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# A guide to $L$-operators

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Riassunto: *In questo articolo intendiamo evidenziare la connessione tra una costruzione di algebre inviluppanti quantizzate data in* [4] *e la teoria di dette algebre come è stata sviluppata più recentemente ad esempio in* [1] *o* [2]*. In* [4] *si realizza l'algebra inviluppante quantizzata individuandone un insieme di generatori, i cosiddetti operatori L. Una costruzione simile è stata descritta in* [2*, Cap.* 9]*; in questo articolo rendiamo esplicita la connessione tra* [4] *e* [2]*. Come applicazione dimostriamo che gli elementi centrali descritti in* [4*, Teorema* 14] *sono gli stessi costruiti in* [3] *nella costruzione dell'inverso dell'omomorfismo di Harish-Chandra.*

Abstract: *Our aim with this work is to prove that the construction of a quantized enveloping algebra* $\mathbf{U}$ *given in* [4] *by means of L-operators is the same described in Chapter* 9 *of* [2]*. An interesting application is the realization of the inverse of the Harish-Chandra homomorphism through the central elements described in Theorem* 14 *of* [4]*.*

## 1 – Introduction

There is little in this paper that is not known. We intend to make explicit the connection between a construction of quantized enveloping algebras given in [4] and the general theory of such algebras as developed for example in [2] or [1]. More precisely, in [4] a quantized enveloping algebra $\mathbf{U}_q$ is recovered from its algebra of matrix coefficients $R_q[G]$ realizing $\mathbf{U}_q$ as the subalgebra of $R_q[G]^*$ generated by certain elements, the so called $L$-operators.

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A similar construction is described in [2 Ch. 9] and in this paper we show that the two constructions are indeed the same. This fact is probably known but, since in [2] no reference to [4] is made, we feel that making the connection between the two constructions explicit can be useful.

Once the connection is made, we translate and reprove in the general setting of the theory of quantized algebras some of the formulas described in [4]. In particular we show that equation (2.3) of [4] is equivalent to the quantum double construction, another interesting application is the realization of the inverse of the Harish-Chandra homomorphism as done in [3] through the central elements described in Theorem 14 of [4].

In our exposition we draw extensively from [1], Ch. 4–8: our methods are elementary precisely because they depend on some deeper results that are described in [1].

The paper is organized as follows: in § 3 we define the $L$-operators as described in [2] and realize explicitly the isomorphism between $\mathbf{U}_q$ and the algebra generated by the $L$-operators. In § 4 we prove that the $L$-operators constructed in the previous section are those introduced in [4], in § 5 we highlight the connection between some of the formulas given in [4] and the quantum double construction. Finally, in § 6, we prove that the central elements of $\mathbf{U}_q$ described in [4, Theorem 14] are those constructed in Theorem 8.6 of [3].

## 2 – Notations

Let $(\pi, (\ ,\ ))$ be a root system of finite type and let $\Phi$ be a set of simple roots for $\pi$. Let $\mathbb{Z}\Phi$ denote the root lattice and $\Lambda$ the weight lattice of $\pi$. We assume that $(\lambda, \alpha) \in \mathbb{Z}$ for any $\lambda \in \Lambda$ and $\alpha \in \mathbb{Z}\Phi$.

Let $\mathbb{K}$ be a field of characteristic zero and fix a nonzero element $q \in \mathbb{K}$ such that $q$ is not a root of unity. Let $\mathbf{U} = \mathbf{U}_q(\pi)$ be the quantized enveloping algebra corresponding to $q$ and $\pi$. This is the algebra generated by elements $E_\alpha$, $F_\alpha$, and $K_\alpha$ ($\alpha \in \Phi$) subject to the relations given in [1], § 4.3.

The algebra $\mathbf{U}$ when equipped with the comultiplication $\Delta$ defined in [1, Proposition 4.11] becomes a Hopf algebra whose counit and antipode are denoted respectively by $\epsilon$ and $S$. We denote by $\mathbf{U}^+$ (resp.

$\mathbf{U}^-$) the subalgebra of $V$ generated by the elements $E_\alpha$ (resp. $F_\alpha$). We set $\mathbf{U}^{\geq}$ (resp. $\mathbf{U}^{\leq}$) to be the Hopf subalgebra of $\mathbf{U}$ generated by the elements $E_\alpha$, $K_\alpha$ ($F_\alpha$, $K_\alpha$). The Hopf subalgebra generated by the elements $K_\alpha$ is denoted by $\mathbf{U}^0$.

Since $\mathbf{U}$ is a Hopf algebra, then there is a natural action Ad of $\mathbf{U}$ on itself, the so called adjoint action: if $X \in \mathbf{U}$ and $\Delta(X) = \sum X_{(1)} \otimes X_{(2)}$ then $\mathrm{Ad}(X)(u) = \sum X_{(1)} u S(X_{(2)})$.

If $M$ is a $\mathbf{U}^0$-module and $\lambda \in \Lambda$ then we set $M_\lambda = \{m \in M \mid K_\alpha \cdot m = q^{(\lambda,\alpha)} m\}$, and we call $M_\lambda$ the weight space of weight $\lambda$ and its elements are called weight vectors. We write $\lambda(m) = \lambda$ for saying that $m \in M_\lambda$. In particular, if $x \in \mathbf{U}$, then $\lambda(x) = \mu$ says that $\mathrm{Ad}(K_\lambda)(x) = q^{(\lambda,\mu)} x$.

We adopt the convention of [1] of calling a $\mathbf{U}$-module $M$ of type **1** if $M = \oplus_{\lambda \in \Lambda} M_\lambda$. In particular all finite dimensional modules are assumed to be of type **1** if not otherwise specified.

We denote by $\check{\mathbf{U}}^0$ the group algebra of $\Lambda$. If $\mathbf{A}$ is a Hopf subalgebra of $\mathbf{U}$ containing $\mathbf{U}^0$ then we can extend the adjoint action Ad of $\mathbf{U}^0$ on $\mathbf{A}$ to $\check{\mathbf{U}}^0$: if $X \in \mathbf{A}$ and $\lambda(X) = \mu$ then we set $\mathrm{Ad}(K_\lambda)(X) = q^{(\lambda,\mu)} X$. We denote by $\check{\mathbf{A}}$ the Hopf algebra $\check{\mathbf{A}} = \mathbf{A} \otimes_{\mathbf{U}^0} \check{\mathbf{U}}^0$ with multiplication defined by

$$(X \otimes K_\lambda)(Y \otimes K_\mu) = X \mathrm{Ad}(K_\lambda)(Y) \otimes K_\lambda K_\mu$$

and comultiplication defined by

$$\Delta(X \otimes K_\lambda) = \tau_{23}(\Delta(X) \otimes K_\lambda \otimes K_\lambda).$$

Let $r$ be the smallest positive integer such that $r \cdot (\lambda, \mu) \in \mathbb{Z}$ for all $\lambda, \mu \in \Lambda$. We set $\mathrm{I\!K_e} = \mathrm{I\!K}[\mathrm{q}^{1/\mathrm{r}}]$ to be the extension of the field $\mathrm{I\!K}$ by a $r$-th root of $q$. If $V$ is a $\mathrm{I\!K}$ vector space, we set $V_e = \mathrm{I\!K_e} \otimes_{\mathrm{I\!K}} \mathrm{V}$.

## 3 – $L$-operators according to [2]

We use as a starting point the bilinear pairing between $\mathbf{U}^{\leq}$ and $\mathbf{U}^{\geq}$ given in [1, Proposition 6.12].

Let ( , ) denote the unique bilinear pairing between $\mathbf{U}^{\leq}$ and $\mathbf{U}^{\geq}$ such that, for all $y, y' \in \mathbf{U}^{\leq}$, all $x, x' \in \mathbf{U}^{\geq}$ and all $\mu, \lambda \in \mathbb{Z}\Phi$,

$$(3.1)\qquad (y, xx') = (\Delta(y), x' \otimes x) \qquad (yy', x) = (y \otimes y', \Delta(x))$$

$$(3.2)\qquad (K_\mu, K_\nu) = q^{-(\mu,\nu)} \qquad (F_\alpha, E_\alpha) = -\delta_{\alpha\beta}(q_\alpha - q_\alpha^{-1})^{-1}$$

$$(3.3)\qquad (K_\mu, E_\alpha) = (F_\alpha, K_\mu) = 0.$$

It follows immediately from the definition that

$$(3.4)\qquad (1, x) = \epsilon(x)$$

$$(3.5)\qquad (y, 1) = \epsilon(y)$$

$$(3.6)\qquad (\mathrm{Ad}(K_\mu)(y), \mathrm{Ad}(K_\mu)(x)) = (y, x).$$

Hence, by (3.6), we obtain that, if $y \in \mathbf{U}^{\leq}_\mu$ and $x \in \mathbf{U}^{\geq}_\nu$ with $\nu \neq -\mu$, then $(y, x) = 0$.

More difficult to prove is the following lemma:

Lemma 3.1. *If $y \in \mathbf{U}^-$, $x \in \mathbf{U}^+$, and $\mu, \lambda \in \mathbb{Z}\Phi$, then*

$$(yK_\lambda, xK_\mu) = q^{-(\lambda,\mu)}(y, x).$$

Proof. Let $\langle\ ,\ \rangle$ be the pairing between $\mathbf{U}^{\leq}$ and $\mathbf{U}^{\geq}$ defined by

$$\langle yK_\lambda, xK_\mu\rangle = q^{-(\lambda,\mu)}(y, x)$$

for $y \in \mathbf{U}^-$, $x \in \mathbf{U}^+$, and $\mu, \nu \in \mathbb{Z}\Phi$. Clearly this pairing satisfies (3.2) and (3.3), so we are done if we prove that it satisfies also (3.1).

Suppose then that $y \in \mathbf{U}^-$, $x, x' \in \mathbf{U}^+$, and $\lambda, \mu, \mu' \in \mathbb{Z}\Phi$. We have that

$$\begin{aligned}\langle yK_\lambda, xK_\mu x'K_{\mu'}\rangle &= \langle yK_\lambda, x\mathrm{Ad}(K_\mu)(x')K_{\mu+\mu'}\rangle = \\ &= q^{-(\lambda,\mu+\mu')}(y, x\mathrm{Ad}(K_\mu)(x')) = \\ &= q^{-(\lambda,\mu+\mu')}(\Delta(y), \mathrm{Ad}(K_\mu)(x') \otimes x).\end{aligned}$$

Without loss of generality we can assume that $y \in \mathbf{U}^-_{-\sigma}$, $x \in \mathbf{U}^+_\omega$, and $x' \in \mathbf{U}^+_\tau$ with $\tau + \omega = \sigma$. Hence

$$\Delta(y) = \sum_{\nu+\eta=\sigma} y_{-\eta} \otimes y_{-\nu} K_{-\eta}.$$

It follows that

$$\langle yK_\lambda, xK_\mu x' K_{\mu'} \rangle = q^{-(\lambda,\mu+\mu')} q^{(\mu,\tau)} \sum_{\nu+\eta=\sigma} (y_{-\eta}, x')(y_{-\nu} K_{-\eta}, x).$$

By (3.6), we get that

$$\begin{aligned}\langle yK_\lambda, xK_\mu x' K_{\mu'} \rangle &= q^{-(\lambda,\mu+\mu')} q^{(\mu,\tau)} (y_{-\tau}, x')(y_{-\omega} K_{-\tau}, x) = \\ &= q^{-(\lambda,\mu+\mu')} q^{(\mu,\tau)} (y_{-\tau}, x')(y_{-\omega} \otimes K_{-\tau}, \Delta(x)) = \\ &= q^{-(\lambda,\mu+\mu')} q^{(\mu,\tau)} (y_{-\tau}, x') \sum_{\eta+\nu=\omega} (y_{-\omega}, x_\eta K_\nu)(K_{-\tau}, x_\nu) = \\ &= q^{-(\lambda,\mu+\mu')} q^{(\mu,\tau)} (y_{-\tau}, x')(y_{-\omega}, x) = \\ &= \langle y_\tau K_\lambda, x' K_{\mu'} \rangle \langle y_{-\omega} K_{-\tau} K_\lambda, xK_\mu >\rangle = \\ &= \langle \Delta(yK_\lambda), x' K_{\mu'} \otimes xK_\mu \rangle\end{aligned}$$

This ends the proof of the first part of (3.1). The second part of (3.1) is proved similarly. □

A consequence of this lemma is the fact that one can extend ( , ) to a pairing with values in $\mathbb{K}_e$ between $\check{\mathbf{U}}^{\leq}$ and $\check{\mathbf{U}}^{\geq}$. We now show that this pairing is nondegenerate.

If $\mathbb{F}$ is a field, $z$ a nonzero element of $\mathbb{F}$, and $N \in \mathbb{Z}^h$, then, if $N = (n_1, \ldots, n_h)$, we set $z^N = (z^{n_1}, \ldots, z^{n_h})$. We have the following lemma:

Lemma 3.2. *Fix $z \in \mathbb{F}$, $z \neq 0$, such that $z$ is not a root of unity. Suppose that $F \in \mathbb{F}[\mathrm{x_1}, \ldots, \mathrm{x_h}, \mathrm{x_1^{-1}}, \ldots, \mathrm{x_h^{-1}}]$ has the property that $F(z^N) = 0$ for each $N \in \mathbb{Z}^h$.*

*Then $F = 0$.*

PROOF. The proof is by induction on $h$. The case $h = 1$ is simply the assertion that a Laurent polynomial can not have infinite roots. Assume that $h > 0$. Suppose that there is $F$ as in the statement and write

$$F(x_1, \dots, x_h) = \sum_I c_I x^I.$$

In particular we can write

$$F(x_1, \dots, x_h) = \sum_k c_k(x_2, \dots, x_h) x_1^k$$

where $c_k(x_2, \dots, x_h) = \sum_{i_1=k} c_I x_2^{i_2} \dots x_h^{i_h}$. For each $N' = (n_2, \dots, n_h)$ the Laurent polynomial $F_{N'}(x) = \sum_k c_k(z^{N'}) x^k$ has infinite roots, namely $z^n$ for all $n \in \mathbb{Z}$. Hence $c_k(z^{N'}) = 0$ for all $N'$. Now apply the induction hypothesis. ∎

It is known that the form ( , ), when restricted to $\mathbf{U}^-_{-\mu} \times \mathbf{U}^+_{\mu}$, is nondegenerate (see [1, Corollary 8.30]). For each $\nu \in \mathbb{Z}\Phi$ we fix a basis $\{u_i^\nu\}$ of $\mathbf{U}^+_\nu$ and let $\{v_i^\nu\}$ be the corresponding dual basis of $\mathbf{U}^-_\nu$.

COROLLARY 3.3. *The pairing ( , ) between* $\check{\mathbf{U}}^{\leq}$ *and* $\check{\mathbf{U}}^{\geq}$ *is nondegenerate.*

PROOF. Suppose that $y \in \check{\mathbf{U}}^{\leq}$ is such that $(y, x) = 0$ for all $x \in \check{\mathbf{U}}^{\geq}$. We can write $y = \sum_{\nu,i} v_i^\nu p_{\nu,i}(K)$ where $p_{\nu,i}(K) = \sum c_\lambda K_\lambda$. If $\mu \in \mathbb{Z}\Phi$, then, for all $\eta \in \Lambda$,

$$0 = (y, v_j^\mu K_\eta) = p_{\mu,j}(q^{-\eta}), \tag{3.7}$$

where $p_{\mu,j}(q^{-\eta}) = \sum c_\lambda q^{-(\eta,\lambda)}$.

Let $\omega_1, \dots, \omega_r$ be the fundamental weights of $\Lambda$. If $I \in \mathbb{N}^r$, then we set $K^I = \prod_j K_{\omega_j}^{i_j}$. Clearly, if we write $\lambda = \sum i_j \omega_j$, then $K_\lambda = K^I$. Hence $p_{\mu,j}(K) = \sum c_I K^I$. Set $d_i = (\alpha_i, \alpha_i)/2$, $d = l.c.m.(d_i)$ and let $k_i = d/d_i$. If $N = (n_1, \dots, n_r)$, then choosing $\eta = \sum -k_i n_i \alpha_i$ and setting $z = q^d$, we find that

$$p_{\mu,j}(q^{-\eta}) = \sum_I c_I \prod_s q^{k_s n_s d_s i_s} = \sum_I c_I \prod_s z^{n_s i_s} = p_{\mu,j}(z^N).$$

By (3.7) and Lemma 3.2 we deduce that $p_{\mu,j} = 0$, henceforth $y = 0$. If $(y, x) = 0$ for all $y \in \check{\mathbf{U}}^{\leq}$, then the same proof yields $x = 0$. ∎

The same argument gives the following result:

COROLLARY 3.4. *The pairing* $(\ ,\ )$ *between* $\check{\mathbf{U}}^{\leq} \otimes \check{\mathbf{U}}^{\leq}$ *and* $\check{\mathbf{U}}^{\geq} \otimes \check{\mathbf{U}}^{\geq}$ *defined by* $(y \otimes y', x \otimes x') = (y, x)(y', x')$ *is nondegenerate.*

We can now introduce the $L$-operators. If $M$ is a finite dimensional $\mathbf{U}$-module, we extend the action of $\mathbf{U}$ to an action of $\check{\mathbf{U}}$ on $M_e$ as follows: if $m \in M_\mu$ then we set $K_\lambda \cdot m = (q^{1/r})^{r(\lambda,\mu)} m$. Fix $v \in M$, if $f \in M^*$, then we can extend $f$ linearly to $M_e$. We can therefore define $c_{f,v} \in (\check{\mathbf{U}}^*)_e$ by setting $c_{f,v}(X) = f(Xv)$. We call the functional $c_{f,v}$ a matrix coefficient of the module $M$. We define an action of $\mathbf{U}$ on $M^*$ by setting $(X \cdot f)(m) = f(S(X) \cdot m)$. With this definition it follows that, if we identify $M$ and $M^{**}$ in the usual way, then $c_{f,v}(S(X)) = c_{v,f}(X)$.

THEOREM 3.5. *Fix* $v \in M$ *and* $f \in M^*$. *Then*

1. *There is a unique element* $\ell^+_{f,v} \in \check{\mathbf{U}}^{\geq}$ *such that*

$$c_{f,v}(y) = (y, \ell^+_{f,v})$$

*for all* $y \in \check{\mathbf{U}}^{\leq}$.

2. *There is a unique element* $\ell^-_{f,v} \in \check{\mathbf{U}}^{\geq}$ *such that*

$$c_{f,v}(S(x)) = (\ell^-_{f,v}, x)$$

*for all* $x \in \check{\mathbf{U}}^{\geq}$.

PROOF. If $y \in \check{\mathbf{U}}^{\leq}$ we can assume that $y = y_0 K_\lambda$ with $y_0 \in \mathbf{U}^-_\mu$ and $\lambda \in \Lambda$. We can also assume that $\lambda(v) = \eta$.
Define $\Theta_\nu = \sum_i v_i^\nu \otimes u_i^\nu$ as in [1, 7.1 (1)] and let

$$\ell^+_{f,v} = \sum_{\nu,i} c_{f,v}(v_i^\nu) u_i^\nu K_{-\eta}. \tag{3.8}$$

We have that

$$\begin{aligned}(y, \ell^+_{f,v}) &= \sum_{\nu,i} c_{f,v}(v_i^\nu)(y, u_i^\nu K_-\eta) = \\ &= q^{(\lambda,\eta)} \sum_{\nu,i} c_{f,v}(v_i^\nu)(y_0, u_i^\nu) = \\ &= q^{(\lambda,\eta)} c_{f,v}\Big(\sum_i v_i^\mu (y_0, u_i^\mu)\Big) = \\ &= c_{f,v}(y_0 q^{(\lambda,\eta)}) = c_{f,v}(y).\end{aligned}$$

The proof of 2 is identical: if $\lambda(f) = \eta$ we define

$$\ell^-_{f,v} = \sum_{\nu,i} c_{f,v}(S(u_i^\nu)) v_i^\nu K_{-\eta}. \tag{3.9}$$

Then, if $x = x_0 K_\lambda$ with $x_0 \in \mathbf{U}^+_\mu$ and $\lambda \in \Lambda$,

$$\begin{aligned}(\ell^-_{f,v}, x) &= \sum_{\nu,i} c_{f,v}(S(u_i^\nu))(v_i^\nu K_{-\eta}, x) = \\ &= q^{(\lambda,\eta)} \sum_{\nu,i} c_{f,v}(S(u_i^\nu))(v_i^\nu, x_0) = \\ &= q^{(\lambda,\eta)} c_{f,v}\Big(\sum_i S(u_i^\mu)(v_i^\mu, x_0)\Big) = \\ &= q^{(\lambda,\eta)} c_{f,v}\Big(\sum_i S(u_i^\mu)(S(v_i^\mu), S(x_0))\Big) = \\ &= q^{(\lambda,\eta)} c_{f,v}(S(x_0)) = \\ &= c_{f,v}(S(K_\lambda)S(x_0)) = c_{f,v}(S(x)).\end{aligned}$$

Here we used the fact that $(S(y), S(x)) = (y, x)$ (see [1, exercise 6.16] or use the uniqueness of the form). The uniqueness of $\ell^\pm_{f,v}$ follows immediately from Corollary 3.3. ∎

Let $R_q[G]$ denote the algebra of matrix coefficients of finite dimensional (type **1**) **U**-modules. Note that $R_q[G]$ is a $\mathbb{K}$-Hopf algebra. The algebra structure on $R_q[G]$ is given by $c_{f,v} c_{g,w} = c_{f\otimes g, v\otimes w}$, while the coalgebra structure is given by the following formula: fix a basis $\{m_i\}$ of $M$ and let $\{f_i\}$ be the dual basis in $M^*$, then

$$\Delta(c_{f,m}) = \sum_i c_{f,m_i} \otimes c_{f_i,m}. \tag{3.10}$$

Let $\check{\mathbf{U}}^{\mathrm{opp}}$ be the bialgebra given by $\check{\mathbf{U}}$ with opposite multiplication and the same comultiplication. Let $L^{\pm} : R_q[G] \to \check{\mathbf{U}}^{\mathrm{opp}}$ be defined by

$$L^{\pm}(c_{f,v}) = \ell^{\pm}_{f,v}.$$

Theorem 3.6. *$L^{\pm}$ are bialgebra homomorphisms.*

Proof. We need to prove that

1. (a)
$$\ell^{+}_{f,v}\ell^{+}_{f',v'} = \ell^{+}_{f'\otimes f,v'\otimes v}.$$
(b)
$$L^{+} \otimes L^{+}(\Delta(c_{f,v})) = \Delta(L^{+}(c_{f,v})).$$
2. (a)
$$\ell^{-}_{f,v}\ell^{-}_{f',v'} = \ell^{-}_{f'\otimes f,v'\otimes v}.$$
(b)
$$L^{-} \otimes L^{-}(\Delta(c_{f,v})) = \Delta(L^{-}(c_{f,v})).$$

Proof of 1. (a): if $y \in \check{\mathbf{U}}^{\leq}$ then,

$$(y, \ell^{+}_{f,v}\ell^{+}_{f',v'}) = (\Delta(y), \ell^{+}_{f',v'} \otimes \ell^{+}_{f,v}).$$

If we write $\Delta(y) = \sum y_{(1)} \otimes y_{(2)}$ then

$$\begin{aligned}(y, \ell^{+}_{f,v}\ell^{+}_{f',v'}) &= \sum (y_{(1)}, \ell^{+}_{f',v'})(y_{(2)}, \ell^{+}_{f,v}) = \\ &= \sum c_{f',v'}(y_{(1)})c_{f,v}(y_{(2)}) = \\ &= c_{f'\otimes f,v'\otimes v}(y) = \\ &= (y, \ell^{+}_{f'\otimes f,v'\otimes v})\end{aligned}$$

The result now follows from Corollary 3.3.

Proof of 1. (b): if $y_1, y_2 \in \check{\mathbf{U}}^{\leq}$, then

$$\begin{aligned}(y_1 \otimes y_2, \Delta(L^{+}(c_{f,v}))) &= (y_1 y_2, \ell^{+}_{f,v}) = \\ &= c_{f,v}(y_1 y_2) = \\ &= \Delta(c_{f,v})(y_1 \otimes y_2) = \\ &= (y_1 \otimes y_2, L^{+} \otimes L^{+}(\Delta(c_{f,v}))).\end{aligned}$$

The result now follows from Corollary 3.4.

The proofs of 2. (a) and 2. (b) are similar. ∎

THEOREM 3.7.

$$L^+(R_q[G]) = \check{\mathbf{U}}^{\geq} \tag{3.11}$$

$$L^-(R_q[G]) = \check{\mathbf{U}}^{\leq}. \tag{3.12}$$

*In particular,* $\check{\mathbf{U}} = L^-(R_q[G])L^+(R_q[G])$.

PROOF. We prove only (3.11).

By (3.8) and Theorem 3.6 it is clear that $L^+(R_q[G])$ is a subalgebra of $\check{\mathbf{U}}^{\geq}$, so it remains only to check that $L^+(R_q[G])$ contains the generators of $\check{\mathbf{U}}^{\geq}$.

If $v \in M$, $\lambda(v) = \lambda$ and $\lambda(f) = -\lambda$, then, by (3.8), $\ell^+_{f,v} = K_{-\lambda}$. Moreover, if $\alpha \in \Pi$, and $F_\alpha v \neq 0$ then we can choose $g \in M^*$ such that $\lambda(g) = -\lambda + \alpha$ and $g(F_\alpha v) = 1$. By (3.8) we find that

$$\ell^+_{g,v} = -(q_\alpha - q_\alpha^{-1})E_\alpha K_{-\lambda}$$

so $E_\alpha \in L^+(R_q[G])$.

Using (3.9) one proves in the same way that $F_\alpha \in L^-(R_q[G])$. □

REMARK. we could have avoided the hassle of using $\check{\mathbf{U}}$ instead of just plain $\mathbf{U}$: let $\mathcal{F}_{\text{int}}$ denote the category of all finite dimensional $\mathbf{U}$-modules whose weights are in $\mathbb{Z}\Phi$ and let $R_q[G]_{\text{int}}$ denote the corresponding algebra of matrix coefficients. $R_q[G]_{\text{int}}$ is a sub-bialgebra of $R_q[G]$ and $L^\pm$ map $R_q[G]_{\text{int}}$ in $\mathbf{U}$.

## 4 – $L$-operators according to [4]

We now show that with an appropriate definition of the $R$-matrix our $L$-operators satisfy formula (2.1) of [4].

Consider the pairing $\langle\ ,\ \rangle$ with values in $\mathbb{K}_e$ between $R_q[G]$ and $\check{\mathbf{U}}_e$ defined by

$$\langle F, kX\rangle = kF(X).$$

where $F \in R_q[G]$, $X \in \check{\mathbf{U}}$ and $k \in \mathbb{K}_e$. The proof of [1, Proposition 5.11] applies to $\check{\mathbf{U}}_e$, therefore this pairing is nondegenerate. Moreover, it is easy to check that

$$\langle F \otimes G, \Delta(X) \rangle = \langle FG, X \rangle \qquad \langle \Delta(F), X \otimes Y \rangle = \langle F, XY \rangle$$

for all $X, Y \in \check{\mathbf{U}}$ and $F, G \in R_q[G]$.

Let $M, N$ be type **1** finite dimensional **U**-modules and define $R^{\pm} : M_e \otimes N_e \to M_e \otimes N_e$ as follows: if $m \in M$, $n \in N$, $f \in M^*$, and $g \in N^*$ we set

$$f \otimes g(R^+(m \otimes n)) = c_{g,n}(\ell^+_{f,m}) \tag{4.1}$$

and

$$f \otimes g(R^-(m \otimes n)) = c_{g,n}(\ell^-_{f,m}) \tag{4.2}$$

Moreover, we define $R : M_e \otimes N_e \to M_e \otimes N_e$ by setting

$$f \otimes g(R(m \otimes n)) = c_{f,m}(\ell^+_{g,n}) \tag{4.3}$$

Recall that, if $\Delta(X) = \sum X_{(1)} \otimes X_{(2)}$ then the opposite comultiplication $\Delta^{\text{opp}}$ is defined by $\Delta^{\text{opp}}(X) = \sum X_{(2)} \otimes X_{(1)}$.

Theorem 4.1.

1. *$R$ is a $R$-matrix i.e.*

$$\Delta^{\text{opp}}(X) R(m \otimes n) = R \Delta(X)(m \otimes n). \tag{4.4}$$

2. $R^- = R^{-1}$.

Proof. The first statement is Theorem 7.3 of [1].

We now prove that $R^- = R^{-1}$. Indeed, let $\{m_i\}$ (resp. $\{n_i\}$) be a basis of $M$ (resp. $N$) and set $\{f_i\}$, $\{g_i\}$ to be their respective dual bases. Then

$$R^-(m \otimes n) = \sum f_i \otimes g_j (R^-(m \otimes n)) m_i \otimes n_j$$

so

$$\begin{aligned} f\otimes g(R\circ R^-(m\otimes n)) &= \sum_{i,j} f_i\otimes g_j(R^-(m\otimes n))f\otimes g(R(m_i\otimes n_j)) = \\ &= \sum_{i,j} c_{g_j,n}(\ell^-_{f_i,m})c_{f,m_i}(\ell^+_{g,n_j}) = \\ &= \sum_{i,j}(\ell^-_{f_i,m},\ell^+_{g_j,n})(S(\ell^-_{f,m_i}),\ell^+_{g,n_j}) = \\ &= \sum_i (S(\ell^-_{f,m_i})\otimes \ell^-_{f_i,m},\Delta(\ell^+_{g,n})) = \\ &= \sum_i (S(\ell^-_{f,m_i})\ell^-_{f_i,m},\ell^+_{g,n}) = \\ &= (\epsilon(\ell^-_{f,m}),\ell^+_{g,n}). \end{aligned}$$

Now

$$\epsilon(\ell^-_{f,m}) = (\ell^-_{f,m},1) = c_{f,m}(S(1)) = c_{f,m}(1) = f(m)$$

so

$$(\epsilon(\ell^-_{f,m}),\ell^+_{g,n}) = f(m)(1,\ell^+_{g,n}) = f(m)c_{g,n}(1) = f(m)g(n)$$

hence

$$f\otimes g(R\circ R^-(m\otimes n) = f(m)g(n) = f\otimes g(m\otimes n)$$

as we wished to prove. ∎

Suppose now that $M = N$. Fix a basis $\{m_i\}$ of $M$ and set $\{f_i\}$ to be its dual basis. If we set $t_{ij} = c_{f_i,m_j}$, then (4.4) is equivalent to equation (1.1) of [4]. In particular, if the matrix coefficients $t_{ij}$ generate $R_q[G]$, then there is a homomorphism of bialgebras $\pi$ from the bialgebra $A_R$ defined in [4,§ 1.1] onto $R_q[G]_e$. This implies that $\pi^*$ is an embedding of $(R_q[G]_e)'$ into $A'_R$ (here $V'$ denotes the $\mathbb{K}_e$ dual). On the other hand the fact that $\langle\ ,\ \rangle$ is nondegenerate implies that there is an injective map $i:\check{\mathbf{U}}\to(R_q[G]_e)'$ defined by $i(X)(k\otimes F) = k < F,X >$. We set

$$\ell^\pm_{i,j} = \pi^*(i(\ell^\pm_{f_i,m_j})). \tag{4.5}$$

We now show that this definition of $\ell^\pm_{i,j}$ agrees with the definition of the $L$-operators given by [4, (2.1)]. Set

$$R^\pm_{ij,hk} = f_i\otimes f_j(R^\pm(m_h\otimes m_k)),$$

and let $P : M_e \otimes M_e \to M_e \otimes M_e$ be the map defined by $P(m \otimes n) = n \otimes m$. It is obvious that (4.1) and (4.3) say that $R^+ = P \circ R \circ P$. This observation together with 2. of Theorem 4.1 imply that our matrices $(R^{\pm}_{ij,hk})$ coincide with the matrices $R^{(\pm)}$ of [4, § 2.1]. It follows that, in order to to check that $\ell^{\pm}_{i,j}$ satisfy the equation in § 2.1 of [4], we need only to check that

$$\langle t_{h_1k_1} t_{h_2k_2} \dots t_{h_nk_n}, \ell^{\pm}_{f_i,m_j} \rangle = R^{\pm}_1 R^{\pm}_2 \dots R^{\pm}_n$$

*i.e.*, that

$$\langle t_{h_1k_1} t_{h_2k_2} \dots t_{h_nk_n}, \ell^{\pm}_{f_i,m_j} \rangle = \sum_{r_1,\cdots,r_n} R^{\pm}_{h_0h_1,r_1k_1} R^{\pm}_{r_1h_2,r_2k_2} \dots R^{\pm}_{r_{n-1}h_n,k_0k_n} \tag{4.6}$$

for all $n$.

If $n = 1$ then (4.6) reduces to

$$\langle t_{hk}, \ell^{\pm}_{f_i,m_j} \rangle = R^{\pm}_{ih,jk}.$$

which are precisely formulas (4.1) and (4.2). If $n > 1$, then it follows from Theorem 3.6 that

$$\Delta(\ell^{\pm}_{f_i,m_j}) = \sum_r \ell^{\pm}_{f_i,m_r} \otimes \ell^{\pm}_{f_r,m_j}$$

so, using induction on $n$, we see that

$$\begin{aligned}\langle t_{h_1k_1} t_{h_2k_2} \dots t_{h_nk_n}, \ell^{\pm}_{f_i,m_j} \rangle &= \langle t_{h_1k_1} \otimes t_{h_2k_2} \dots t_{h_nk_n}, \Delta(\ell^{\pm}_{f_i,m_j}) \rangle = \\ &= \sum_r \langle t_{h_1k_1}, \ell^{\pm}_{f_i,m_r} \rangle \langle t_{h_2k_2} \dots t_{h_nk_n}, \ell^{\pm}_{f_r,m_j} \rangle = \\ &= \sum_{r_1,\cdots,r_{n-1}} R^{\pm}_{ih_1,r_1k_1} R^{\pm}_{r_1h_2,r_2k_2} \dots R^{\pm}_{r_{n-1}h_n,jk_n}.\end{aligned}$$

which is precisely (4.6).

## 5 – The quantum double

In this section we wish to prove Theorem 9 of [4]. The first equation in [4, (2.3)] is derived as in [1, § 7.12] using Theorem 3.6.

The second equation of [4, (2.3)] is more intricate and says that $\mathbf{U}_e$ is the quotient of a quantum double. If $M, N$ are finite dimensional $\mathbf{U}$-modules then we set

$$\Phi^{\pm} : (M \otimes N) \otimes (M^* \otimes N^*) \to \check{\mathbf{U}}$$

by setting

$$\Phi^+(m \otimes n \otimes f \otimes g) = \ell^+_{f,m}\ell^-_{g,n} \tag{5.1}$$

and

$$\Phi^-(m \otimes n \otimes f \otimes g) = \ell^-_{g,n}\ell^+_{f,m} \tag{5.2}$$

We extend linearly the maps $\Phi^{\pm}$ to maps from $(M_e \otimes N_e) \otimes ((M^*)_e \otimes (N^*)_e)$ to $\check{\mathbf{U}}_e$ that we still denote by $\Phi^{\pm}$.

If $\{m_i\}$ is a basis of $M$, $\{n_i\}$ is a basis of $N$, and $\{f_i\}$, $\{g_i\}$ are their respective dual bases, then

$$\begin{aligned}\Phi^-(R^+(m_i \otimes n_j) \otimes f_h \otimes g_k) &= \Phi^-\Big(\Big(\sum_{r,s} R^+_{rs,ij} m_r \otimes n_s\Big) \otimes f_h \otimes g_k\Big) = \\ &= \sum_{r,s} R^+_{rs,ij}\ell^-_{g_k,n_s}\ell^+_{f_h,m_r}\end{aligned}$$

while

$$\begin{aligned}\Phi^+(m_i \otimes n_j \otimes (R^+)^t(f_h \otimes g_k)) &= \Phi^+\Big(m_i \otimes n_j \otimes \Big(\sum_{r,s} R^+_{hk,rs} f_r \otimes g_s\Big)\Big) = \\ &= \sum_{r,s} R^+_{hk,rs}\ell^+_{f_r,m_i}\ell^-_{g_s,n_j}.\end{aligned}$$

Since the map $\pi^* : R_q[G]'_e \to A'_R$ is injective, these formulas and the definition of $\ell^{\pm}_{i,j}$ given in (4.5) imply that we can prove the second equation of [4, (2.3)] by showing that

$$\Phi^-(R^+(m \otimes n) \otimes f \otimes g) = \Phi^+(m \otimes n \otimes (R^+)^t(f \otimes g)). \tag{5.3}$$

We now check (5.3). We can write

$$R^+(m\otimes n)=\sum_{i,j} f_i\otimes g_j(R^+(m\otimes n))m_i\otimes n_j$$

so

$$\begin{aligned}\Phi^-(R^+(m\otimes n)\otimes f\otimes g)&=\sum_{i,j}\Phi^-(f_i\otimes g_j(R^+(m\otimes n))m_i\otimes n_j\otimes f\otimes g)\\&=\sum_{i,j} f_i\otimes g_j(R^+(m\otimes n))\ell^-_{g,n_j}\ell^+_{f,m_i}\\&=\sum_{i,j} c_{g_j,n}(\ell^+_{f_i,m})\ell^-_{g,n_j}\ell^+_{f,m_i}.\end{aligned}$$

Analogously, since

$$(R^+)^t(f\otimes g)=\sum_{i,j} f\otimes g(R^+(m_i\otimes n_j))f_i\otimes g_j,$$

we obtain that

$$\Phi^+(m\otimes n\otimes (R^+)^t(f\otimes g))=\sum_{i,j} c_{g,n_j}(\ell^+_{f,m_i})\ell^+_{f_i,m}\ell^-_{g_j,n}.$$

In order to prove (5.3), using the fact that $\langle\ ,\ \rangle$ is nondegenerate, we need only to check that for any matrix coefficient $c_{h,x}$ we have that

$$\langle c_{h,x},\sum_{i,j} c_{g_j,n}(\ell^+_{f_i,m})\ell^-_{g,n_j}\ell^+_{f,m_i}\rangle=\langle c_{h,x},\sum_{i,j} c_{g,n_j}(\ell^+_{f,m_i})\ell^+_{f_i,m}\ell^-_{g_j,n}\rangle. \tag{5.4}$$

By writing $\Delta(c_{h,x})=\sum_r c_{h,x_r}\otimes c_{h_r,x}$ we can write the l.h.s. of (5.4) as

$$\begin{aligned}&\sum_{i,j,r} c_{g_j,n}(\ell^+_{f_i,m})c_{h,x_r}(\ell^-_{g,n_j})c_{h_r,x}(\ell^+_{f,m_i})=\\&\qquad=\sum_{j,r} g\otimes h(R^-(n_j\otimes x_r))c_{h_r,x}\otimes c_{g_j,n}(\Delta(\ell^+_{f,m}))=\\&\qquad=\sum_{j,r} g\otimes h(R^-(n_j\otimes x_r))h_r\otimes g_j(\Delta(\ell^+_{f,m})(x\otimes n))=\\&\qquad=\sum_{j,r} g\otimes h(R^-(n_j\otimes x_r))g_j\otimes h_r(\Delta^{\mathrm{opp}}(\ell^+_{f,m})(n\otimes x))=\\&\qquad=g\otimes h(R^-\circ\Delta^{\mathrm{opp}}(\ell^+_{f,m})(n\otimes x))=\\&\qquad=g\otimes h(\Delta(\ell^+_{f,m})\circ R^-(n\otimes x)).\end{aligned}$$

On the other hand, the r.h.s. of (5.4) becomes

$$\begin{aligned}\sum_{i,j,r} c_{g,n_j}(\ell^+_{f,m_i})c_{h,x_r}(\ell^+_{f_i,m})c_{h_r,x}(\ell^-_{g_j,n}) &=\\ &= \sum_{j,r} g_j\otimes h_r(R^-(n\otimes x))c_{g,n_j}\otimes c_{h,x_r}(\Delta(\ell^+_{f,m})) =\\ &= \sum_{j,r} g_j\otimes h_r(R^-(n\otimes x))g\otimes h(\Delta(\ell^+_{f,m})(n_j\otimes x_r)) =\\ &= g\otimes h(\Delta(\ell^+_{f,m})\circ R^-(n\otimes x)).\end{aligned}$$

This concludes the proof of (5.3).

Equation (5.3) can be rewritten as

$$\Phi^- = \Phi^+\circ((R^+)^{-1}\otimes(R^+)^t). \tag{5.5}$$

We claim that this equation is the same of [1, Remark 6.17]. Indeed, if we fix basis $\{m_i\}$, $\{n_i\}$ of $M$ and $N$ and their respective dual basis $\{f_i\}$, $\{g_i\}$ then, by Theorem 3.6,

$$\Delta(\ell^+_{f,m}) = \sum \ell^+_{f,m_i}\otimes\ell^+_{f_i,m}$$

and

$$\Delta(\ell^-_{g,m}) = \sum \ell^-_{g,n_i}\otimes\ell^-_{g_i,n}$$

Since

$$\begin{aligned}(R^+)^{-1}(m\otimes n)\otimes(R^+)^t(f\otimes g) &=\\ &= \sum_{h,k} f_k\otimes g_h(PR^-P(m\otimes n))m_k\otimes n_h)\otimes(R^+)^t(f\otimes g),\end{aligned}$$

we have that

$$\begin{aligned}\Phi^+((R^+)^{-1}(m\otimes n)\otimes(R^+)^t(f\otimes g)) &=\\ &= \sum_{h,k,i,j} f_k\otimes g_h(PR^-P(m\otimes n))c_{g,n_j}(\ell^+_{f,m_i})\ell^+_{f_i,m_k}\ell^-_{g_j,n_h} =\\ &= \sum_{h,k,i,j} c_{f_k,m}(\ell^-_{g_h,n})c_{g,n_j}(\ell^+_{f,m_i})\ell^+_{f_i,m_k}\ell^-_{g_j,n_h} =\\ &= \sum_{h,k,i,j} (S(\ell^-_{g,n_j}),\ell^+_{f,m_i})(\ell^-_{g_h,n},\ell^+_{f_k,m})\ell^+_{f_i,m_k}\ell^-_{g_j,n_h}.\end{aligned}$$

hence (5.5) reads

$$\ell^-_{g,n}\ell^+_{f,m} = \sum_{h,k,i,j} (S(\ell^-_{g,n_j}), \ell^+_{f,m_i})(\ell^-_{g_h,n}, \ell^+_{f_k,m})\ell^+_{f_i,m_k}\ell^-_{g_j,n_h} \tag{5.6}$$

which is precisely equation (3) of [1, 6.17].

## 6 – $L$-operators and the adjoint action

In this section we want to describe the adjoint action of $\check{\mathbf{U}}$ on itself by means of the $L$-operators.

We begin by recalling briefly the quantum double construction as it is described in [2].

There is a Hopf algebra structure on $\mathcal{D}(\check{\mathbf{U}}^{\geq}, \check{\mathbf{U}}^{\leq}) = (\check{\mathbf{U}}^{\geq} \otimes \check{\mathbf{U}}^{\leq})_e$ defined as follows: if $u \in \check{\mathbf{U}}$ write $(\Delta \otimes 1) \circ \Delta(u) = \sum u_{(1)} \otimes u_{(2)} \otimes u_{(3)}$. Then the multiplication on $\mathcal{D}(\check{\mathbf{U}}^{\geq}, \check{\mathbf{U}}^{\leq})$ is defined by

$$(X \otimes Y)(X' \otimes Y') = \sum (S(Y_{(1)}), X'_{(1)})(Y_{(3)}, X_{(3)})XX'_{(2)} \otimes Y_{(2)}Y'$$

while the comultiplication is given by

$$\Delta(X \otimes Y) = \tau_{23}(\Delta(X) \otimes \Delta(Y)).$$

Moreover the unit is $1 \otimes 1$, the counit is defined by $\epsilon(X \otimes Y) = \epsilon(X)\epsilon(Y)$ and the antipode is defined by $S(X \otimes Y) = (1 \otimes S(Y))(S(X) \otimes 1)$.

Consider the Rosso form on $\mathcal{D}(\check{\mathbf{U}}^{\geq}, \check{\mathbf{U}}^{\leq})$; this is the $\mathbb{K}_e$-bilinear map $(\ ,\ )$ such that

$$(X \otimes Y, X' \otimes Y') = (S(Y), X')(Y', S(X)).$$

Then, as shown in [2, Lemma 3.3.1], this form is Ad-invariant, meaning by this that, if $a \in \mathcal{D}(\check{\mathbf{U}}^{\geq}, \check{\mathbf{U}}^{\leq})$, then

$$(\mathrm{Ad}(a)(X \otimes Y), X' \otimes Y') = (X \otimes Y, \mathrm{Ad}(S(a))(X' \otimes Y')).$$

It can also be proved as in Corollary 3.3 that the form is nondegenerate.

Let $\mathcal{M} : \mathcal{D}(\check{\mathbf{U}}^{\geq}, \check{\mathbf{U}}^{\leq}) \to \check{\mathbf{U}}_e$ denote the multiplication map $\mathcal{M}(X \otimes Y) = XY$. It can easily be proved using formula (5.6) that $\mathcal{M}$ is a Hopf algebra map.

If $M$ is a $\mathbf{U}$-module of type $\mathbf{1}$ then we define a map $F_M : M_e^* \otimes M_e \to \mathcal{D}(\check{\mathbf{U}}^{\geq}, \check{\mathbf{U}}^{\leq})$ by setting

$$F_M(f \otimes m) = (L^+ \otimes S \circ L^-) \circ \Delta(c_{f,m})$$

We have also an action of $\check{\mathbf{U}}^{\geq}$ and of $\check{\mathbf{U}}^{\leq}$ on $M_e^* \otimes M_e$.

Theorem 6.1.

1. *Set* $u_{f,m} = F_M(f \otimes m)$. *Then* $u_{f,m}$ *is the unique element of* $\mathcal{D}(\check{\mathbf{U}}^{\geq}, \check{\mathbf{U}}^{\leq})$ *such that*
$$(X \otimes Y, u_{f,m}) = c_{f,m}(S(XY))$$
*for all* $X \in \check{\mathbf{U}}^{\geq}$ *and* $Y \in \check{\mathbf{U}}^{\leq}$.
2. *If* $X \in \check{\mathbf{U}}^{\geq}$, *then*
$$F_M(X \cdot f \otimes m) = \mathrm{Ad}(X) F_M(f \otimes m)$$
*and, if* $Y \in \check{\mathbf{U}}^{\leq}$,
$$F_M(Y \cdot f \otimes m) = \mathrm{Ad}(Y) F_M(f \otimes m).$$

Proof. For 1. we write $\Delta(c_{f,m}) = \sum_i c_{f,m_i} \otimes c_{f_i,m}$ so

$$\begin{aligned}(X \otimes Y, u_{f,m}) &= \sum_i (S(Y), \ell^+_{f,m_i})(S(\ell^-_{f_i,m}), S(X)) = \\ &= \sum_i c_{f,m_i}(S(Y)) c_{f_i,m}(S(X)) = \\ &= c_{f,m}(S(XY)).\end{aligned}$$

The uniqueness follows from the fact that the Rosso form ( , ) is nondegenerate.

For 2. we observe that, since the form ( , ) is Ad-invariant then, if $X \in \check{\mathbf{U}}^{\geq}$,

$$\begin{aligned}(X' \otimes Y', \mathrm{Ad}(X)(F_M(f \otimes m))) &= (\mathrm{Ad}(S^{-1}(X))(X' \otimes Y'), F_M(f \otimes m)) = \\ &= (\mathrm{Ad}(S^{-1}(X))(X' \otimes Y'), u_{f,m}) = \\ &= c_{f,m}(S(\mathcal{M}(\mathrm{Ad}(S^{-1}(X))(X' \otimes Y')))).\end{aligned}$$

Using the fact that $\mathcal{M}$ is a a Hopf algebra map we deduce that

$$\mathcal{M}(\mathrm{Ad}(S^{-1}(X))(X' \otimes Y')) = \mathrm{Ad}(S^{-1}(X))(X'Y')$$

so

$$(X' \otimes Y', \mathrm{Ad}(X)(F_M(f \otimes m))) = c_{f,m}(S(\mathrm{Ad}(S^{-1}(X))(X'Y')))$$

hence, if we write $\Delta(X) = \sum X_{(1)} \otimes X_{(2)}$, we get

$$\begin{aligned}(X' \otimes Y', \mathrm{Ad}(X)(F_M(f \otimes m))) &= \sum c_{f,m}(S(X_{(1)})S(X'Y')X_{(2)})) = \\ &= \sum c_{X_{(1)}\cdot f, X_{(2)}\cdot m}(S(X'Y')) = \\ &= (X' \otimes Y', \sum u_{X_{(1)}\cdot f, X_{(2)}\cdot m}) = \\ &= (X' \otimes Y', F_M(X \cdot f \otimes m)).\end{aligned}$$

Using the fact that the form is nondegenerate we can conclude the proof of 2.

We wish to use Theorem 6.1 to prove that the central elements of $\check{\mathbf{U}}$ defined in [4, Theorem 14] are the same constructed in [3], Theorem 8.6.

First of all we need to identify the trivial isotypic component of $M^* \otimes M$: consider $tr : M^* \otimes M \to \mathbb{K}$ to be the map defined by $tr(f \otimes m) = f(m)$. It is easy to convince ourselves that $tr$ is an intertwining between $M^* \otimes M$ and the trivial representation. It follows that $\mathbb{K}\mathrm{tr}$ is the isotypic component for the trivial representation in $(M^* \otimes M)^*$. The map $T : M^* \otimes M \to (M^* \otimes M)^*$ defined by $T(f \otimes m)(g \otimes n) = f(n)g(K_{-2\rho}m)$ is an injective $\check{\mathbf{U}}$-map. Hence $T^{-1}(\mathbb{K}\mathrm{tr})$ is the trivial isotypic component in $M^* \otimes M$. Fixing a basis $\{m_i\}$ of $M$ and letting $\{f_i\}$ be the dual basis one finds, unwinding the definitions, that

$$T^{-1}(tr) = \sum_i f_i \otimes K_{2\rho} m_i. \tag{6.1}$$

Since $\mathcal{M} : \mathcal{D}(\check{\mathbf{U}}^{\geq}, \check{\mathbf{U}}^{\leq}) \to \check{\mathbf{U}}_e$ is a Hopf algebra map, it follows from Theorem 6.1 that $\mathcal{M} \circ F_M(X \cdot (f \otimes m)) = \mathrm{Ad}(X)(\mathcal{M} \circ F_M(f \otimes m))$, hence $z_M = \mathcal{M} \circ F_M(T^{-1}(tr))$ is a central element of $\check{\mathbf{U}}_e$. Using (6.1), we find that

$$z_M = \sum_{i,j} \ell^+_{f_i, m_j} S(\ell^-_{f_j, K_{2\rho} m_i})$$

so $z_M$ is a central element of $\check{\mathbf{U}}$. If one uses a basis of weight vectors, then, if $\lambda_i = \lambda(m_i)$, it turns out that

$$z_M = \sum_{i,j} q^{(2\rho,\lambda_i)} \ell^+_{f_i,m_j} S(\ell^-_{f_j,m_i}) \tag{6.2}$$

which is (I believe) the central element $c_1$ of [4, Theorem 14].

Suppose now that $\lambda \in \Lambda$ is a dominant integral weight and suppose furthermore that $2\lambda$ is a sum of roots. Let $V(\lambda)$ be the irreducible $\mathbf{U}$-module of highest weight $\lambda$. We set $F_\lambda = F_{V(\lambda)}$ and $z_\lambda = z_{V(\lambda)}$.

We now prove that $z_\lambda$ is the central element constructed in [3, Theorem 8.6]. For this we need a technical result:

Lemma 6.2. *The map from* $\mathbf{U} \otimes \mathbf{U}^{\leq} \otimes \mathbf{U}^{\geq}$ *to* $\mathbf{U} \otimes \mathbf{U}$ *defined by*

$$u \otimes Y \otimes X \mapsto \Delta(u)(Y \otimes X)$$

*is onto. Here the product is the ordinary product of the algebra* $\mathbf{U} \otimes \mathbf{U}$.

Proof. Set $\mathbf{V}$ to be the image of our map in $\mathbf{U} \otimes \mathbf{U}$. The result follows from the following two statements:

1. If $u \in \mathbf{U}$ is such that $u \otimes X \in \mathbf{V}$ for all $X \in \mathbf{U}^{\geq}$, then $K_{-\alpha}E_\alpha u \otimes X \in \mathbf{V}$ for all $X \in \mathbf{U}^{\geq}$.
2. If $u \in \mathbf{U}$ is such that $Y \otimes u \in \mathbf{V}$ for all $Y \in \mathbf{U}$, then $Y \otimes F_\alpha K_\alpha u \in \mathbf{V}$ for all $Y \in \mathbf{U}$.

Indeed, by triangular decomposition, 1. implies that $\mathbf{U} \otimes \mathbf{U}^{\geq} \subset \mathbf{V}$, and this together with 2. implies that $\mathbf{U} \otimes \mathbf{U} \subset \mathbf{V}$.

Let us prove the two statements:

$$K_{-\alpha}E_\alpha u \otimes X = \Delta(K_{-\alpha}E_\alpha)(u \otimes K_\alpha X) - u \otimes K_{-\alpha}E_\alpha K_\alpha X$$

while

$$Y \otimes F_\alpha K_\alpha u = \Delta(F_\alpha K_\alpha)(K_{-\alpha} Y \otimes u) - F_\alpha Y \otimes u.$$ ∎

Let $\{v_0, \dots, v_r\}$ be a basis of weight vectors of $V(\lambda)$ and assume that $v_0$ is a highest weight vector. Let $\{f_0, \dots, f_r\}$ be the dual basis and set $P_0 = f_0 \otimes v_0$. An immediate consequence of Lemma 6.2 is

Corollary 6.3.

$$\mathbf{U} \cdot P_0 = V(\lambda)^* \otimes V(\lambda).$$

We now compute $\mathcal{M}(F_\lambda(P_0))$. Since $\ell^-_{\overline{f_i},m_0} = \delta_{i0} K_\lambda$, by the definition of $F_\lambda$ we find that

$$\mathcal{M}(F_\lambda(P_0)) = \sum_i \ell^+_{f_0,m_i} S(\ell^-_{\overline{f_i},m_0}) = K_{-2\lambda} \in \mathbf{U}.$$

By Theorem 6.1, it follows that $\mathcal{M}(F_\lambda(\mathbf{U} \cdot P_0)) = \mathrm{Ad}(\mathbf{U})(K_{-2\lambda})$. Because of Corollary 6.3, we conclude that $\mathcal{M}(F_\lambda(V(\lambda)^* \otimes V(\lambda)) = \mathrm{Ad}(\mathbf{U})(K_{-2\lambda})$. In particular $z_\lambda \in \mathrm{Ad}(\mathbf{U})(K_{-2\lambda})$ and $\mathbb{K}z_\lambda$ is the trivial isotypic component of $\mathrm{Ad}(\mathbf{U})(K_{-2\lambda})$, *i.e.* $z_\lambda$ is precisely the central element constructed in [3], Theorem 8.6, providing an inverse to the Harish-Chandra isomorphism.

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