrak{s})$, where $\mathfrak{s}$ is the Lie algebra of a Damek-Ricci space and $m = \dim \mathfrak{z}$.
We calculate first the norm $\|J^P\|_P$ corresponding to the spaces $(S, \langle\,,\rangle_P)$. If $Z \in \mathfrak{z}$ with $|Z|_P = 1$ then

$$\|J_Z^P\|_P^2 = \max\{|J_Z^P X|_P^2 : |X|_P = 1\} = \max\{|J_{PZ}X|^2 : |X| = 1\} = \\ = \max\{\langle PZ, PZ\rangle\langle X, X\rangle : |X| = 1\} = \langle PZ, PZ\rangle,$$

so we have

$$\|J^P\|_P^2 = \max\{\langle PZ, PZ\rangle : |Z|_P = 1\} = \max\{\langle PZ, Z\rangle_P : |Z|_P = 1\} = \\ = \max\{\lambda : \lambda \text{ eigenvalue of } P\}.$$

We then obtain $\|J^P\|_P = \max(P)^{\frac{1}{2}}$, where $\max(P)$ denotes the greatest eigenvalue of P. It is clear that $\langle\,,\rangle_P \in \mathrm{AM}(\mathfrak{s})$ for any P, then by proposition 3.7 we have

$$\mathcal{P} = \{\langle\,,\rangle_P : \max(P) \leq 1\} \subset \mathrm{QP}(\mathfrak{s}) \cap \mathrm{AM}(\mathfrak{s}).$$

Let $\mathcal{P}^*$ denote the isometry classes of solvmanifolds $(S, \langle,\rangle_P)$ with $\langle,\rangle_P \in \mathcal{P}$. Using Theorem 3.2, (2) and Theorem 3.2, (2), we obtain that $\mathcal{P}^*$ can be identified (homeomorphically) with the set

$$\Delta_m = \{(\lambda_1, \ldots, \lambda_m) : 1 \geq \lambda_1 \geq \ldots \geq \lambda_m > 0\},$$

where $m = \dim \mathfrak{z}$ and $\{\lambda_i\}$ are the corresponding eigenvalues.

Thus we have proved the following result.

THEOREM 3.5. *Let s be the 3-step Carnot algebra corresponding to a Damek-Ricci space, i.e.* $\mathfrak{s} = \mathbb{R}A \oplus \mathfrak{v} \oplus \mathfrak{z}$ *with* $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ *an H-type algebra. Thus* $\mathrm{QP}^*(\mathfrak{s})$ *contain a subset* $\mathcal{P}^*$ *homeomorphic to* Δ_m*, where* $m = \dim \mathfrak{z}$.

REFERENCES

[1] J. Boggino: *Generalized Heisenberg groups and solvmanifolds naturally associated*, Rend. Sem. Mat. Univers. Politecn. Torino, **43** (1985), 529-547.

[2] J. Berndt – F. Ricci – L. Vanhecke: *Weakly symmetric groups of Heisenberg type*, to appear in Diff. Geom. Appl.

[3] J. Berndt J. – F. Tricerri – L. Vanhecke: *Generalized Heisenberg groups and Damek-Ricci harmonic spaces*, Lecture Notes in Mathematics **1598**, Springer Verlag, Berlin-Heidelberg, 1995.

[4] M. Cowling – A. Dooley – A. Korányi – F. Ricci: *H-type groups and Iwasawa decompositions*, Advances in Math., **87** (1991), 1-41.

[5] E. Damek: *Curvature of a semidirect extension of a Heisenberg type nilpotent group*, Coll. Math., **53** (1987), 249-253.

[6] E. Damek – F. Ricci: *Harmonic analysis on solvable extensions of H-type groups*, Journal of Geometric Analysis, **2** (1992), 213-248.

[7] I. Dotti: *On the curvature of certain extensions of H-type groups*, to appear in Proc. Amer. Math. Soc.

[8] P. Eberlein: *Geometry of 2-step nilpotent Lie groups with a left invariant metric*, Ann. Sci. Ecole Norm. Sup. (4), **27** (1994), 611-660.

[9] P. Eberlein: *Geometry of 2-step nilpotent Lie groups with a left invariant metric II*, Trans. Amer. Math. Soc., **343** (1994), 805-828.

[10] P. Eberlien – J. Heber: *Quarter pinched homogeneous spaces of negative curvature*, Internat. J. Math., **7** (1996), 441-500.

[11] C. Gordon: *Isospectral closed riemannian manifolds which are not locally isometric*, Jour. Diff. Geom., **37** (1993), 639-649.

[12] E. Heintze: *On homogeneous manifolds of negative curvature*, Math. Ann., **211** (1974), 23-34.

[13] A. Kaplan: *Fundamental solutions for a class of hypoelliptic operators*, Trans. Amer. Math. Society, **258** (1980), 147-153.

[14] M. Lanzendorf: *Einstein metrics with nonpositive sectional curvature on extensions of Lie algebras of Heisenberg type*, preprint, July 1995.

[15] J. Lauret: *Symmetric-like Riemannian spaces on 2-step nilpotent groups*, Phd. thesis, FaMAF, Univ. Nac. de Córdoba (1997).

[16] F. Ricci: *Spherical Functions on Certain Non-symmetric Harmonic manifolds*, Workshop on Representation Theory of Lie groups 1993, I.C.T.P. Trieste.

[17] Z. Szabo: *Spectral theory for operator families on riemannian manifolds*, Proceedings of Symposia in Pure Mathematics, **54** (1993), 615-665.

[18] T. Wolter: *Einstein metrics on solvable groups*, Math. Z., **206** (1991), 457-471.

Lavoro pervenuto alla redazione il 23 aprile 1997
ed accettato per la pubblicazione il 3 dicembre 1997.
Bozze licenziate il 3 marzo 1998

Indirizzo degli Autori:

I. Dotti – J. Lauret – Universidad Nacional de Córdoba – 5000 Córdoba – Argentina
e-mail:idotti@mate.uncor.edu – lauret@mate.uncor.edu