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Heat content and mean curvature

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RIASSUNTO: *In questo articolo si illustrano dei risultati sulla funzione che misura la quantità di calore all'interno di un dominio di una varietà riemanniana. In particolare, si dà un algoritmo per il calcolo ricorsivo della serie asintotica di tale funzione (per tempi piccoli) su un dominio a bordo liscio, e i primi tre termini di tale espansione per poliedri convessi di dimensione arbitraria. Si dànno inoltre delle stime ottimali del calore nel caso in cui sia la curvatura di Ricci del dominio che la curvatura media del bordo sono positive o nulle.*

ABSTRACT: *We use distance function methods to obtain results on the heat content of a domain in an arbitrary Riemannian manifold. In particular, we give an algorithm for the calculation of the complete asymptotic series (for small times) of the heat content of a domain with smooth boundary, and the first three terms of this expansion for convex polyhedrons of arbitrary dimensions. We also write optimal upper and lower bounds of the heat content when the Ricci curvature of the domain and the mean curvature of its boundary are non-negative.*

– Introduction

Let (M^n, g) be a Riemannian manifold, and let Ω be an open domain in M with piecewise smooth boundary and compact closure. We consider the following problem of heat diffusion. Assume that Ω has constant

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temperature equal to one at time $t = 0$, and that the boundary $\partial\Omega$ is kept at zero temperature at all times. The question is:

What is the amount of heat remaining in Ω at time t?

This function of time, which we denote by $H(t)$ and call the *heat content* of Ω, can be written as:

$$H(t) = \int_\Omega u_t(x)\, dx \tag{1}$$

where $u_t(x)$ is the *temperature* at time t, at the point $x \in \Omega$: u_t is the solution of the *heat equation* on Ω: $\Delta u + \frac{\partial u}{\partial t} = 0$ with Dirichlet boundary conditions (which means $u_t = 0$ on $\partial\Omega$ for all t), and with unit initial conditions: $u_0 = \mathbf{1}_\Omega$. Here Δ is the Laplace- Beltrami operator of M (it depends on the Riemannian metric g: if the manifold is the Euclidean space $\mathbb{R}^n$ then $\Delta = -\frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2}$), and integration in (1) is taken with respect to the measure dx induced from the Riemannian metric.

This paper is a survey of some of the results contained in [13] and [14], to which we refer for complete proofs. Some of the results were announced in [15].

Here is a description of the contents of the paper. In Section 1, which is standard, we relate the heat content with the spectrum of the domain, and we observe that the heat content decreases to zero exponentially as time tends to infinity, with speed given by the first Dirichlet eigenvalue of the Laplace operator on Ω. In Section 2 we give the main technical tool, based on distance function methods, which allows to essentially reduce the study of the heat content from dimension n to dimension 1, the parameter being the distance from the boundary; as the Laplacian of this distance is the mean curvature (but considered in a distributional sense), we are able to relate the heat content with the mean curvature in a direct way (see formula (9)). Some of the previous results on the heat content have focused on its asymptotic behavior for small time (see [1]-[5], [10]) and we will generalize some of them by a uniform method in Section 3 and 4, while in Section 5 we give optimal bounds which are valid for all times.

Let us assume that the boundary is smooth. Then van den Berg and Le Gall showed in [4] that, if Ω is a domain in $\mathbb{R}^n$, then, as $t \to 0$:

$$H(t) \sim \mathrm{vol}(\Omega) - \frac{2}{\sqrt{\pi}}\mathrm{vol}(\partial\Omega)\sqrt{t} + \frac{n-1}{2}\int_{\partial\Omega} \eta \cdot t + O(t^{3/2}) \tag{2}$$

where η is the mean curvature. Shortly after, van den Berg and Gilkey observed in [3] that the heat content of a domain in a arbitrary manifold admits, as $t \to 0$, an asymptotic series in powers of $\sqrt{t}$:

$$H(t) \sim \mathrm{vol}(\Omega) - \sum_{k=1}^{\infty} \beta_k t^{k/2} \tag{3}$$

and then proceeded to compute the coefficients β_k up to $k = 4$.

In Section 3 we sketch the argument which, using distance function methods, will lead to the recursive calculation of the entire series (3). We show that for each k, there exists a differential operator $\tilde{D}_k$ belonging to an algebra with only two generators, such that $\beta_k = \int_{\partial\Omega} \tilde{D}_k \rho$, where ρ is the distance function to the boundary, and we give an algorithm for the determination of $\tilde{D}_k$ knowing all $\tilde{D}_i, i < k$. The algorithm is proved in [14].

Polyhedral boundaries are considered in Section 4, where we compute the third term $\beta_2 t$ of the expansion of the heat content of a convex polyhedron in $\mathbb{R}^n$ for small times:

$$\beta_2 = 4 \sum_E \mathrm{vol}_{n-2}(E) \int_0^\infty \left(1 - \frac{\tanh(\gamma x)}{\tanh(\pi x)}\right) dx$$

The sum is extended over all $(n-2)-$dimensional faces of Ω (the edges if $n = 3$), and γ is the interior angle at E. This generalizes to any dimension, in the convex case, the calculation done in [5] for polygonal regions in the plane.

Finally, in Section 5, we give an optimal upper and lower bound of the heat content, in terms of the volume of the parallel domains $\Omega(r) = \{x \in \Omega : d(x, \partial\Omega) > r\}$:

$$\frac{4}{\sqrt{\pi t}} \int_0^\infty e^{-r^2/t} \mathrm{vol}(\Omega(r)) dr - \mathrm{vol}(\Omega) \le H(t) \le \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-r^2/4t} \mathrm{vol}(\Omega(r)) dr.$$

These bounds apply when the Ricci curvature of the domain and the mean curvature of its boundary are both non-negative. Estimating the function $r \mapsto \mathrm{vol}(\Omega(r))$ first for domains with smooth boundary, and then for arbitrary convex domains, we obtain explicit bounds of the heat content in those cases.

1 – Heat content and the spectrum

To illustrate the link with the spectrum, let us introduce the Dirichlet *heat kernel* of Ω (also called the fundamental solution of the heat equation on Ω). This is the function $k(t,x,y)$ which is smooth on $(0,\infty)\times\Omega\times\Omega$, and which represents the temperature at time t, at the point x, assuming that one unit of heat is placed at y at time $t=0$, and that the boundary is kept at temperature zero at all times. It is called "fundamental" because the solution $\phi_t(x)$ of the heat equation with initial data $\phi(x)$ and Dirichlet boundary conditions is then given by convolution with the heat kernel: $\phi_t(x) = \int_\Omega k(t,x,y)\phi(y)\,dy$. Taking $\phi = \mathbf{1}_\Omega$ and inserting in (1), we observe that the heat content can be written as:

$$H(t) = \int_{\Omega\times\Omega} k(t,x,y)\,dx\,dy \tag{4}$$

Next, the eigenvalues of the Dirichlet problem on Ω ($\Delta\phi = \lambda\phi$, $\phi = 0$ on $\partial\Omega$) form a non-decreasing sequence going to infinity:

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \dots$$

(each eigenvalue is repeated here according to its multiplicity; the first eigenvalue is easily shown to be simple). Let $\{\phi_k\}$ be a corresponding orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions. One proves that: $k(t,x,y) = \sum_{k=1}^{\infty} e^{-\lambda_k t}\phi_k(x)\phi_k(y)$ (in fact the series converges to a smooth function satysfying the properties of the heat kernel). One then gets the following Fourier series representation of the heat content:

$$H(t) = \sum_{k=1}^{\infty} \Big\{ \int_\Omega \phi_k \Big\}^2 e^{-\lambda_k t} \tag{5}$$

1.1 – Large time behavior of the heat content

It is determined by the *lowest eigenvalue* λ_1 of the domain Ω. First note that, expanding the constant function $\mathbf{1}_\Omega$ in a Fourier series, we obtain $\mathrm{vol}(\Omega) = \sum_{k=1}^{\infty} \big\{ \int_\Omega \phi_k \big\}^2$. Therefore, for all t:

$$\Big\{ \int_\Omega \phi_1 \Big\}^2 e^{-\lambda_1 t} \le H(t) \le \mathrm{vol}(\Omega) e^{-\lambda_1 t} \tag{6}$$

As $t \to \infty$ the lower bound is sharp, because, from (5):

$$e^{\lambda_1 t} H(t) = \Big\{ \int_\Omega \phi_1 \Big\}^2 + O(e^{-(\lambda_2 - \lambda_1)t}) \tag{7}$$

Finally, note that:

$$e^{-\lambda_1} = \lim_{t \to \infty} H(t)^{1/t}$$

2 – Reduction to a one dimensional heat problem

2.1 – The distance function and the mean-value lemma

Let Ω be an open domain with compact closure and piecewise-smooth boundary $\partial\Omega$. For $x \in \Omega$, denote by $\rho(x)$ the *minimum distance* of x from $\partial\Omega$. This defines the distance function $\rho : \Omega \to (0, \infty)$. How regular is it? Since $|\rho(x) - \rho(y)| \le d(x, y)$ (triangle inequality) we see that ρ is *Lipschitz regular* , hence continuous, on Ω. However, ρ is not differentiable all over Ω. This can be seen as follows. Denote by $Cut(\partial\Omega)$ (the *cut-locus* of $\partial\Omega$) the closure (in $\bar{\Omega}$) of the set A of all points of Ω which are equidistant from at least two different points of the boundary. A moment's thought shows that ρ has a jump at any point of A. It turns out, however, that the distance function is actually C^∞–smooth on the set of its *regular points* $\Omega \setminus Cut(\partial\Omega)$ (which follows from the smoothness of the normal exponential map), and that the singular set $Cut(\partial\Omega)$ has always *zero-measure in* Ω (this fact is classical for the cut-locus of a point; see [13, App.D] for a detailed proof in our case).

At any regular point x we have $\|\nabla\rho(x)\| = 1$, hence the level set (parallel submanifold) $\rho^{-1}(\rho(x))$ is locally a *smooth submanifold* near x. A straightforward calculation shows that, if x is a regular point:

$$\Delta\rho(x) = \text{trace of the second fundamental form of the level set through } x$$

If the boundary is *smooth*, then the cut-locus is at positive distance from the boundary; this distance, called the *injectivity radius* and denoted by Inj, is not greater than the minimum distance of $\partial\Omega$ from its focal set (the set of critical values of the normal exponential map). We conclude that if r is less than the injectivity radius, then the level set $\rho^{-1}(r)$ is a

$C^\infty-$smooth hypersurface, and $\Delta\rho$, restricted to it, is $(n-1)$ times the *mean curvature function* of this level set.

As the distance function is globally only Lipschitz, its Laplacian must be taken in the sense of distributions. The distributional Laplacian of ρ is by definition the element of the dual of $C_c^\infty(\Omega)$ (the space of smooth, compactly supported functions on Ω), which takes the test function ϕ to the number $\int_\Omega \rho\Delta\phi$.

It turns out (see [13, lemma 1.3]) that $\Delta\rho$ splits as a sum $\Delta_{reg}\rho+\Delta_{cut}\rho$ where $\Delta_{reg}\rho$ is the Laplacian of the restriction of ρ to the set of its regular points (hence this regular part gives the mean curvature of the level sets), and where $\Delta_{cut}\rho$ is a *positive Dirac measure* supported on the cut-locus. As $\Delta_{reg}\rho$ is summable on Ω, we then conclude that $\Delta\rho$ itself is a *measure on* Ω.

We now state the fundamental technical lemma, to be extensively used in the sequel. This lemma holds more generally for the distance function to any piecewise-smooth submanifold of M, and has been used in [13] to obtain sharp estimates of eigenvalues.

In what follows, $\Omega(r)$ will denote the set of points of Ω which are at distance greater than r from the boundary of Ω.

MEAN-VALUE LEMMA ([13, thm. 1.9]). *Let $u \in C^2(\Omega)$, and for $r > 0$, let $F(r) = \int_{\Omega(r)} u\, dv_n$. Then the following equality holds as measures on $(0,\infty)$:*

$$F''(r) = -\int_{\Omega(r)} \Delta u + \rho_*(u\Delta\rho)(r)$$

where $\rho_*(u\Delta\rho)$ is the push-forward measure of $u\Delta\rho$ by ρ, naturally defined by the formula: $\int_0^\infty \psi\rho_*(u\Delta\rho) = \int_\Omega u(\psi\circ\rho)\Delta\rho$.

REMARK. For example, if Ω is a rectangle with sides a and b, with $a \geq b$, and if $u = \mathbf{1}_\Omega$, so that $F(r) = \mathrm{vol}(\Omega(r))$, then

$$F''(r) = 8 + 2(a-b)\delta_{b/2}(r)$$

where $\delta_{b/2}$ is the Dirac measure supported at $b/2$. (This can be checked directly by an elementary calculation).

Next, we give a local version of the mean-value lemma (used in Section 3). Assume that the boundary is smooth, so that $Inj > 0$. Then

$\rho^{-1}(r)$ is a smooth hypersurface for $r \in (0, Inj)$, and $F(r)$ will be smooth on that interval.

Mean-value lemma (local version). *If $\partial\Omega$ is smooth, and if $r < Inj(\partial\Omega)$:*

$$F''(r) = -\int_{\Omega(r)} \Delta u \, dv_n + \int_{\rho^{-1}(r)} u\Delta\rho \, dv_{n-1}$$

We give the proof of this result. In what follows, we make use of the co-area formula, which states that:

$$\int_{\Omega} u \, dv_n = \int_0^\infty \int_{\rho^{-1}(r)} u(y) \, dH_{n-1}(y) \, dr \tag{8}$$

where H_{n-1} is the $(n-1)-$dimensional Hausdorff measure on $\rho^{-1}(r)$. When the level sets are regular submanifolds, the Hausdorff measure on them does coincide with the induced Riemannian measure, which we denote by dv_{n-1}, or which we simply omit denoting at all.

Then let $r < Inj$. From the co-area formula we obtain immediately $F'(r) = -\int_{\rho^{-1}(r)} u \, dv_{n-1}$. Next, the gradient $\nabla\rho$ is of course orthogonal to the level sets and has unit length. Therefore, by Green's and co-area formulas:

$$\begin{aligned}\int_{\rho^{-1}(r+\epsilon)} u \, dv_{n-1} - \int_{\rho^{-1}(r)} u \, dv_{n-1} &= \int_{\Omega(r)\setminus\Omega(r+\epsilon)} (\nabla u \cdot \nabla\rho - u\Delta\rho) \, dv_n \\ &= \int_r^{r+\epsilon} \int_{\rho^{-1}(s)} (\nabla u \cdot \nabla\rho - u\Delta\rho) \, dv_{n-1} \, ds\end{aligned}$$

Dividing by ϵ, passing to the limit as $\epsilon \to 0$ and applying Green's formula again, we obtain the assertion.

Remark. Why the name "mean-value lemma"? Let Ω be a ball in a Euclidean space, and let u be harmonic. Let $M(r)$ be the mean value of u on the sphere $\rho^{-1}(r)$; as the mean curvature (hence $\Delta\rho$) is constant on each sphere we easily conclude by applying the local formula that the derivative of $M(r)$ vanishes identically. Hence the mean value of a harmonic function on each sphere equals the value of the function at its center: this is the classical mean-value lemma. The same proof in fact shows that, if the mean curvature is constant on each level set of a distance function, the mean value of a harmonic function is the same on all level sets.

Remark. The global formula follows from the local one with the push-forward operator replacing the operator of integration on level sets (which does not make sense for distributions). In fact, if $T = T(x)$ is a genuine function on Ω then $\rho_*(T)$ is the function (regular distribution) given by integration on the level sets:

$$\rho_*(T)(r) = \int_{\rho^{-1}(r)} T(x)\, dH_{n-1}(x)$$

which follows immediately from the co-area formula.

All applications to heat diffusion stem from the following consequence of the mean-value lemma:

Corollary: reduction to dimension one. *Let $\phi_t(x)$ be any solution of the heat equation on Ω, and let $F(t,r) = \int_{\Omega(r)} \phi_t(x)\, dx$ be the corresponding heat content. Then $F(t,r)$ satisfies the following heat equation on the half-line:*

$$-\frac{\partial^2 F}{\partial r^2} + \frac{\partial F}{\partial t} = -\rho_*(\phi_t \Delta\rho)$$

The advantage of dealing with one-dimensional heat equations (even with a potential, like the above), lies in the fact that their solutions can be explicitly written down in terms of usual exponentials, by using Duhamel principle (see [9]). Taking $w_t = 1 - u_t$ in the corollary, one gets the following representation of the heat content:

$$\begin{aligned} H(t) &= \mathrm{vol}(\Omega) - \frac{2}{\sqrt{\pi}}\mathrm{vol}(\partial\Omega)\sqrt{t} + \\ &+ \int_0^t \frac{1}{\sqrt{\pi(t-\tau)}} \int_0^\infty e^{-r^2/4(t-\tau)} \rho_*((1-u_\tau)\Delta\rho)\, dr\, d\tau \end{aligned} \tag{9}$$

We stress the fact that (9) holds for any domain with piecewise-smooth boundary.

3 – Small-time asymptotics of the heat content: smooth boundaries

3.1 – The principle of not feeling the boundary

While the heat content of Ω, when time is very large, depends on the global invariant λ_1, we should expect that when time is very small only the geometry of Ω near its boundary will affect the heat distribution. The reason is clear: for small times, a point far from the boundary can't feel the cooling effect of $\partial\Omega$. This is basically the "principle of not feeling the boundary" formulated in this language by Kac in [12], and, when Ω is a domain in Euclidean space, formally expressed by the inequality, due to Paul Levy:

$$
\begin{aligned}
1 - u_t(x) &\leq \frac{2}{(4\pi t)^{n/2}} \int_{\|y\| \geq d(x,\partial\Omega)} e^{-\|y\|^2/4t}\, dy \\
&\leq 2n e^{-d(x,\partial\Omega)^2/4nt}
\end{aligned}
\tag{10}
$$

(for domains in arbitrary Riemannian manifolds, see [14, App.B]).

We assume in this section that $\partial\Omega$ is smooth, and we consider the asymptotic series (3). From the principle of not feeling the boundary we derive the following:

FACT. *The coefficients β_k of the asymptotic series* (3) *of the heat content depend only on the geometry of Ω near $\partial\Omega$.*

In particular, domains which are locally isometric near the respective boundaries will give rise to the same sequence of coefficients β_k. To see how this follows from (10), let A be a proper, open subset of our domain Ω, and let ϕ be a smooth function which is zero on A and which is identically equal to one on some neighborhood of $\partial\Omega$ in Ω. Then:

$$
\sum_{k=1}^{\infty} \beta_k t^{k/2} \sim \mathrm{vol}(\Omega) - H(t) = \int_\Omega (1-u_t) = \int_\Omega (1-u_t)\phi + \int_\Omega (1-u_t)(1-\phi)
$$

By (10), the second integral on the right is bounded above by a constant times the exponentially decreasing function $e^{-\alpha/t}$ where $\alpha = \frac{1}{4n} d(A,\partial\Omega)^2 > 0$; therefore the second integral decreases to zero faster than any power of t, and does not contribute to the asymptotics on the left. As the open set A fills Ω, the first integral is supported arbitrarily near $\partial\Omega$, thus proving the assertion.

3.2 – The calculation of the series

We now allow arbitrary initial conditions $\phi \in C^\infty(\bar{\Omega})$, and sketch how to obtain the complete asymptotics for small time of the more general integral:

$$H_\phi(t) = \int_\Omega \phi_t(x)\,dx \tag{11}$$

where ϕ_t is the solution of the heat equation on Ω satysfying Dirichlet conditions on the boundary, and having initial conditions ϕ. The corresponding asymptotic series will be written:

$$H_\phi(t) \sim \int_\Omega \phi - \sum_{k=1}^\infty \beta_k(\phi) t^{k/2} \tag{12}$$

and we observe that the coefficients β_k of the series (3) are obtained by setting $\phi = \mathbf{1}_\Omega$: $\beta_k = \beta_k(1)$.

We iterate the Corollary of Section 2 to the integral $F(t,r) = \int_{\Omega(r)}(1-u_t(x)\phi(x)\,dx$, because then $\sum_{k=1}^\infty \beta_k(\phi)t^{k/2} = F(t,0)$. Thanks to the principle of not feeling the boundary, the asymptotics of $F(t,0)$ are unchanged if we replace the initial data ϕ by a function $\tilde{\phi}$ which agrees with ϕ near $\partial\Omega$, and is identically zero outside a small neighborhood U of $\partial\Omega$ which does not meet the cut-locus; and so we can assume from the start that ϕ is actually supported on U. We do so because then $F(t,r)$ will be $C^\infty-$smooth on $(0,\infty)\times[0,\infty)$, and we have an elementary method of writing the asymptotics of $F(t,0)$ in that case. In what follows, L will denote the operator $-\dfrac{\partial^2}{\partial r^2} + \dfrac{\partial}{\partial t}$.

LEMMA 4.5 (Iterated Duhamel principle) (see [14, lemma 4.5]). *Let $F(t,r)$ be smooth on $(0,\infty)\times[0,\infty)\to\mathbb{R}$, and assume that:*

(i) $L^kF(0,r) = \lim_{t\to 0} L^kF(t,r)$ *exists in the sense of distributions for each $k \geq 0$;*

(ii) *As $t\to 0$, all $L^kF(t,0)$ and $\dfrac{\partial}{\partial r}L^kF(t,0)$ converge to a finite limit.*

Then, for all $m \in \mathbb{N}$, and $t > 0$:

$$
\begin{aligned}
F(t,0) = \sum_{k=0}^{m} \frac{t^k}{k!} \int_0^\infty & e(t,r,0) L^k F(0,r)\, dr - \\
&+ \frac{1}{\sqrt{\pi}} \sum_{k=0}^{m} \frac{1}{k!} \int_0^t \frac{\partial}{\partial r} L^k F(\tau,0)(t-\tau)^{k-1/2}\, d\tau + \\
&+ \frac{1}{m!} \int_0^t \int_0^\infty e(t-\tau,r,0) L^{m+1} F(\tau,r)(t-\tau)^m\, dr\, d\tau.
\end{aligned}
$$

One then shows that in our case the integral $\int_0^t\int_0^\infty e(t-\tau,r,0)L^{m+1}F(\tau,r)(t-\tau)^m dr\, d\tau$ is small of order $t^{\frac{m+1}{2}}$ as $t \to 0$. It follows that:

$$
\begin{aligned}
\sum_{k=1}^{\infty} \beta_k(\phi) t^{k/2} \sim \sum_{k=0}^{m} \frac{t^k}{k!} \int_0^\infty & e(t,r,0) L^k F(0,r)\, dr - \\
&+ \frac{1}{\sqrt{\pi}} \sum_{k=0}^{m} \frac{1}{k!} \int_0^t \frac{\partial}{\partial r} L^k F(\tau,0)(t-\tau)^{k-1/2}\, d\tau
\end{aligned}
\tag{13}
$$

It then remains to calculate the distributions $L^k F(0,r)$ and the functions $\frac{\partial}{\partial r} L^k F(t,0)$. By the local mean-value lemma and Green's formula:

$$
LF(t,r) = \int_{\rho^{-1}(r)} (1-u(t,x)) N\phi(t,x)\, dx + \int_{\Omega(r)} (1-u(t,x)) \Delta\phi(x)\, dx
$$

where N is the first order differential operator acting on $C^\infty(U)$, and defined by $N\phi = 2\nabla\phi \cdot \nabla\rho - \phi\Delta\rho$, and Δ is the Laplacian of the ambient manifold. It is now clear why $L^k F(0,r)$ and $\frac{\partial}{\partial r} L^k F(t,0)$ can be expressed in terms of the algebra $\mathcal{A}$ generated by the operators N and Δ. Substituting in (13) we obtain, after a good amount of technical work, the complete asymptotics of $F(t,0)$.

3.3 – The recursive algorithm

Let us define the operators R_{kj} and S_{kj}, for $k \geq 1$ and $j \geq 0$, inductively by:

$$
\begin{cases}
R_{kj} = -(N^2+\Delta) R_{k-1,j} + N S_{k-1,j} \\
S_{kj} = N R_{k-1,j-1} + \Delta N R_{k-1,j} - \Delta S_{k-1,j}
\end{cases}
$$

and set $R_{00} = Id, S_{00} = 0$, and $R_{kj} = S_{kj} = 0$ whenever k or j is a negative integer. Then set: $\{a,b\} = \frac{\Gamma(a+b+1/2)}{(a+b)!\Gamma(a+1/2)}$, and define the operators $Z_n, \alpha_n \in \mathcal{A}$ by: $Z_{n+1} = \sum_{j=0}^{n}\{n+1, j-1\}R_{n+j,j}$, and $\alpha_n = \sum_{j=0}^{n+1}\{n,j\}S_{n+j,j}$.

THEOREM. *Let $\beta_k(\phi)$ be the coefficient of $t^{k/2}$ in the asymptotic series of the heat content with initial data ϕ. Then, for each $k \geq 1$, we have $\beta_k(\phi) = \int_{\partial\Omega} D_k\phi$, where D_k is a homogeneous polynomial of degree $k-1$ in the operators N and Δ. The D_k's are determined by the following recursive formulas:*

$$D_1 = \frac{2}{\sqrt{\pi}} Id$$

$$D_{2n} = \frac{1}{\sqrt{\pi}} \sum_{i=1}^{n} \frac{\Gamma(i+\frac{1}{2})\Gamma(n-i+\frac{1}{2})}{n!} D_{2i-1}\alpha_{n-i}$$

$$D_{2n+1} = \frac{1}{\sqrt{\pi}} Z_{n+1} + \frac{1}{\sqrt{\pi}} \sum_{i=1}^{n} \frac{i!\Gamma(n-i+\frac{1}{2})}{\Gamma(n+\frac{3}{2})} D_{2i}\alpha_{n-i}$$

where Γ is the gamma function.

The operators $D_1, \ldots, D_8$ have been explicited in [14, Table 1.4]. Setting $\phi = 1$ we obtain the following expression of the coefficients $\beta_1, \ldots, \beta_8$ of (3):

$$\beta_1 = \frac{2}{\sqrt{\pi}}\mathrm{vol}(\partial\Omega); \quad \beta_2 = -\frac{1}{2}\int_{\partial\Omega} \Delta\rho; \quad \beta_3 = -\frac{1}{6\sqrt{\pi}}\int_{\partial\Omega} N\Delta\rho;$$

$$\beta_4 = \frac{1}{16}\int_{\partial\Omega} \Delta^2\rho; \quad \beta_5 = \frac{1}{240\sqrt{\pi}}\int_{\partial\Omega} (N^3 + 8N\Delta)\Delta\rho;$$

$$\beta_6 = -\frac{1}{768}\int_{\partial\Omega} (\Delta N^2 + N\Delta N - N^2\Delta + 8\Delta^2)\Delta\rho;$$

$$\beta_7 = -\frac{1}{6720\sqrt{\pi}}\int_{\partial\Omega} (N^5 + 4N^3\Delta + 4N^2\Delta N + 4N\Delta N^2 + 40N\Delta^2 + \\ + 8\Delta N\Delta - 8\Delta^2 N)\Delta\rho;$$

$$\beta_8 = \frac{1}{24576}\int_{\partial\Omega} (40\Delta^3 + 8N\Delta^2 N - 8N^2\Delta^2 + 4\Delta^2 N^2 + 4\Delta N\Delta N + \\ + 4\Delta N^2\Delta + N\Delta N^3 + \Delta N^4 - N^4\Delta)\Delta\rho.$$

In general β_k, for $k \geq 2$, is given by integration on $\partial\Omega$ of $\tilde{D}_k\Delta\rho$, where $\tilde{D}_k$ is a differential operator of order $k-2$ in N and Δ.

The above expressions of β_k can be converted into expressions involving the classical invariants (curvature and second fundamental form). For example:

$$\beta_3 = -\frac{1}{6\sqrt{\pi}} \int_{\partial\Omega} (2\|S\|^2 + Ric(\nabla\rho, \nabla\rho) - (trS)^2)$$

where S is the second fundamental form.

Special calculations. Let Ω be a domain in $\mathbb{R}^3$. We can then write the coefficients β_3, β_4 and β_5 in terms of the mean curvature η and the Gaussian curvature K of $\partial\Omega$. Actually, the coefficients are better expressed in terms of η and the function $W = \eta^2 - K$; let $\tilde{\nabla}$ denote the gradient in $\partial\Omega$. Then:

$$\begin{aligned}
\beta_3 &= -\frac{2}{3\sqrt{\pi}} \int_{\partial\Omega} W \\
\beta_4 &= -\frac{1}{2} \int_{\partial\Omega} \eta W \\
\beta_5 &= -\frac{1}{3\sqrt{\pi}} \int_{\partial\Omega} (4\eta^2 + W)W + \frac{2}{15\sqrt{\pi}} \int_{\partial\Omega} \|\tilde{\nabla}\eta\|^2
\end{aligned}$$

We show how to derive these expressions.

Let U be a small neighborhood of $\partial\Omega$ so that ρ is smooth on U. Then, for r small, the level surface $\rho^{-1}(r)$ will be smooth. For $x \in U$, we define $\eta(x)$ to be the mean curvature at x of the level surface passing through x, and we extend K and W in a similar way. Note that $\Delta\rho = 2\eta$. For a smooth function f on U we denote by f' the normal derivative of f (that is $f' = \nabla f \cdot \nabla\rho$). Then we saw that $\frac{d}{dr} \int_{\rho^{-1}(r)} f = \int_{\rho^{-1}(r)} (f' - f\Delta\rho)$. One has, classically: $\eta_i' = \eta_i^2$ for the principal curvatures of the level surfaces. Using this, one can write any derivative of η and any power $N^k\Delta\rho$ in terms of η and W; for example, $N\Delta\rho = 4W$, and we immediately get β_3.

For $f \in C^\infty(U)$, we can split the Laplacian Δf into its *radial part* $\Delta_r f = -f'' + f'\Delta\rho$ and its tangential part $\tilde{\Delta} f$ which is just the Laplacian on the level surfaces. As the tangential Laplacian integrates to zero on $\partial\Omega$ (and on each level surface), a straightforward calculation of $\Delta_r\Delta\rho$ now gives β_4. As for β_5, for any function f we have $\int_{\partial\Omega} N\tilde{\Delta} f = 2\int_{\partial\Omega} \tilde{\nabla}\eta \cdot \tilde{\nabla} f$, which can be verified by differentiating the identity $\int_{\rho^{-1}(r)} \tilde{\Delta} f =$

0 with respect to r, and applying Green' formula. This formula and straightforward work give the required coefficient. $\square$

On the 2-sphere the function W is identically zero, and η is constant. Hence, for a ball in $\mathbb{R}^3$, $\beta_3 = \beta_4 = \beta_5 = 0$. We have in fact in that case $\beta_k = 0$ for all $k \geq 3$ (this fact has already been observed in [3] and [8]). More generally, for balls in a 3-dim. space form, our algorithm simplifies drastically, and we can write the asymptotics in closed form:

PROPOSITION (see [14, prop. 4.21]). *Let* Ω *be a ball* (*or an annulus*) *in the simply connected* $3-$*dim. manifold of constant curvature* K. *Then* $\beta_2 = \int_{\partial\Omega} \eta$, $\beta_{2n} = 0$ *for all* $n \geq 2$, *and:*

$$\beta_{2n+1} = -\frac{2\mathrm{vol}(\partial\Omega)}{\sqrt{\pi}} \cdot \frac{K^n}{n!(4n^2-1)}$$

for all $n \geq 0$.

The three dimensional case is simpler because in that case the operators N and Δ commute, when applied to functions which depend only on the distance from the boundary (radial functions).

4 – Asymptotics of the heat content on a convex polyhedron

What happens to the asymptotic expansion of the heat content when the boundary is no longer smooth, but only piecewise-smooth? Let us first observe that the coefficient β_2 of the term in t is not continuous under smooth approximations of domain: for example, if Ω is the unit square in the plane, and if we round off the corners a little bit, any approximating domain will have $\beta_2 = \pi$, while the exact value of β_2 is $\frac{16}{\pi}$. This phenomenon may be explained by observing that, if the boundary is piecewise-smooth, the cut-locus hits the boundary and therefore the singular part of the Laplacian of the distance function (which we called $\Delta_{cut}\rho$), contributes with a non-neglectable term to the double integral in (9).

In this section, it is exactly the contribution of this singular part $\Delta_{cut}\rho$ which we want to evaluate. We restrict ourselves to the case where Ω is a convex polyhedral body in $\mathbb{R}^n$, in which case $\Delta_{reg}\rho = 0$ identically. This leads to the following theorem, which generalizes the result of [5] referred to in the Introduction. The rough scheme of the proof is carried below; the main points are the explicit description of the measure $\rho_*(u_t\Delta\rho)$, and the fact that, on the cut-locus near an $(n-2)-$face, u_t is suitably approximated by the temperature on the infinite wedge bounded by the two hyperplanes which meet at the given face: this temperature can be explicited by special functions.

Theorem (see [13, thm. 3.3]). *If Ω is a convex polyhedral body in $\mathbb{R}^n$, then:*

$$\int_\Omega u_t(x)\,dx = \mathrm{vol}(\Omega) - \frac{2}{\sqrt{\pi}}\mathrm{vol}(\partial\Omega)\sqrt{t} + \beta_2 t + l(t)$$

with:

$$\beta_2 = 4\sum_E \mathrm{vol}_{n-2}(E)\cdot\int_0^\infty\left(1-\frac{\tanh(\gamma(E)x)}{\tanh(\pi x)}\right)dx$$

where E runs through the set of all $(n-2)-$dimensional faces of Ω (the "edges"f if $n=3$), and $\gamma(E)$ is the interior angle of the two $(n-1)-$faces whose intersection is E. The remainder $l(t)$ is bounded, in absolute value, for all t, by $Ct^{3/2}+h(t)$ for a constant C, and for a function $h(t)$ which is exponentially decreasing as $t\to 0$ (see [14] for an explicit expression of C and $h(t)$).

We sketch the proof of the theorem.

Let us first fix some notation. The closure of Ω is a *polytope*, i.e. is the intersection of a finite family $I=\{1,\dots m\}$ of closed half-spaces $\mathcal{H}_i$, where $\mathcal{H}_i=\{x\in\mathbb{R}^n : \rho_{\pi_i}(x)\geq 0\}$ and where ρ_{π_i} denotes the distance, taken with sign, from the oriented affine hyperplane π_i of $\mathbb{R}^n$. The $(n-1)$-*dimensional faces* of $\bar\Omega$ are the subsets of $\partial\Omega$ defined by: $\mathcal{F}_i=\pi_i\cap\bar\Omega$ for $i\in I$. By $\mathrm{vol}_d(P)$ we denote the Lebesgue measure in $\mathbb{R}^d$, and by γ_{ij} we denote the interior angle at $\mathcal{F}_i\cap\mathcal{F}_j$. Note that, if $\mathcal{F}_i$ and $\mathcal{F}_j$ are incident faces, then $0<\gamma_{ij}<\pi$. Our aim is then to prove that the expansion in Theorem 4 holds with:

$$\beta_2 = 2\sum_{i\neq j}\mathrm{vol}_{n-2}(\mathcal{F}_i\cap\mathcal{F}_j)\cdot\int_0^\infty\left(1-\frac{\tanh(\gamma_{ij}x)}{\tanh(\pi x)}\right)dx \tag{14}$$

For the proof, we let ρ denote the distance from $\partial\Omega$, and we will use representation (9) of the heat content; so we need to determine the behavior of the integral:

$$\int_0^t \frac{1}{\sqrt{\pi(t-\tau)}} \int_0^\infty e^{-r^2/4(t-\tau)} \rho_* \left((1-u_\tau)\Delta\rho\right)(r)\, dr\, d\tau$$

as $t \to 0$, and show that in fact this behavior is given by $\beta_2 t + O(t^{3/2})$.

Description of the cut-locus. The first thing to observe is that, since each level set $\rho^{-1}(r)$ is piecewise-linear (because of the convexity of the polyhedron), we have that $\Delta_{reg}\rho = 0$; hence $\Delta\rho = \Delta_{cut}\rho$ is *purely singular*. The cut-locus is the closure of the set of all points of Ω which can be joined to $\partial\Omega$ by at least two minimizing line segments. Therefore:

$$Cut(\partial\Omega) = \bigcup_{i\neq j} Cut_{ij} \quad \text{where} \quad Cut_{ij} = \{x \in \bar{\Omega} : \rho(x) = \rho_{\pi_i}(x) = \rho_{\pi_j}(x)\}$$

Then it is not difficult to show that for each $i \neq j$, Cut_{ij} is a polytope in the hyperplane $\pi_{ij} = \{x \in \Omega : \rho_{\pi_i}(x) = \rho_{\pi_j}(x)\}$ (the "bisecting hyperplane" of π_i and π_j). The next proposition shows that $\Delta\rho$ is indeed a Dirac measure supported on the cut-locus.

Proposition (see [13], Prop. 3.4]). *Let $\phi \in C^0(\bar{\Omega})$, and $\psi \in C^0([0,\infty))$. Then:*

$$\int_\Omega \phi\Delta\rho = \sum_{i\neq j} \cos(\frac{\gamma_{ij}}{2}) \int_{Cut_{ij}} \phi(x)\, dx;$$
$$\int_0^\infty \psi\rho_*(u\Delta\rho) = \sum_{i\neq j} \cos(\frac{\gamma_{ij}}{2}) \int_{Cut_{ij}} u(x)\, \psi(\rho(x))\, dx.$$

dx denoting Lebesgue measure on the hyperplane π_{ij} of $\mathbb{R}^n$.

By the Proposition and formula (9):

$$\begin{aligned} &\int_0^t \frac{1}{\sqrt{\pi(t-\tau)}} \int_0^\infty e^{-r^2/4(t-\tau)} \rho_* \left((1-u_\tau)\Delta\rho\right)(r)\, dr\, d\tau = \\ &= \sum_{i\neq j} \cos(\frac{\gamma_{ij}}{2}) \int_0^t \frac{1}{\sqrt{\pi(t-\tau)}} \int_{Cut_{ij}} e^{-\rho(x)^2/4(t-\tau)} (1-u_\tau(x))\, dx\, d\tau \end{aligned} \tag{15}$$

One reduces the right-hand side to $\beta_2 t + O(t^{3/2})$ in four steps.

Step 1. It is clear that a pair (i,j) for which Cut_{ij} is at positive distance ϵ from $\partial\Omega$ will contribute to the sum in (15) with a term (depending on ϵ) which is exponentially decreasing as $t \to 0$. For the computation of β_2 we can then restrict the sum in (15) to the pairs (i,j) for which $\mathcal{F}_i$ and $\mathcal{F}_j$ are intersecting faces.

Step 2. Approximation of $u(t,x)$. One can show that, modulo terms of order $t^{3/2}$ and higher, we can replace $1-u(\tau,x)$ on Cut_{ij} in (13) by the function $1-u_{ij}(\tau,x)$, where u_{ij} is the *temperature function relative to the infinite open wedge in* $\mathbb{R}^n$ *bounded by the oriented hyperplanes* π_i *and* π_j. This is in fact the most delicate step in the proof.

Step 3. We observe that, when restricted to $Cut_{ij} \subseteq \pi_{ij}$, the temperature function $u_{ij}(\tau,x)$ depends only on $\rho_{ij}(x) =$ distance of x from $\pi_i \cap \pi_j$, so that it can be written as $\tilde{u}_{ij}(\tau, \rho_{ij}(x))$ for a function $\tilde{u}_{ij} = \tilde{u}_{ij}(\tau, r)$. By the formula of co-area, applied to the function $\rho_{ij} : Cut_{ij} \to \mathbb{R}$, and by straightforward work, this implies that, modulo terms of order $t^{3/2}$ or higher, the right-hand side of (15) is given by:

$$
\begin{aligned}
&\sum_{(i,j)} \mathrm{vol}_{n-2}(\mathcal{F}_i \cap \mathcal{F}_j) \cos(\gamma_{ij}/2) \cdot \\
&\cdot \int_0^t \frac{1}{\sqrt{\pi(t-\tau)}} \int_0^\infty e^{-r^2 \sin^2(\gamma_{ij}/2)/4(t-\tau)} (1-\tilde{u}_{ij}(\tau,r))\, dr\, d\tau
\end{aligned}
\tag{16}
$$

Step 4. Take the Laplace transform (with respect to time) of (16). Evaluated at $s>0$, this is equal to:

$$
\frac{1}{\sqrt{s}} \sum_{(i,j)\in I_2} \mathrm{vol}_{n-2}(\mathcal{F}_i \cap \mathcal{F}_j) \cos(\gamma_{ij}/2) \int_0^\infty e^{-\sqrt{s} r \sin(\gamma_{ij}/2)} \left(\frac{1}{s} - \tilde{U}_{ij}(s,r) \right) dr
$$

where $\tilde{U}_{ij}(s,r)$ is the Laplace transform, at $s>0$, of $\tilde{u}_{ij}(\cdot, r)$. This function is computable: in fact, using Kontorovich-Lebedev's explicit expression of the Green's function of an infinite open wedge in the plane (already used in [5]), one has:

$$
\frac{1}{s} - \tilde{U}_{ij}(s,r) = \frac{2}{\pi s} \int_0^\infty K_{ix}(\sqrt{s} r) \frac{\cosh(\pi x/2)}{\cosh(\gamma_{ij} x/2)}\, dx
$$

Substituting, and using integral tables, one obtains the quantity $\frac{\beta_2}{s^2}$; taking inverse Laplace transform, one obtains the theorem.

We remark that, if $\dim(\Omega) = 2$, the proof simplifies considerably (steps 2 and 3 are in fact immediate), and we can easily extend it to cover the (not necessarily convex) polygonal case (see [13]), thus re-obtaining van den Berg-Srisatkunarajah's calculation.

REMARKS. (i) for a convex polyhedron in $\mathbb{R}^n$ one should have:

$$H(t) = \mathrm{vol}(\Omega) + \sum_{k=1}^{n} \beta_k t^{k/2} + \text{exponentially decreasing terms}$$

and β_k is supported on the $(n-k)-$dimensional skeleton of Ω. We have no expression of the coefficients β_k for $k \geq 3$.

(ii) As for the arbitrary, piecewise-smooth case, the following fact should be true: let $\gamma(y)$ denote the interior angle of the tangent spaces of the two smooth pieces of $\partial\Omega$ meeting at the singular point y, and assume that $\gamma(y) > 0$ (that is, the intersections are transversal). Then the coefficient of the term in t in the asymptotics of the heat content should be given by:

$$4\int_{\mathrm{Sk}_{n-2}} \int_0^\infty \left(1 - \frac{\tanh(\gamma(y)x)}{\tanh(\pi x)}\right) dx\, dv_{n-2}(y) + \frac{1}{2}\int_{\partial_{reg}\Omega} \eta(y)\, dv_{n-1}(y)$$

where Sk_{n-2} is the union of all pieces of dimension $n-2$ in the cellular decomposition of $\partial\Omega$. This should follow naturally from the splitting of $\Delta\rho$ into its regular and singular parts, by a process similar to the one carried above.

5 – Uniform estimates

We assume in this section that Ω is an open set with piecewise-smooth boundary which satisfies the condition that the measure $\Delta\rho$ is *non-negative* on Ω, where ρ is as usual the distance function from the boundary. As $\Delta\rho = \Delta_{reg}\rho + \Delta_{cut}\rho$, and as $\Delta_{cut}\rho$ is always non-negative, we see that:

Fact. $\Delta\rho \geq 0$ if and only if the mean curvature of (the regular part of) each level set is non-negative.

Sufficient condition. *If $\partial\Omega$ is smooth, and if both the mean curvature of $\partial\Omega$ and the Ricci curvature of Ω are non-negative, then so is $\Delta\rho$.*

If $\partial\Omega$ is merely piecewise-smooth, we add the condition that the foot of any geodesic segment which minimizes the distance from the boundary is a regular point of $\partial\Omega$.

The sufficient condition is an immediate consequence of Bochner formula:

$$\nabla(\Delta\rho)\cdot\nabla\rho = \|\mathrm{Hessian}(\rho)\|^2 + \mathrm{Ric}(\nabla\rho,\nabla\rho)$$

which implies that the mean curvature of the level hypersurfaces does not decrease in the normal direction $\nabla\rho$.

As $u_t \leq 1$, we immediately have from (9) the following inequality:

$$\int_\Omega u_t(x)\,dx \geq \mathrm{vol}(\Omega) - \frac{2}{\sqrt{\pi}}\mathrm{vol}(\partial\Omega)\sqrt{t}$$

We actually have a sharper estimate. First, we note the following upper bound of the temperature function:

$$u_t(x) \leq \frac{1}{\sqrt{\pi t}}\int_0^{\rho(x)} e^{-r^2/4t}\,dr$$

Using this bound, the non-negativity of u_t and formula (9), one proves the main estimate:

Theorem ([14, thm. 3.2]). *If $\Delta\rho \geq 0$, then:*

$$\frac{4}{\sqrt{\pi t}}\int_0^\infty e^{-r^2/t}\mathrm{vol}(\Omega(r))\,dr - \mathrm{vol}(\Omega) \leq \int_\Omega u(t,x)\,dx \leq$$
$$\leq \frac{1}{\sqrt{\pi t}}\int_0^\infty e^{-r^2/4t}\mathrm{vol}(\Omega(r))\,dr.$$

These estimates can be extended, by polyhedral approximation, to any bounded, convex subset of $\mathbb{R}^n$.

Both bounds are *optimal.* Let in fact C_R be a flat cylinder: $C_R = N \times (0, 2R)$ where N is a closed manifold and the metric is given by the product. Then $\text{vol}(\Omega(r)) = \text{vol}(\Omega) - r\text{vol}(\partial\Omega)$ for $r < R$, and is zero for $r > R$. Then as t is fixed, both the upper and the lower bound approach the *common value* $\text{vol}(\Omega) - \frac{2}{\sqrt{\pi}}\text{vol}(\partial\Omega)\sqrt{t}$, as $R \to \infty$ (the optimality for semi-infinite flat cylinders comes from the fact that $\Delta\rho = 0$ in that case).

But perhaps the main virtue of the above bounds is that they are much sharper than those in (6) when time is small. We say something about that.

Assume first that $\partial\Omega$ is smooth. Then $r \mapsto \text{vol}(\Omega(r))$ is smooth on the interval $(0, Inj)$ and can be expanded in a Taylor series around 0:

$$\text{vol}(\Omega(r)) = \text{vol}(\Omega) - r\text{vol}(\partial\Omega) + \frac{n-1}{2}\int_{\partial\Omega} \eta \cdot r^2 + O(r^3)$$

where $O(r^3)$ can be estimated from both sides in terms of the curvature of M near $\partial\Omega$. From the theorem, one gets:

Corollary. *Let $\partial\Omega$ be smooth, and fix $0 < a < Inj$. Then, for all $t > 0$:*

$$\begin{aligned}\text{vol}(\Omega) - \frac{2}{\sqrt{\pi}}\text{vol}(\partial\Omega)\sqrt{t} + \frac{n-1}{2}\int_{\partial\Omega}\eta \cdot t + c_3 t^{3/2} - g(t) \leq H(t) \leq \\ \leq \text{vol}(\Omega) - \frac{2}{\sqrt{\pi}}\text{vol}(\partial\Omega)\sqrt{t}(1 - e^{-a^2/4t}) + (n-1)\int_{\partial\Omega}\eta \cdot t + 4C_3 \cdot t^{3/2}\end{aligned}$$

where $c_3 = \min\{\inf_{r\in(0,a)} C(r), 0\}$, $C_3 = \max\{\sup_{r\in(0,a)} C(r), 0\}$, $C(r) = \frac{1}{3\sqrt{\pi}}\int_{\rho^{-1}(r)}\left(scal_M - Ricci(\nabla\rho, \nabla\rho) - scal_{\rho^{-1}(r)}\right) dv_{n-1}$ *and* $g(t) = \frac{n-1}{a\sqrt{\pi}}(\int_{\partial\Omega}\eta)(2a^2 t^{1/2} + 2t^{3/2})e^{-a^2/t}$.

Remark. We see from (3) that the lower bound is sharp up and including the term in t, as $t \to 0$.

Next, assume that Ω is an open convex subset of $\mathbb{R}^n$. As $\partial\Omega$ is not always piecewise-smooth, we see that $\text{vol}(\Omega(r))$ is no longer smooth in r, even near $r = 0$. What is smooth is instead the function $r \mapsto \text{vol}(\Omega^+(r))$

where $\Omega^+(r) = \{x \in \mathbb{R}^n : d(x,\Omega) < r\}$. The Steiner-Minkowski formula (see [7]) states that:

$$\mathrm{vol}(\Omega^+(r)) = \sum_{k=0}^{n} V_k(\Omega) r^k$$

for certain coefficients $V_k(\Omega)$. The geometric significance of the coefficients is the following: $V_0(\Omega)$ is the volume of Ω, $V_1(\Omega)$ is the *canonical* $(n-1)-$volume of the boundary (all notions of $(n-1)-$volume: Hausdorff, Minkowski, etc. coincide for convex sets and will be denoted by $\mathrm{vol}(\partial\Omega)$), and $V_2(\Omega)$ is the *integral mean curvature* indeed reducing to $\frac{n-1}{2}\int_{\partial\Omega}\eta$ when the boundary is smooth. In general, $V_k(\Omega)$ is computed by polyhedral approximation: for example $V_1(\Omega) = \sup\{\mathrm{vol}_{n-1}(P) : P \text{ convex polyhedron} \subseteq \Omega\}$.

Using the Steiner-Minkowski formula, one can estimate $\mathrm{vol}(\Omega(r))$. In what follows, R is the *inner radius* of Ω (the radius of the biggest ball included in Ω), and $d = \inf_{x\in\rho^{-1}(R)} \sup_{y\in\partial\Omega}\{d(x,y)\}$. Clearly $R \leq d < \mathrm{diam}(\Omega)$, and $d = R$ if Ω is a ball.

Proposition (see [14, App. A]). *Let Ω, d, R be as above. Then, for all $r \geq 0$:*

$$\begin{aligned}\mathrm{vol}(\Omega) - r\mathrm{vol}(\partial\Omega) + V_2(\Omega)r^2 - c_3 r^3 \leq \\ \leq \mathrm{vol}(\Omega(r)) \leq \mathrm{vol}(\Omega) - r\mathrm{vol}(\partial\Omega) + \frac{d}{R}V_2(\Omega)r^2 + C_3 r^3\end{aligned}$$

where, if $m = 2$: $V_2(\Omega) = \pi$, $C_2 = \frac{\pi d}{R}$, $c_3 = \frac{\pi}{R}$, and $C_3 = 0$; if $m \geq 3$: $C_2 = 2^{m-3}(m-1)\mathrm{vol}(S^{m-1})\frac{d^{m-1}}{R}$; $c_3 = 2^{m-3}(m-1)(m-2)\mathrm{vol}(S^{m-1})\frac{d^{m-2}}{3R}$, and $C_3 = \frac{d}{2R}c_3$.

The above implies the following estimate of the heat content:

Corollary. *Let Ω be a convex subset of $\mathbb{R}^n$. In the above notation we have, for all $t > 0$:*

$$\begin{aligned}\mathrm{vol}(\Omega) - \frac{2}{\sqrt{\pi}}\mathrm{vol}(\partial\Omega)\sqrt{t} + V_2(\Omega)\cdot t - \frac{2}{\sqrt{\pi}}c_3 t^{3/2} \leq H(t) \leq \\ \leq \mathrm{vol}(\Omega) - \frac{2}{\sqrt{\pi}}\mathrm{vol}(\partial\Omega)\sqrt{t} + \frac{2d}{R}V_2(\Omega)\cdot t + \frac{8}{\sqrt{\pi}}C_3 t^{3/2},\end{aligned}$$

A COMPARISON THEOREM. Finally we mention a comparison theorem (proved in [14, Prop. 3.10]) for the quantity $F_\Omega(t) = \mathrm{vol}(\Omega) - \int_\Omega u_t(x)dx$, which is the total heat content of Ω at time t, now assuming *zero* initial temperature, and assuming that the boundary is kept at constant unit temperature at all times. Then, if $\Delta\rho \geq 0$:

At all times, $F_\Omega(t)$ is less than or equal to the corresponding quantity $F_{\bar\Omega}(t)$, where $\bar\Omega$ is a flat cylinder with the same (*or bigger*) *inner radius, and with boundary having the same* (*or bigger*) *volume.*

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