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Spectral comparison between Dirac and Schrödinger operators

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RIASSUNTO: *Si dimostra un teorema generale di confronto fra gli autovalori di due operatori autoaggiunti e semilimitati agenti su due diversi spazi di Hilbert fra i quali sia data un'opportuna applicazione. Come caso particolare, si trovano stime degli autovalori dell'operatore di Dirac classico agente sugli spinori tramite quelli dell'operatore di Laplace-Beltrami. Le stime sono ottimali, nel senso che per il primo autovalore si ritrova la diseguaglianza di Friedrich.*

ABSTRACT: *We show a general comparison theorem for the eigenvalues of two self-adjoint semibounded operators acting on two different Hilbert spaces, which are related by a suitable mapping. As a particular case, we get estimates of the eigenvalues of the classical Dirac operator acting on spinors in terms of the eigenvalues of the Laplace-Beltrami operator. These estimates are sharp, in the sense that they give Friedrich's inequality for the minimal eigenvalue.*

1 – Introduction

One of the most important results on the classical Dirac operator acting on spinors is the *Lichnerowicz formula* which gives, as a simple consequence, a lower bound for the squares of its eigenvalues. Precisely,

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let (M, g, γ) be a n-dimensional boundaryless compact Riemannian spin manifold (g and γ denote the Riemannian metric and the spin structure respectively). For basic facts concerning spin geometry, we refer to [8]. As M admits a spin structure, there exists a bundle of spinors, i.e. a vector bundle $S \longrightarrow M$ on whose fiber S_x the Clifford algebra of (T_xM, g_x) acts via a representation in $End(S_x)$; this representation will be denoted by "$\cdot$", as the Clifford multiplication. The fiber S_x is endowed with a Spin_n-invariant metric $\langle\ ,\ \rangle$ satisfying for any $X \in TM$ and for any spinor section $\varphi \in \Gamma(S)$:

$$\langle X \cdot \varphi, X \cdot \varphi \rangle = g(X, X)\langle \varphi, \varphi \rangle.$$

As the Spin_n-principal bundle is a 2-sheeted covering of the Riemannian principal bundle, the Levi-Civita connection induces a connection, denoted ∇, on the bundle of spinors which is compatible with the metric. The classical *Dirac operator* $\mathcal{D}$ acting on the space $\Gamma(S)$ of sections of the spinor bundle is the firts order elliptic self-adjoint operator defined as the composition of the connection ∇ with the Clifford multiplication. Locally, with respect to any g-orthonormal tangent frame $\{e_1, \ldots, e_n\}$, one has:

$$\mathcal{D}\varphi = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}\varphi. \tag{1.1}$$

The *Lichnerowicz formula* ([9], 1963) gives the relation between $\mathcal{D}$ and the so called *rough Laplacian* $\nabla^*\nabla$:

$$\mathcal{D}^2 = \nabla^*\nabla + \frac{s}{4}\mathrm{Id} \tag{1.2}$$

where ∇^* is the formal adjoint of the connection ∇ with respect to the natural integral product on the spinor bundle, and where s is the *scalar curvature* of (M, g). Applying (1.2) to a spinor field φ, taking the inner product with φ and integrating on M with respect to the canonical measure v_g induced by g, give:

$$\int_M |\mathcal{D}\varphi|^2 dv_g = \int_M |\nabla\varphi|^2 dv_g + \int_M \frac{s}{4}|\varphi|^2 dv_g \geq \frac{s_0}{4}\int_M |\varphi|^2 dv_g$$

where $s_0 = \min_{x\in M} s(x)$. It follows that if λ is an eigenvalue of $\mathcal{D}$, then (*Lichnerowicz inequality*):

$$\lambda^2 \geq \frac{s_0}{4}. \tag{1.3}$$

Clearly, this inequality is interesting only for manifolds with *positive* scalar curvature. It is a fact that, for such manifolds, the minimal value $\frac{s_0}{4}$ cannot be attained. Indeed, if equality holds in (1.3) for some eigenvalue λ of $\mathcal{D}$, then $\nabla\varphi \equiv 0$ and thus, by definition (1.1), $\mathcal{D}\varphi = 0$ and $\lambda = 0$; moreover, the scalar curvature is a constant, $s = s_0 = 0$.

To improve Lichnerowicz inequality in a sharp inequality, one must wait for the year 1980, when T. Friedrich [3] showed that:

$$\lambda^2 \geq \frac{n}{4(n-1)} s_0 \tag{1.4}$$

and that manifolds which admit the minimal eigenvalue are Einstein manifolds (but not Kähler, see O. Hijazi [7]). A simple proof of *Friedrich's inequality* (1.4) comes by considering the modified Riemannian connection ∇^λ acting on spinors:

$$\nabla^\lambda_X \varphi = \nabla_X \varphi + \frac{\lambda}{n} X \cdot \varphi \tag{1.5}$$

where λ is a real constant (this connection was introduced by T. Friedrich [3] and generalized by O. Hijazi [7]). As a direct calculation gives:

$$|\nabla^\lambda \varphi|^2 = |\nabla\varphi|^2 - 2\frac{\lambda}{n}\langle \mathcal{D}\varphi, \varphi\rangle + \frac{\lambda^2}{n}|\varphi|^2\,, \tag{1.6}$$

one gets, by Lichnerowicz formula:

$$0 \leq \int_M |\nabla^\lambda\varphi|^2 dv_g = \int_M \Big(|\mathcal{D}\varphi|^2 - 2\frac{\lambda}{n}\langle\mathcal{D}\varphi,\varphi\rangle + \frac{\lambda^2}{n}|\varphi|^2 - \frac{s}{4}|\varphi|^2\Big) dv_g$$

which gives, when $\mathcal{D}\varphi = \lambda\varphi$, the required inequality.

Spinors which are parallel for ∇^λ are called *real Killing spinors* (because the 1-form ξ_φ defined by $\xi_\varphi(X) = \langle X\cdot\varphi, \varphi\rangle$ is dual of a Killing vector field when $\nabla^\lambda\varphi = 0$). They first appeared in mathematical physics in the context of supergravity and recently in superstring theories. Here, real Killing spinors are related to eigenspinors of the Dirac operator in the sense that $\nabla^\lambda\varphi = 0$ implies $\mathcal{D}\varphi = \lambda\varphi$. Conversely, $\mathcal{D}\varphi = \lambda_0\varphi$ with $\lambda_0^2 = \frac{n}{4(n-1)}s_0$ implies $\nabla^{\lambda_0}\varphi = 0$ and $s = s_0 =$ constant: eigenspinors related to the minimal eigenvalue λ_0 are ∇^{λ_0}-parallel and manifolds with minimal eigenvalue have constant scalar curvature.

Friedrich's and Lichnerowicz's inequalities concern only the minimal eigenvalue of the Dirac operator. The following theorem, due to the Author (see [1], 1994), gives a "spectral comparison" between the Dirac operator and the Laplace-Beltrami operator Δ acting on functions, in the sense that it gives, for any squared eigenvalue of $\mathcal{D}$, a lower bound in terms of a corresponding eigenvalue of Δ and of the scalar curvature s; moreover, it gives a comparison between averages of eigenvalues.

THEOREM 1.7. *Let (M, g, γ) be a n-dimensional compact Riemannian spin manifold whitout boundary and let us consider the bundle $S \longrightarrow M$ of spinors with its Dirac operator $\mathcal{D}$. Then, for any finite set $\{\lambda_i(\mathcal{D})\}_{i\in I}$ of eigenvalues of $\mathcal{D}$, one has:*

$$\text{(i)} \qquad \sup_{i\in I} \lambda_i(\mathcal{D})^2 \geq \frac{n}{n-1}\Big(\frac{s_0}{4} + C\lambda_{k+1}(\Delta)\Big)$$

and

$$\text{(ii)} \qquad \frac{1}{\#I}\sum_{i\in I}\lambda_i(\mathcal{D})^2 \geq \frac{n}{n-1}\Big(\frac{s_0}{4} + \frac{1}{2\#I}\sum_{j=1}^{k}\lambda_j(\Delta) + C\lambda_{k+1}(\Delta)\Big)$$

where s_0 is the minimum of the scalar curvature of (M, g), and where $k =$ integer part of $\frac{\#I}{2^{[n/2]}+1}$, and $C = \frac{1}{8(2^{[n/2]}+1)^2}$.

WARNING. As we quote the spectrum of Δ from 1 to $+\infty$ and not (as usual) from 0 to $+\infty$, the first eigenvalue different from zero is $\lambda_2(\Delta)$.

The remaining sections of this paper are devoted to prove Theorem 1.7 as a particular case of a general theorem of spectral comparison. Before doing this, let us notice that the estimates (i) and (ii) are sharp not only beacuse they give Friedrich's inequality when applied to the minimal eigenvalue of $\mathcal{D}$, but also in the following sense. Inequalities (i) and (ii) imply that there exist at most $2^{[n/2]}$ eigenvalues of $\mathcal{D}^2$ in the interval $[\frac{ns_0}{4(n-1)}, \frac{ns_0}{4(n-1)} + \frac{nC}{n-1}\lambda_2(\Delta)[$. This is sharp in the case of the flat torus, which has exactly $2^{[n/2]}$ eigenvalues of $\mathcal{D}^2$ equal to $\frac{n}{4(n-1)}s_0 = 0$.

Another remark concerns the $\hat{A}$-genus $\hat{A}(M)$ of M, a topological invariant, not to define here, which is actually the index of the Dirac operator. As the $\hat{A}$-genus is bounded by the dimension of the kernel of $\mathcal{D}$, it follows from Theorem 1.7 that:

Corollary 1.8. *Let (M, g) be any compact Riemannian n-manifold whose $\hat{A}$-genus is not trivial. Then the eigenvalues of the Laplace-Beltrami operator Δ of (M, g) satisfy*

$$\frac{1}{2}\sum_{j=1}^{k}\lambda_j(\Delta) + \hat{A}(M)C\lambda_{k+1}(\Delta) \leq -\frac{\hat{A}(M)}{4}s_0$$

where $k = [\frac{\hat{A}(M)}{2^{[n/2]}+1}]$ and C is given in Theorem 1.7.

This result may be read also as $\frac{1}{k}\sum_{j=1}^{k}\lambda_j(\Delta) \leq -\frac{s_0}{4}$, where k is of the order of $\frac{\hat{A}(M)}{2^{[n/2]}+1}$: in other terms, Corollary 1.8 states that a manifold with non trivial topology cannot have too many small eigenvalues.

2 – Spectral comparison theorems

a) settling the problem. For any given Riemannian n-manifold (M, g), denote v_g the canonical measure induced by g and $\langle\langle u, v\rangle\rangle_{L^2(M)} = \int_M u(x)v(x)dv_g(x)$ the integral inner product, defined for every couple of continuous functions u, v on M; we shall also write briefly $\langle\langle u, v\rangle\rangle_{L^2}$ or $\langle\langle u, v\rangle\rangle$ when no ambiguity is possible. We denote by $L^2(M)$ the completion of $C^\infty(M)$ with respect to the norm $\|\ \|_{L^2}$ associated to the integral inner product, and by $H^1(M)$ the first Sobolev space of M, i.e. the completion of $C^\infty(M)$ with respect to the norm associated to the inner product $\langle\langle u, v\rangle\rangle_{H^1} = \int_M u(x)v(x)dv_g(x) + \int_M g(\operatorname{grad} u, \operatorname{grad} v)(x)dv_g(x)$.

We shall consider the spectrum of a quadratic form q or, equivalentely, of the corresponding self-adjoint operator T. Precisely, a quadratic form q, closed on the domain $H^1(M)$, is *semibounded* if there exists a real constant c such that $q(v) \geq c\|v\|_{L^2}$ for any $v \in H^1(M)$ (we may assume $c = 0$ by a shift). Such a form is the quadratic form of a unique self-adjoint operator T which is a Friedrichs extension (see [10], [I], Theorem VIII.15), and the spectrum of q, defined by max-min principle, coincides with the discrete part of the spectrum of T, i.e. with the part lying under the essential spectrum of T (see [10], [IV], Theorem XIII.2). An example is the quadratic form $\|\operatorname{grad} v\|^2_{L^2}$ and the Laplace-Beltrami operator Δ_M or, more generally, the quadratic form $\|\operatorname{grad} v\|^2_{L^2} + \|V^{\frac{1}{2}} v\|^2_{L^2}$ and the corresponding Schrödinger operator $\Delta_M + V$, with V a bounded potential function.

Let us consider two Riemannian manifolds (M', g') and (M, g) and suppose that a mapping $\varpi : H^1(M') \longrightarrow H^1(M)$ is given. Our aim is to compare via ϖ the spectra of two semibounded quadratic forms q' and q closed on the domains $H^1(M')$ and $H^1(M)$ respectively, or equivalentely the spectra of the corresponding self-adjoint operators T' and T. When the manifolds are compact, these spectra are discrete; otherwise, we shall compare their discrete parts, i.e. the parts lying under the essential spectra.

In order to get comparison theorems, we shall assume that ϖ preserves the L^2-norm:

$$\|\varpi u\|_{L^2(M)} = \|u\|_{L^2(M')} \tag{2.1}$$

for any $u \in L^2(M')$ (*Fubini's property*), and that ϖ does not increase the energy of the operators:

$$q(\varpi u) = \langle\langle T(\varpi u), \varpi u \rangle\rangle_{L^2(M)} \leq \langle\langle T' u, u \rangle\rangle_{L^2(M')} = q'(u) \tag{2.2}$$

for any $u \in H^1(M')$ (then we say that the quadratic forms q' and q, or also the operators T' and T, obey *Kato's inequality* with respect to ϖ).

b) THE LINEAR CASE. When the mapping ϖ is linear, it is very easy to obtain a spectral comparison between T' and T. Let $\mathcal{E} \subset H^1(M')$, resp. $\mathcal{K} \subset H^1(M)$, be the subspace spanned by the first N eigenfunctions of T', resp. by the first k eigenfunctions of T.

THEOREM 2.3. *Let $\varpi : H^1(M') \longrightarrow H^1(M)$ be a linear mapping verifying Fubini's property* (2.1), *and let T' and T be two self-adjoint* *semibounded operators, whose quadratic forms are closed on* $H^1(M')$ *and* $H^1(M)$ *respectively, which obey Kato's inequality* (2.2) *with respect to* ϖ. *Suppose that* $\dim \varpi(\mathcal{E}) > \dim \mathcal{K}$*, where* $\mathcal{E}$ *and* $\mathcal{K}$ *are the subspaces defined above. Then we have:*

$$\lambda_N(T') \geq \lambda_{k+1}(T).$$

Proof. As $\dim \varpi(\mathcal{E}) > \dim \mathcal{K}$, there exists at least one $u \in \mathcal{E}\backslash\{0\}$ such that ϖu is L^2-orthonormal to $\mathcal{K}$.

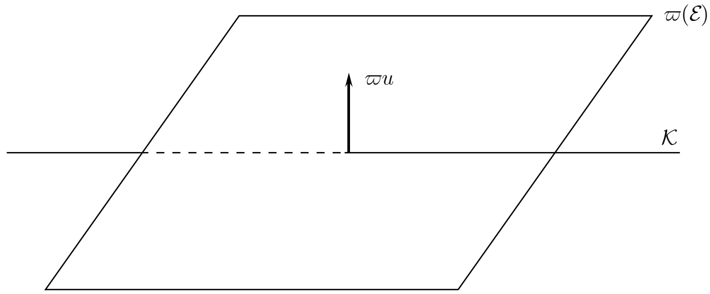

Fig. 1

For such u, one has:

$$\begin{aligned}
\lambda_N(T')\|u\|^2_{L^2(M')} &\geq \langle\langle T'u, u\rangle\rangle_{L^2(M')} && \text{by min-max principle, } \geq \\
&\geq \langle\langle T(\varpi u), \varpi u\rangle\rangle_{L^2(M)} && \text{by Kato's inequality, } \geq \\
&\geq \lambda_{k+1}(T)\|\varpi u\|^2_{L^2(M)} && \text{by max-min, } = \\
&= \lambda_{k+1}(T)\|u\|^2_{L^2(M')} && \text{by Fubini's property.}
\end{aligned}$$

□

Notice that, even if the proof is trivial in the linear case, the inequality stated in Theorem 2.3 is the basis for a lot of spectral comparison theorems (see for instance the results of S.Y. Cheng, S. Gallot, M. Gromov, P. Li and S.T. Yau).

Example 2.4. Let Ω be a regular domain with smooth boundary in a compact boundaryless Riemannian manifold (M,g). The operators to compare are $T' = \Delta_\Omega$ with Dirichlet conditions on the boundary of Ω, and $T = \Delta_M$. For any function $u \in C_0^\infty(\Omega)$ compactly supported in the interior of Ω, define $\varpi u \in C^\infty(M)$ to be the natural extension of u on M by $\varpi u = 0$ on $M\backslash\Omega$. This gives a linear injective mapping $\varpi : H_0^1(\Omega) \longrightarrow H^1(M)$, where $H_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the H^1-norm. Since Fubini's property and Kato's inequality are automatically satisfied, we get $\lambda_N^D(\Omega) \geq \lambda_N(M)$ (trivial!).

c) THE NON LINEAR CASE. When ϖ is not linear, we can try to mimic the proof done in the linear case (Theorem 2.3). For $u \in \mathcal{E}$, denote $(\varpi u)^{\mathcal{K}}$ and $(\varpi u)^\perp$ the orthogonal projection of ϖu on $\mathcal{K}$ and the component orthogonal to $\mathcal{K}$ respectively. The decomposition $\varpi u = (\varpi u)^{\mathcal{K}} + (\varpi u)^\perp$ is at the same time L^2-orthogonal and orthogonal for the quadratic form associated to the operator T. Then, the same sequence of inequalities used for ϖ linear (i.e. min-max, Kato's inequality, max-min and min-max, Fubini's property) gives:

$$
\begin{aligned}
\lambda_N(T')\|u\|^2 &\geq \langle\!\langle T'u, u\rangle\!\rangle \geq \\
&\geq \langle\!\langle T(\varpi u), \varpi u\rangle\!\rangle = \\
&= \langle\!\langle T(\varpi u)^\perp, (\varpi u)^\perp\rangle\!\rangle + \langle\!\langle T(\varpi u)^{\mathcal{K}}, (\varpi u)^{\mathcal{K}}\rangle\!\rangle \geq \\
&\geq \lambda_{k+1}(T)\|(\varpi u)^\perp\|^2 + \lambda_1(T)\|(\varpi u)^{\mathcal{K}}\|^2 = \\
&= (\lambda_1(T) + (\lambda_{k+1}(T) - \lambda_1(T))\frac{\|(\varpi u)^\perp\|^2}{\|\varpi u\|^2})\|u\|^2.
\end{aligned} \tag{2.5}
$$

Now, if the image $\varpi(\mathcal{E})$ is too concentrated around $\mathcal{K}$, the ratio $\frac{\|(\varpi u)^\perp\|^2}{\|\varpi u\|^2}$ is very small. To get a lower bound $\frac{\|(\varpi u)^\perp\|^2}{\|\varpi u\|^2} \geq C > 0$, one must have the possibility to go enough away from $\mathcal{K}$. This is not possible in general, but it is when ϖ is induced by a mapping on manifolds in the following sense.

Let $f : (M', g') \longrightarrow (M, g)$ be any mapping. For any function u on M' for which it makes sense, we set, at $x \in M$:

$$
(\varpi u)(x) = \Big(\int_{f^{-1}(x)} (u|_{f^{-1}(x)}(y))^2 dv_{g'_x}(y)\Big)^{1/2} \tag{2.6}
$$

where g'_x is the metric induced by restriction of g' on the fiber $f^{-1}(x)$. For instance, when f is a Morse function, ϖu is a function defined almost everywhere on M. The mapping $\varpi : u \longmapsto \varpi u$ is not linear, but it is positively homogeneous of degree 1.

Fubini's property (2.1) is verified when f is a Lipschitz mapping whose horizontal Jacobian $J_H f$ satisfies $|J_H f| = 1$ a.e. In fact, in this case, the differential $df(x')$ exists for a.e. $x' \in M'$ and one can define the horizontal Jacobian $J_H f(x')$ as the determinant of the restriction of $df(x')$ to the orthogonal subspace of $T_{x'}(f^{-1}(f(x')))$ in $T_{x'}M'$. The coarea formula (see [2], Theorem 13.4.2 and Corollary 13.4.6) then gives:

$$\int_{M'} (u(x'))^2 |J_H f(x')| dv_{g'}(x') = \int_M \Big(\int_{f^{-1}(x)} (u|_{f^{-1}(x)}(y))^2 dv_{g'_x}(y) \Big) dv_g(x).$$

DEFINITION 2.7. Let $\mathcal{E} \subset L^2(M')$ be a vector subspace of enough regular functions and let $\mathcal{E}_x$ be its image in $L^2(f^{-1}(x))$ by the restriction $u \longmapsto u|_{f^{-1}(x)}$, which is defined for a.e. $x \in M$; we assume $\mathcal{E}_x = \{0\}$ when u is not defined on $f^{-1}(x)$. We define the *rank* of $\mathcal{E}$ to be the essential supremum of the dimensions of $\mathcal{E}_x$, i.e.

$$\operatorname{rank} \mathcal{E} = \inf_{A \in \mathcal{A}} \Big(\sup_{x \in M \setminus A} (\dim \mathcal{E}_x) \Big)$$

where $\mathcal{A}$ is the class of all subsets in M of measure equal to zero.

Notice that this is not the usual definition of the rank, and that the rank of $\mathcal{E}$ may be much smaller than the dimension of $\mathcal{E}$. For instance, a symmetric tensor S of type (0,q) on M induces a function u_S on the total space $M' = U(M)$ of the unit tangent bundle by setting $u_S(v) = S(v, \ldots, v)$. The space $\mathcal{E}$ of such functions is infinite dimensional, but $\mathcal{E}_x$ is the space of symmetric homogeneous polynomials of degree q on $T_xM \cong \mathbb{R}^n$, so the rank of $\mathcal{E}$ is equal to $\binom{n+q-1}{q}$.

We then have the following technical lemma (cf. [5], [1]):

HILBERTIAN LEMMA 2.8. *Let ϖ be the mapping defined by* (2.6). *For any couple of vector subspaces $\mathcal{E} \subset H^1(M')$ and $\mathcal{K} \subset H^1(M)$ such that $\dim \mathcal{K} < \frac{\dim \mathcal{E}}{\operatorname{rank} \mathcal{E}}$, there exists a universal constant $C = C(\dim \mathcal{E}, \dim \mathcal{K})$, $0 < C < 1$, such that*

$$average_{u \in \mathcal{E}, \|u\|=1} \|(\varpi u)^{\perp}\|^2 \geq C\ average_{u \in \mathcal{E}, \|u\|=1} \|u\|^2.$$

The value of the constant C is explicitely calculated in the papers [5], [1].

We can now state the main theorem of spectral comparison:

Theorem 2.9. *Let $f : (M', g') \longrightarrow (M, g)$ be a smooth mapping. Suppose that the mapping ϖ defined by* (2.6) *verifies Fubini's property* (2.1) *and Kato's inequality* (2.2) *with respect to two given self-adjoint semibounded operators T' and T, whose quadratic forms are closed on $H^1(M')$ and $H^1(M)$ respectively. For any positive integer N, the eigenvalues of T' and T satisfy the inequalities:*

$$\text{(i)} \qquad \lambda_N(T') \geq (1 - C(r))\lambda_1(T) + C(r)\lambda_{k+1}(T)\,;$$

$$\text{(ii)} \qquad \sum_{i=1}^{N} \lambda_i(T') \geq (N-k)\lambda_1(T) + \frac{1}{2}\sum_{j=1}^{k} \lambda_j(T) + NC(r)\lambda_{k+1}(T)\,,$$

where r is the rank of the subspace spanned by the first N eigenfunctions of T', and where $k = [\frac{N}{r+1}]$, $C(r) = \frac{1}{8(r+1)^2}$. When the operators have non discrete spectra, the inequalities (i) *and* (ii) *reduce to the discrete parts of spectra* (*i.e. the parts lying under the essential spectra*).

Proof of (i). In the present case, $\varpi(\mathcal{E})$ is a half cone which goes enough away from $\mathcal{K}$: as a consequence of Lemma 2.8, there exists at least a function $u \in \mathcal{E}\backslash\{0\}$ such that $\frac{\|(\varpi u)^{\perp}\|^2}{\|\varpi u\|^2} \geq C$. One achieves the proof of (i) by inserting this last inequality in (2.5). ∎

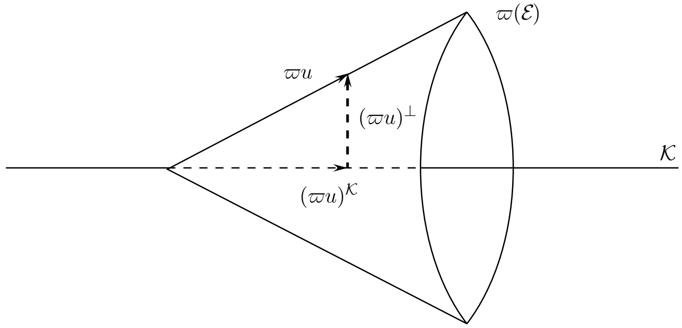

Fig. 2

The proof of (ii) is more technical and we refer the reader to [1] for it.

Remark 2.10. The assumption on the dimension in the Hilbertian Lemma 2.8 and hence in the main Theorem 2.9 is sharp, as the following example proves. Let $\mathcal{K}$ be the space spanned by the L^2-orthonormal functions $\frac{\chi_{U^i}}{\text{Vol } U^i}$, $i = 1, \dots, k$, where U^i are disjoint subsets of M of finite volume different from zero and χ_{U^i} are their characteristic functions. Choose L^2-orthonormal functions $h_1, \dots, h_r$ on a manifold (F, g_F) and let $\mathcal{E}$ be the subspace of $L^2(M \times F)$ spanned by the products $\frac{\chi_{U^i}}{\text{Vol } U^i} h_j$. A direct calculation shows that for any $u \in \mathcal{E}$ one has $(\varpi u)^{\mathcal{K}} = \varpi u$. Hence, in this example we have rank $\mathcal{E} = \frac{\dim \mathcal{E}}{\dim \mathcal{K}}$ and $(\varpi u)^{\perp} = 0$ for every $u \in \mathcal{E}$.

We end this section with some examples of geometrical interest of manifolds, mappings and operators verifying the assumptions of Theorem 2.9.

Example 2.11. Let $f : (\tilde{M}, \tilde{g}) \longrightarrow (M, g)$ be a regular ℓ-sheeted Riemannian covering of compact boundaryless Riemannian manifolds, and let us consider the operators $T' = \Delta_{\tilde{M}} + V'$ and $T = \Delta_M + V$, where V' is a continuous function on $\tilde{M}$ and where, at $x \in M$, $V(x) = \inf_{x' \in f^{-1}(x)} V'(x')$. For any $u \in C^\infty(\tilde{M})$, the mapping $\varpi : u \mapsto \varpi u \in C^\infty(M)$ defined by $(\varpi u)(x) = (\sum_{x' \in f^{-1}(x)} u(x')^2)^{\frac{1}{2}}$, verifies Fubini's property and Kato's inequality with respect to T' and T. The rank of any subspace $\mathcal{E} \subset H^1(\tilde{M})$ is in this case boundel by the degree ℓ of the covering.

Example 2.12. Let $f : (M', g') \longrightarrow (M, g)$ be a Riemannian submersion of compact boundaryless Riemannian manifolds, and let us consider again the operators $T' = \Delta_{M'} + V'$ and $T = \Delta_M + V$. The mapping ϖ defined by (2.6) verifies Fubini's property. To satisfy Kato's inequality with respect to T' and T, we must assume that the fibers are minimal submanifolds of M'. The rank of $\mathcal{E}$ is in this case bounded by a function of the dimension of the eigenspaces of the fibers (see [1]).

3 – Application to vector bundles

Let us consider a compact Riemannian manifold without boundary (M, g) and a real vector bundle E of rank r on M; $f : E \longrightarrow M$ is the bundle projection. Suppose that E is a Riemannian bundle, i.e. that E is endowed with an inner product $\langle \ , \ \rangle_x$ on each fiber $E_x = f^{-1}(x)$ which varies continuously when x ranges over M.

For any smooth section $s \in \Gamma(M, E)$ of the bundle, define on M the *pointwise norm* function $|s|$ by $|s|(x) = \langle s(x), s(x) \rangle_x^{\frac{1}{2}}$. Denote $L^2(M, E)$

the Hilbert space of sections s such that $|s| \in L^2(M)$, i.e. such that $|s|$ is measurable on M and $\|s\|_{L^2(M,E)} = (\int_M |s|(x)^2 dv_g(x))^{\frac{1}{2}} < +\infty$. The mapping defined on smooth sections by $s \mapsto |s|$, extends to a mapping $L^2(M,E) \longrightarrow L^2(M)$ and Fubini's property (2.1) is automatically satisfied.

THEOREM 3.1. *Let $(E, \langle\ ,\ \rangle)$ be any metric vector bundle on a compact boundaryless Riemannian manifold (M,g). Let T' and T be two self-adjoint semibounded operators, acting respectively on sections and on functions, which obey Kato's inequality* (2.2) *with respect to the mapping $s \mapsto |s|$. Then for any positive integer N, the eigenvalues of T' and T satisfy the inequalities:*

$$\text{(i)} \qquad \lambda_N(T') \geq (1 - C(r))\lambda_1(T) + C(r)\lambda_{k+1}(T)\,;$$

$$\text{(ii)} \qquad \sum_{i=1}^{N} \lambda_i(T') \geq (N-k)\lambda_1(T) + \frac{1}{2}\sum_{j=1}^{k} \lambda_j(T) + NC(r)\lambda_{k+1}(T)\,,$$

where r is the dimension of the fibers of E, $k = [\frac{N}{r+1}]$, and $C(r) = \frac{1}{8(r+1)^2}$.

PROOF. As the unitary bundle $M' = U(E)$ associated to E is endowed with a canonical Riemannian metric g', we may consider f as a mapping $f : (U(E), g') \longrightarrow (M,g)$. Then, the mapping ϖ defined by (2.6) satisfies automatically Fubini's property (2.1). One has an isometric injection $\psi : L^2(M,E) \longrightarrow L^2(M)$ which maps a section s on the function ψ_s defined by $\psi_s(v) = \sqrt{r}\langle s(f(v)), v\rangle$ at $v \in U(E)$ (r is the rank of the bundle E). According to the definition (2.6), one has

$$(\varpi\psi_s(v))(x) = \Big(\int_{U(E_x)} r\langle v, s(f(v))\rangle_x dv\Big)^{\frac{1}{2}} = |s|(x)$$

where dv is the canonical Lebesgue probability measure on the sphere $U(E_x)$.

Let us consider any vector subspace $\mathcal{E} \subset L^2(M,E)$. As ψ is linear and isometric, the rank of $\psi(\mathcal{E})$ (as defined in 2.7) is the essential supremum of the dimension of the image of the mapping $s \mapsto s(x)$ from $\mathcal{E}$ to E_x, so it is always bounded by r. Noticing that $\|\psi(s)\|_{L^2(U(E_x))} = |s(x)| = |s|(x)$ and applying Theorem 2.9, we have the claim. ∎

Theorem 3.1 applies in particular to the case when the vector bundle E is endowed with a connection D acting on sections and compatible with the metric $\langle\ ,\ \rangle$. Denote $H^1(M,E)$ the space of L^2-sections such that $|Ds| \in L^2(M)$. The mapping $\varpi : s \mapsto |s|$ extends to a mapping $H^1(M,E) \longrightarrow H^1(M)$.

Definition 3.2. The *rough Laplacian* is the operator D^*D acting on sections, where D^* is the formal adjoint of D with respect to the integral product of sections. We shall call *natural Laplacian*, according to J.P. Bourguignon, the operator $T' = D^*D + \mathcal{R}$, where $\mathcal{R}$ is a field of symmetric endomorphisms of the fibers: T' is a second order differential operator with properties similar to the ones of the usual Laplacian; $T' = D^*D + \mathcal{R}$ is called *Weitzenböck formula*. The *Dirac operator* $\mathcal{D}$ associated to T' is, *when it exists*, a first order self-adjoint operator acting on sections such that $\mathcal{D}^2 = T' = D^*D + \mathcal{R}$.

The typical example is the bundle of differential p-forms with $T' = \Delta^p$, the Hodge-de Rham operator acting on p-forms. In this case, the relation $\Delta^p = D^*D + \mathcal{R}$ is the classical Weitzenböck formula, where $\mathcal{R}$ is explicitely expressed in terms of the curvature of the manifold (M,g): for instance, when $p = 1$, $\mathcal{R}$ is the Ricci curvature (see [4]). The corresponding Dirac operator acting on forms is $\mathcal{D} = d + \delta$, where d is the differential and δ is the codifferential.

Let us consider a *Schrödinger operator* $T = \Delta_M + V$ acting on functions, where V is a given potential function. If we suppose that $\langle \mathcal{R}_x s(x), s(x)\rangle_x \geq V(x)\langle s(x), s(x)\rangle_x$ at any $x \in M$ and for any section s, Kato's inequality (2.2) follows from the *classical Kato's inequality* $|d|s|| \leq |Ds|$ (which is natural, because in Ds one has the component $d|s|$, which gives the derivative of the length of s, plus the orthogonal component, which gives the "rotational derivative" of s; for a complete proof, see [6]). In the sequel, we shall take $V(x) = \mathcal{R}_{\min}(x) =$ smallest eigenvalue of $\mathcal{R}_x$.

Applying Theorem 3.1 to the operators $T' = \mathcal{D}^2$ and $T = \Delta_M + V$, we get:

Corollary 3.3 (spectral comparison between Dirac and Schrödinger operators). *Let $(E, \langle\ ,\ \rangle, D)$ be a vector Riemannian bundle of rank r on a compact boundaryless Riemannian manifold (M,g), endowed with a* *compatible connection D. Let $\mathcal{R}$ be any field of symmetric endomorphisms of the fibers, and let $\mathcal{R}_{\min}(x)$ be the minimal eigenvalue of $\mathcal{R}_x$ at $x \in M$. Then the estimates* (i) *and* (ii) *of Theorem 3.1 hold when applied to the operators $T' = D^*D + \mathcal{R}$ and $T = \Delta_M + \mathcal{R}_{\min}$. In particular, if there exists a Dirac operator $\mathcal{D}$ such that $\mathcal{D}^2 = D^*D + \mathcal{R}$,* (i) *and* (ii) *are estimates for the squared eigenvalues of $\mathcal{D}$ in terms of the eigenvalues of Δ_M.*

4 – Application to the classical Dirac operator

We come back now to the situation described in section 1: (M, g, γ) is a n-dimensional compact Riemannian spin manifold whitout boundary, $S \longrightarrow M$ is the bundle of spinors, endowed with its Spin_n-invariant metric $\langle\ ,\ \rangle$ and with the compatible connection ∇ induced by the Levi-Civita connection; the rank of S is $2^{[\frac{n}{2}]}$. Let us consider the *classical Dirac operator* $\mathcal{D}$ acting on spinors: the corresponding Weitzenböck formula is in this case the *Lichnerowicz formula* (1.2),

$$\mathcal{D}^2 = \nabla^*\nabla + \frac{s}{4}\mathrm{Id}\,,$$

where s is the *scalar curvature* of (M, g).

Applying direcly Theorem 3.1 or Corollary 3.3 to the operators $T' = \mathcal{D}^2$ and $T = \Delta_M + \frac{s}{4}$, we get estimates of type (i), (ii), which are not sharp: they give indeed *Lichnerowicz inequality* (1.3) for the minimal eigenvalue of $\mathcal{D}$:

$$\lambda_1(\mathcal{D})^2 \geq \frac{s_0}{4}$$

where $s_0 = \min_{x \in M} s(x)$. But we know the best inequality $\lambda_1(\mathcal{D})^2 \geq \frac{n}{4(n-1)} s_0$ (*Friedrich's inequality* (1.4)). To obtain inequalities of type (i), (ii) which give this sharp result on the minimal eigenvalue, i.e. to get Theorem 1.7, we must consider another operator, built up with the modified connection (1.5):

THEOREM 4.1. *Let (M, g, γ) be a compact boundaryless Riemannian spin manifold of dimension n. For any set $\{\lambda_i(\mathcal{D})\}_{i \in I}$ of eigenvalues of* *the Dirac operator $\mathcal{D}$, one has:*

(i) $\sup_{i\in I} \lambda_i(\mathcal{D})^2 \geq \frac{n}{n-1}\Big((1-C(r))\lambda_1\Big(\Delta_M+\frac{s}{4}\Big)+C(r)\lambda_{k+1}\Big(\Delta_M+\frac{s}{4}\Big)\Big);$

(ii) $$\sum_{i\in I}\lambda_i(\mathcal{D})^2 \geq \frac{n}{n-1}\Big((\#I-k)\lambda_1\Big(\Delta_M+\frac{s}{4}\Big)+ + \frac{1}{2}\sum_{j=1}^{k}\lambda_j\Big(\Delta_M+\frac{s}{4}\Big)+\#IC(r)\lambda_{k+1}\Big(\Delta_M+\frac{s}{4}\Big)\Big).$$

where $r=2^{[\frac{n}{2}]}$, $k=[\frac{\#I}{2^{[n/2]}+1}]$ *and* $C(r)=\frac{1}{8(2^{[n/2]}+1)^2}$.

Proof. Let us consider the connection (1.5), $\nabla^\lambda_X\varphi=\nabla_X\varphi+\frac{\lambda}{n}X\cdot\varphi$, and recall that the construction of ∇^λ is such that any real Killing spinor (i.e. ∇^λ-parallel spinor) is an eigenspinor of $\mathcal{D}$ related to the eigenvalue λ. A direct calculation gives $|\nabla\varphi|^2=|\nabla^\lambda\varphi|^2+2\frac{\lambda}{n}\langle\mathcal{D}\varphi,\varphi\rangle-\frac{\lambda^2}{n}|\varphi|^2$ (see (1.6)). Injecting this in Lichnerowicz formula (1.2) and integrating on M, we get

$$\int_M\Big(\langle\mathcal{D}^2\varphi,\varphi\rangle-2\frac{\lambda}{n}\langle\mathcal{D}\varphi,\varphi\rangle+\frac{\lambda^2}{n}|\varphi|^2\Big)=\int_M\Big\langle\Big(\nabla^{\lambda^*}\nabla^\lambda+\frac{s}{4}\mathrm{Id}\Big)\varphi,\varphi\Big\rangle.$$

It follows that the operators $\mathcal{D}^2-2\frac{\lambda}{n}\mathcal{D}+\frac{\lambda^2}{n}\mathrm{Id}$ and $\nabla^{\lambda^*}\nabla^\lambda+\frac{s}{4}\mathrm{Id}$ have same quadratic forms and hence same eigenvalues. One verifies easily that $\mathcal{D}\varphi=\lambda\varphi$ implies that $\frac{n-1}{n}\lambda^2$ is an eigenvalue for the above operators. Then it suffices to apply Theorem 3.1 to the operators $T'=\nabla^{\lambda^*}\nabla^\lambda+\frac{s}{4}\mathrm{Id}$ and $T=\Delta_M+\frac{s}{4}$ to achieve the proof. □

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