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The Reidemeister number as a homotopy equalizer

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Riassunto: *L'obiettivo di questo lavoro è l'introduzione ad una nuova interpretazione del numero di Reidemeister di una automappa utilizzando un equalizzatore debole nella categoria omotopica* hTop. *Questa nuova definizione è totalmente indipendente dallo spazio usato e permette una facile generalizzazione dalla teoria dei punti fissi alla teoria della coincidenza. Le proprietà standard del numero di Reidemeister risultano come proprietà funtoriali*

Abstract: *The objective of this paper is to introduce a new way of looking at the Reidemeister number of a self-map using a particular weak equalizer in the homotopy category* hTop. *This new definition is totally independent of the spaces used and moreover allows a generalization from Fixed Point Theory to Coincidence Theory. The standard properties of the Reidermeister number follow from functorial properties.*

– Introduction

Given a topological space X and a self-map $f\colon X \to X$, the subspace $\mathrm{fix}(f) = \{x \in X\colon f(x) = x\} \hookrightarrow X$ may be easily interpreted as an equalizer in the category Top of topological spaces and maps. What does this interpretation lead to when we look at the Fixed Point Theory from a topological point of view? The most suitable universe in which to work is the category hTop of topological spaces and homotopy classes of maps, but it is well known that hTop is not complete because it just lacks equalizers. We have to give up the idea of a (strong) limit and to inquire

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into "fixed point classes" in a weak equalizer E_f of the pair $([f],[1_X])$, that is a space E_f and a homotopy class $[e]\colon E_f \to X$ with $[fe]=[e]$ such that, for every space W and every class $[h]\colon W \to X$ with $[fh]=[h]$, there exists a (not necessarily unique) class $[h']$ such that $[eh']=[h]$.

A key-element of Topological Fixed Point Theory is the Reidemeister number which is generally defined in two (obviously equivalent) ways.

"*Geometric*" *definition* (see [3, p. 6]): the Reidemeister number of a self-map f of a space with a universal covering space is the cardinality of the set of conjugacy classes $[\tilde{f}]=\{d\tilde{f}d^{-1} : d \in \mathcal{D}\ \text{(covering group)}\}$ of liftings of f.

"*Algebraic*" *definition* (see [2, p. 269]): the Reidemeister number of a self-map f of a path-connected space X is the cardinality of the set of orbits of the left action of $\pi_1(X,x_0)$ on itself given by $(\langle\alpha\rangle,\langle\lambda\rangle) \mapsto \langle\alpha+\lambda+f(-\alpha)\rangle$.

The aim of this paper is to show that the Reidemeister number and its properties reside in a particular weak equalizer E_f in hTop of the pair $([f],[1_X])$.

In order to make the contents of the paper more understandable, the author presents the work in inverted order as to its natural development. The space E_f is constructed in section 1; the number R_f is defined as the cardinality of the set of path-components of the space E_f. Some immediate properties are given and then it is proved that R_f is the Reidemeister number. The proof that E_f is a weak equalizer is given in section 2; the approach taken for this has a possible extension to Coincidence Theory. The results of the ending part of this section may also be worked out from the more technical section 3, in which the space E_f is described as homotopy equalizer.

1 – The space E_f and the Reidermeister number

Given a topological space X and a self-map $f\colon X \to X$, we define the space E_f as the pullback in Top

$$\begin{array}{ccc} E_f & \xrightarrow{\ \langle\overline{1,f}\rangle\ } & X^I \\ {\scriptstyle \overline{\varepsilon}_{0,1}}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon_{0,1}} \\ X & \xrightarrow[\ \langle 1,f\rangle\ ]{} & X \times X \end{array}$$

where $\varepsilon_{0,1}(\lambda)=(\lambda(0),\lambda(1))$ and $\langle 1,f\rangle(x)=(x,f(x))$. In other words, E_f is the space $X_{\langle 1,f\rangle}\sqcap_{\varepsilon_{0,1}} X^I \approx \{(x,\lambda)\in X\times X^I : x=\lambda(0), f(x)=\lambda(1)\}$ with the initial topology with respect to $\overline{\langle 1,f\rangle}$ and $\overline{\varepsilon}_{0,1}$ defined as $\overline{\langle 1,f\rangle}(x,\lambda)=\lambda$, $\overline{\varepsilon}_{0,1}(x,\lambda)=x$.

We state now a lemma which will be useful for the development of this section; it is a particular case of a more general result (Lemma 2.1) and so its proof is here omitted.

LEMMA 1.1. *E_f and $E_{f'}$ have the same homotopy type whenever f and f' are homotopic.*

Consider a path $\alpha\colon I\to E_f$ with $\alpha(0)=(x_0,\lambda_0)$ and $\alpha(1)=(x_1,\lambda_1)$. Referring to the previous pullback diagram and calling $\alpha_1=\overline{\varepsilon}_{0,1}\alpha$, $\alpha_2=\overline{\langle 1,f\rangle}\alpha$ and $\alpha_2^\flat$ the adjoint of α_2 in $-\times I\dashv(-)^I$(1), we have

$$\alpha_2^\flat(0,-)=\lambda_0,\quad \alpha_2^\flat(-,0)=\alpha_1,\quad \alpha_2^\flat(1,-)=\lambda_1,\quad \alpha_2^\flat(-,1)=f\alpha_1.$$

From these facts, the proof of the following result is obtained routinely.

LEMMA 1.2. *Given two points (x_0,λ_0) and (x_1,λ_1) of E_f, there exists a path $\alpha\colon I\to E_f$ with $\alpha(0)=(x_0,\lambda_0)$ and $\alpha(1)=(x_1,\lambda_1)$ if and only if there exists a path $\alpha_1\colon I\to X$ from x_0 to x_1 such that $\lambda_0\sim_{\partial I}\alpha_1+\lambda_1+f(-\alpha_1)$.*

COROLLARY 1.3. *Let be λ_0 and λ_1 two paths in X from x_0 to $f(x_0)$. If $\lambda_0\sim_{\partial I}\lambda_1$ then there exists a path in E_f from (x_0,λ_0) to (x_0,λ_1).*

This corollary allows us to speak about the path-component of E_f of the homotopy class $\langle\lambda\rangle$ (rel. $\{0,1\}$).

We shall write $E_f(x,\lambda)$ for the path-component of E_f containing the point (x,λ) and $\pi_0=h\mathrm{T}op(*,-):h\mathrm{Top}\to\mathrm{Set}$ will be the hom-functor which assigns the set π_0X of the path-components to the space X and the obviously inducted function $f_0\colon\pi_0X\to\pi_0Y$ to the homotopy class $[f]\colon X\to Y$.

DEFINITION 1.4. Given a topological space X and a self-map f on X, we define R_f to be the cardinality of the set π_0E_f (notation: $R_f=\sharp\,\pi_0E_f$).

(1) "$F\dashv G$" means that the functor F is left adjoint to G.

Some basic properties of the number R_f follow immediately:

1. R_f is a homotopy invariant.
2. $R_f \geq \sharp\,\mathrm{fix}\,(f_0)$.
3. If for every $x_0 \in X$ is $\pi_1(X, x_0) = 0$, then $R_f = \sharp\,\mathrm{fix}\,(f_0) \leq \sharp\,\pi_0 X$.
4. If $H_1 X = 0$, then $R_f = \sharp\,\mathrm{fix}\,(f_0) \leq \sharp\,\pi_0 X$.

In some cases (for example when f is the antipodal map on the 0-dimensional sphere $S^0 = \{-1, +1\}$), the space E_f is empty and R_f is zero, but if we consider X a path-connected space (this is the most interesting situation in Fixed Point Theory), we have

2.′ $R_f \geq 1$.

3.′ If X is simply-connected then $R_f = 1$.

THEOREM 1.5. *Given a space X with a universal covering space $p: \tilde{X} \to X$ and a self-map f on X, R_f is the Reidemeister number of f.*

PROOF. Choose points $x_0 \in X$ and $\tilde{x}_0 \in p^{-1}(x_0) \subset \tilde{X}$ and identify:

i.) an element d of the covering group $\mathcal{D}$ with $\langle\delta\rangle \in \pi_1(X, x_0)$, using the well-known isomorphism;

ii.) $\tilde{x} \in \tilde{X}$ with $\langle c\rangle \in \Pi X(x_0, p(\tilde{x}))$, using the 1–1 correspondence between points of $\tilde{X}$ and path classes in X starting from x_0;

iii.) a lifting $\tilde{f}$ of f (i.e. a map $\tilde{f}: \tilde{X} \to \tilde{X}$ such that $p\tilde{f} = fp$) with $\langle w\rangle \in \Pi X(x_0, f(x_0))$; $\tilde{f}$ is unique determined by its value $\tilde{f}(\tilde{x}_0) \in p^{-1}(f(x_0))$ and therefore by $\langle w\rangle \in \Pi X(x_0, f(x_0))$ corresponding to $\tilde{f}(\tilde{x}_0)$.

Observe that under these identifications, $\tilde{f}(\tilde{x}) = \langle w + fc\rangle$ and $(d\tilde{f}d^{-1})(\tilde{x}) = \langle\delta + w + f(-\delta) + fc\rangle$.

Consider the set L of the lifting classes $[\tilde{f}] = \{d\tilde{f}d^{-1} : d \in \mathcal{D}\}$ and the function $\phi: L \to \pi_0 E_f$ defined by $\phi([\tilde{f}]) = E_f(x_0, w)$.

ϕ is well-defined and injective: $[\tilde{f}'] = [\tilde{f}]$ iff $\exists\, d \in \mathcal{D} : \tilde{f}' = d\tilde{f}d^{-1}$ iff $\exists\,\langle\delta\rangle \in \pi_1(X, x_0) : \forall\,\langle c\rangle \in \Pi X(x_0, -)\langle w' + fc\rangle = \langle\delta + w - f\delta + fc\rangle$ iff $\exists\,\delta \in \Omega_{x_0} X : w' \sim_{\partial I} \delta + w + f(-\delta)$ iff $E_f(x_0, w') = E_f(x_0, w)$.

ϕ is surjective: consider $(x, \lambda) \in E_f(x, \lambda)$ and a path α_1 in X from x_0 to x; $\langle w\rangle = \langle\alpha_1 + \lambda + f(-\alpha_1)\rangle$ selects a lifting class $[\tilde{f}]$ such that $E_f(x_0, w) = E_f(x, \lambda)$. □

The previous theorem establishes a "geometric" link between the space E_f and the Reidemeister number of the self-map f. However, it may be more meaningful to find out a relationship through algebraic methods. With this objective in mind, we suppose that the path-connected space X has a point x_0 such that $i\colon \{x_0\} \hookrightarrow X$ is a cofibration. The homotopy extension property of i assures every map $g\colon X \to X$ is homotopic to a map f based on x_0.

Make the following choice of base points:

$$(X, x_0)\,, \quad (X \times X, (x_0, x_0))\,, \quad (X^I, c_{x_0})\,, \quad (E_f, (x_0, c_{x_0}))\,.$$

Consider now the exact homotopy sequence of the fibration $\overline{\varepsilon}_{0,1}$ (see [6, § 3.1]):

$$\ldots \to \pi_{q+1}(X, x_0) \to \pi_q(\Omega X, c_{x_0}) \to \pi_q(E_f, (x_0, c_{x_0})) \to \pi_q(X, x_0) \to \ldots$$

An analysis of the morphisms leads to the exact sequence

$$\begin{aligned} \ldots \to \pi_q(E_f, (x_0, c_{x_0})) \xrightarrow{(\overline{\varepsilon}_{0,1})_q} \pi_q(X, x_0) \xrightarrow{1-f_q} \pi_q(X, x_0) \\ \xrightarrow{j_{q-1}} \pi_{q-1}(E_f, (x_0, c_{x_0})) \to \ldots \end{aligned}$$

where $(\overline{\varepsilon}_{0,1})_q\langle\alpha\rangle = \langle\overline{\varepsilon}_{0,1}\alpha\rangle$; $(1 - f_q)\langle\alpha\rangle = \langle\alpha - f\alpha\rangle$; $j_{q-1}\langle\alpha\rangle = \langle j\alpha^\sharp\rangle$ with $j\colon \Omega_{x_0}X \hookrightarrow E_f$ and $\alpha^\sharp$ the adjoint of α in $\Sigma \dashv \Omega$.

Observe now that the exactness of

$$\pi_1(X, x_0) \xrightarrow{1-f_1} \pi_1(X, x_0) \xrightarrow{j_0} \pi_0(E_f, (x_0, c_{x_0})) \longrightarrow *$$

leads to the standard "algebraic" definition of the Reidemeister number.

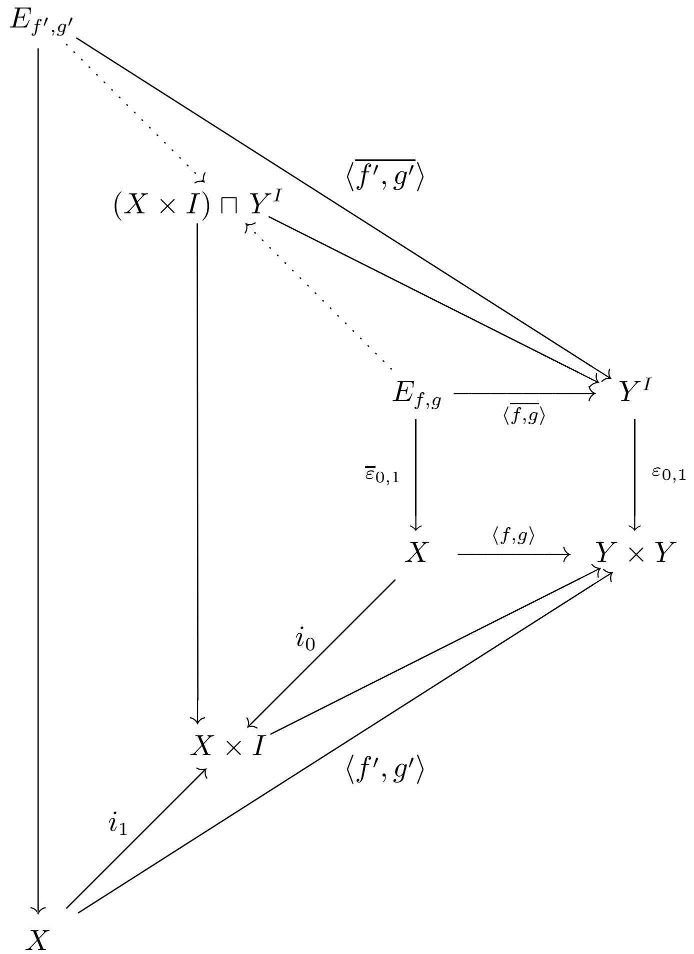
2 – The space $E_{f,g}$

Given two topological spaces X, Y and two maps $f, g\colon X \to Y$, we define the space $E_{f,g}$ as the pullback in Top

$$\begin{array}{ccc} E_{f,g} & \xrightarrow{\overline{\langle f,g\rangle}} & Y^I \\ {\scriptstyle \overline{\varepsilon}_{0,1}}\downarrow & \lrcorner & \downarrow{\scriptstyle \varepsilon_{0,1}} \\ X & \xrightarrow[\langle f,g\rangle]{} & Y \times Y \end{array}$$

Lemma 2.1. *The homotopy type of the space* $E_{f,g}$ *depends only on the homotopy classes* $[f]$ *and* $[g]$.

Proof. Consider two homotopies $G\colon g \sim g'$, $F\colon f \sim f'$. From the universal property of the product $Y \times Y$, we have a homotopy $H\colon \langle f, g\rangle \sim \langle f', g'\rangle$. Consider the maps i_0, $i_1\colon X \to X \times I$ defined by $i_j(x) = (x, j)$. Since i_0 and i_1 are homotopy equivalences and $\varepsilon_{0,1}$ is a fibration, the *cogluing lemma* (see [6, pp. 65,71] or [1, p. 313]) applied to the following diagram assures that $E_{f,g} \sim (X \times I)_H \sqcap_{\varepsilon_{0,1}} Y^I \sim E_{f',g'}$. ∎



Proposition 2.2. $E_{f,g} \xrightarrow{\overline{\varepsilon}_{0,1}} X$ *is a weak equalizer in* $h\mathrm{Top}$ *of the arrows* $[f], [g]\colon X \to Y$.

Proof. *i.*) The homotopy $A\colon E_{f,g} \times I \to Y$ defined as $A((x,\lambda),t) = \lambda(t)$, gives us the equality $[f][\overline{\varepsilon}_{0,1}] = [g][\overline{\varepsilon}_{0,1}]$.

ii.) For every arrow $[h]\colon W \to X$ in hTop such that $[f][h] = [g][h]$ there exists an arrow $[h']$ such that $[\overline{\varepsilon}_{0,1}][h'] = [h]$; in fact: let $H\colon W \times I \to Y$ be a homotopy from fh to gh and $H^\sharp\colon W \to Y^I$ its adjoint in $- \times I \dashv (-)^I$. For the pullback universal property, there exists a unique map $h'\colon W \to E_{f,g}$ such that $\overline{\varepsilon}_{0,1}h' = h$ and $\langle\overline{f,g}\rangle h' = H^\sharp$ and so $[\overline{\varepsilon}_{0,1}][h'] = [h]$ (and $[\langle\overline{f,g}\rangle][h'] = [H^\sharp]$).

Remark. The uniqueness of h' is under the choice of H. ∎

Let $\mathrm{Top}^{\rightrightarrows}$ be the category of the $\rightrightarrows$-type diagrams in Top, where an object is a pair of parallel maps $f,g\colon X \to Y$, an arrow $(\alpha,\beta)\colon (f,g) \to (f',g')$ is the diagram on left of next fig. (1)[2] and the composition of two arrows is $(\alpha,\beta)(\alpha',\beta') = (\alpha\alpha',\beta\beta')$.

The functor $E\colon \mathrm{Top}^{\rightrightarrows} \to \mathrm{Top}$ is now defined as the weak equalizer $E_{f,g}$ on the object (f,g) and by $(\alpha \sqcap \beta^I)(x,\lambda) = (\alpha(x),\beta\lambda)$ on the morphism (α,β).

$$\begin{array}{ccc} X & \overset{f}{\underset{g}{\rightrightarrows}} & Y \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ X' & \overset{f'}{\underset{g'}{\rightrightarrows}} & Y' \end{array} \qquad \overset{E}{\longmapsto} \qquad \begin{array}{c} E_{f,g} \\ \big\downarrow{\scriptstyle\alpha\sqcap\beta^I} \\ E_{f',g'} \end{array} \tag{1}$$

As a consequence of the functoriality of E, we can state that if in diagram (1) α and β are homeomorphisms (i.e. (α,β) is an isomorphism in $\mathrm{Top}^{\rightrightarrows}$) then $\alpha \sqcap \beta^I\colon E_{f,g} \to E_{f',g'}$ is a homeomorphism.

Consider the functor $D\colon \mathrm{Top} \to \mathrm{Top}^{\rightrightarrows}$ defined on the object W as the pair (i_0,i_1), where $i_j\colon W \to W \times I$ is the map $i_j(w) = (w,j)$, and on the morphism $\alpha\colon W \to W'$ as the arrow $(\alpha,\alpha\times I)\colon (i_0,i_1) \to (i_0,i_1)$.

$$\begin{array}{c} W \\ {\scriptstyle\alpha}\big\downarrow \\ W' \end{array} \qquad \overset{D}{\longmapsto} \qquad \begin{array}{ccc} W & \overset{i_0}{\underset{i_1}{\rightrightarrows}} & W \times I \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\alpha\times I} \\ W' & \overset{i_0}{\underset{i_1}{\rightrightarrows}} & W' \times I \end{array}$$

[2] The commutativity requires both $\beta f = f'\alpha$ and $\beta g = g'\alpha$, but neither $\beta f = g'\alpha$ nor $\beta g = f'\alpha$.

Proposition 2.3. *D is left adjoint to E.*

Proof. Given an arrow $(\alpha,\beta): DW \to (f,g)$, consider $\langle \alpha, \beta^\sharp \rangle: W \to E_{f,g}$, where $\beta^\sharp$ is the adjoint of β in $- \times I \dashv (-)^I$. Looking over the pullback defining $E_{f,g}$ it turns out that this assignment gives us the required natural bijection between the hom-sets $\mathrm{Top}^{\rightrightarrows}(DW,(f,g))$ and $\mathrm{Top}(W, E_{f,g})$. ∎

As a consequence of this proposition, we can state that the functor E preserves all limits; in particular we have the homeomorphism $E_{f\times f', g\times g'} \approx E_{f,g} \times E_{f',g'}$.

Some basic properties of the number $R_{f,g} = \sharp\, \pi_0 E_{f,g}$ are given below.

1. $R_{f,g}$ is a homotopy invariant.
2. If in diagram (1) α and β are homeomorphisms, then $R_{f,g} = R_{f',g'}$; in particular: given two maps $f: X \to X, f': X' \to X'$ and a homeomorphism $\alpha: X \to X'$ such that $\alpha f = f' \alpha$, then $R_f = R_{f'}$.[3]
3. $R_{f\times f', g\times g'} = R_{f,g} R_{f',g'}$; in particular: $R_{f\times f'} = R_f R_{f'}$.
4. Given two maps $f: X \to Y, g: Y \to X$, then $R_{fg} = R_{gf}$; in fact, for every map $\varphi: W \to W$ it is $(\varphi \sqcap \varphi^I)_0 = 1_{\pi_0 E_\varphi}$ $(\lambda \sim_{\partial I} \lambda + \varphi\lambda - \varphi\lambda$ and so $(\varphi \sqcap \varphi^I)_0 E_\varphi(x,\lambda) = E_\varphi(\varphi(x), \varphi\lambda) = E_\varphi(x,\lambda))$.
   $1_{\pi_0 E_{fg}} = (fg \sqcap (fg)^I)_0 = (f \sqcap f^I)_0 (g \sqcap g^I)_0$.
   $1_{\pi_0 E_{gf}} = (gf \sqcap (gf)^I)_0 = (g \sqcap g^I)_0 (f \sqcap f^I)_0$.

3 – The homotopy equalizer $E_{f,g}$

Our aim is now to extend the functoriality of $E_{f,g}$ to a category in which diagrams of maps are homotopically commutative.

The following lemmas allow us to define the category $lh\mathrm{Top}^{\rightrightarrows}$ in which an arrow $[\alpha,\beta,F,G]: (f,g) \to (f',g')$ is a class of homotopically commutative diagrams $(\alpha,\beta): (f,g) \to (f',g')$ together with the homotopies $F: \beta f \sim f'\alpha$ and $G: \beta g \sim g'\alpha$. We shall prove that a weak equivalence in $\mathrm{Top}^{\rightrightarrows}$ (i.e. a homotopically commutative diagram $(\alpha,\beta): (f,g) \to (f',g')$ with α and β homotopy equivalences) is an isomorphism in $lh\mathrm{Top}^{\rightrightarrows}$. Then we shall introduce a functor $E: lh\mathrm{Top}^{\rightrightarrows} \to h\mathrm{Top}$ defined as the weak equalizer $E_{f,g}$ on the object (f,g), which will give the right adjoint of the constant functor $\Delta: h\mathrm{Top} \to lh\mathrm{Top}^{\rightrightarrows}$.

[3] See also the more general Corollary 3.6.

These constructions allow us to consider $E_{f,g}$ and $R_{f,g}$ as objects of study in “Abstract” Homotopy Theory (see [7]).

Let us first introduce some notations.

Given the maps $f, g: X \to Y$ and the homotopies $A, A': f \sim g$, we say that A and A' are equivalent, and write $A \simeq A'$, when there exists a map $\mathcal{A}: X \times I \times I \to Y$ such that

$$\mathcal{A}(-,-,0) = A\,, \quad \mathcal{A}(-,-,1) = A'\,, \quad \mathcal{A}(-,0,t) = f\,, \quad \mathcal{A}(-,1,t) = g\,.$$

Given $A, B: X \times I \to Y$ such that $A(-,1) = B(-,0)$ we shall write $A + B$, $-A$ and A_x for the homotopies

$$(A+B)(x,t) = \begin{cases} A(x,2t) & 0 \le t \le 1/2 \\ B(x,2t-1) & 1/2 \le t \le 1\,, \end{cases} \qquad (-A)(x,t) = A(x,1-t)$$

and the path $A_x(t) = A(x,t)$.

Lemma 3.1. *Given the maps $f, f': X \to Y$, $g, g': Y \to W$ and the homotopies $F: f \sim f'$ and $G: g \sim g'$, the sums $G(F \times I) + g'F$ and $gF + G(f' \times I)$ are equivalent.*

Proof. Consider the maps $\mathcal{K}, \mathcal{A}, \mathcal{B}: X \times I \times I \to W$ defined as follows: $\mathcal{K}(x,t,u) = G(F(x,t),u)$,

$$\mathcal{A}(x,t,u) = \begin{cases} \mathcal{K}\left(x, (1-u)\dfrac{t}{1-t}, 2tu\right) & 0 \le t \le 1/2 \\ \mathcal{K}\left(x, 2tu - 2u + 1, u + (1-u)\dfrac{2t-1}{t}\right) & 1/2 \le t \le 1\,, \end{cases}$$

$$\mathcal{B}(x,t,u) = \begin{cases} \mathcal{K}\left(x, u\dfrac{t}{1-t} + 2t - 2tu, 0\right) & 0 \le t \le 1/2 \\ \mathcal{K}\left(x, 1, (1-u)(2t-1) + u\dfrac{2t-1}{t}\right) & 1/2 \le t \le 1\,; \end{cases}$$

it is easy to verify that $G(f \times I) + g'F = \mathcal{A}(-,-,1) \simeq \mathcal{A}(-,-,0) = \mathcal{B}(-,-,1) \simeq \mathcal{B}(-,-,0) = gF + G(f' \times I)$. ∎

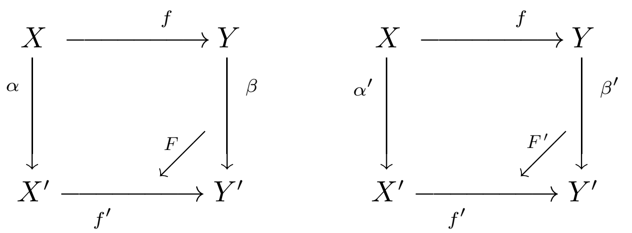
The proof of the following result is obtained routinely and thus omitted.

LEMMA 3.2. *Given* $A, B, C, D\colon X \times I \to Y$ *such that* $A+B \simeq C+D$, *there exists a map* $\mathcal{W}\colon X \times I \times I \to Y$ *such that*

$$\begin{aligned} \mathcal{W}(x,0,u) &= A(x,u) \quad \mathcal{W}(x,t,1) = B(x,t) \\ \mathcal{W}(x,t,0) &= C(x,t) \quad \mathcal{W}(x,1,u) = D(x,u)\,. \end{aligned}$$

A homotopically commutative diagram is said to be *labelled* when the homotopies realizing the commutativity are fixed.

We say that two labelled squares (α, β, F), $(\alpha', \beta', F')\colon f \to f'$



are equivalent, and write $(\alpha, \beta, F) \sim (\alpha', \beta', F')$, when there exist two homotopies $A\colon X \times I \to X'$, $A\colon \alpha \sim \alpha'$ and $B\colon Y \times I \to Y'$, $B\colon \beta \sim \beta'$ such that $F + f'A \simeq B(f \times I) + F'$. Here $[\alpha, \beta, F]$ stands for the equivalence class of (α, β, F).

Given two labelled squares $(\alpha, \beta, F)\colon f \to g$, $(\gamma, \delta, G)\colon g \to h$ consider the "vertical union" $(\gamma, \delta, G)(\alpha, \beta, F) = (\gamma\alpha, \delta\beta, \delta I + G(\alpha \times I))$, and, for a map $\varphi\colon X \to Y$, let $S_\varphi\colon X \times I \to Y$ be the "static" homotopy on φ, $S_\varphi(x,t) = \varphi(x)$.

LEMMA 3.3. *The composition law of equivalence classes of labelled squares*

$$[\gamma, \delta, G][\alpha, \beta, F] = [\gamma\alpha, \delta\beta, \delta F + G(\alpha \times I)]$$

is well-defined, associative and has as identity the class $[1, 1, S]$.

PROOF. *i.)* Given $(\alpha', \beta', F') \in [\alpha, \beta, F]$, let $A\colon \alpha \sim \alpha', B\colon \beta \sim \beta'$ be the homotopies realizing the equivalence $F + gA \simeq B(f \times I) + F'$. We are going to prove that $(\gamma\alpha, \delta\beta, \delta F + G(\alpha \times I)) \sim (\gamma\alpha', \delta\beta', \delta F' + G(\alpha' \times I)$ showing the equivalence $(\delta F + G(\alpha \times I)) + h(\gamma A) \simeq (\delta B)(f \times I) + (\delta F + G(\alpha \times I))$.

First of all observe that $\delta g A + G(\alpha' \times I)\colon (\delta g)\alpha \sim (\delta g)\alpha' \sim (h\gamma)\alpha'$ and $G(\alpha \times I) + h\gamma A\colon (\delta g)\alpha \sim (h\gamma)\alpha \sim (h\gamma)\alpha'$, and so, from Lemma 3.1, we have $\delta F + (G(\alpha \times I) + h\gamma A) \simeq \delta F + (\delta g A + G(\alpha' \times I)) \simeq \delta(F + gA) + G(\alpha' \times I) \simeq \delta(B(f \times I) + F') + G(\alpha' \times I) \simeq \delta B(f \times I) + \delta F' + G(\alpha' \times I)$.

In the same way one can prove that $(\gamma, \delta, G)(\alpha', \beta', F') \sim (\gamma', \delta', G')$ (α', β', F'), whenever $(\gamma, \delta, G) \sim (\gamma', \delta', G')$.

ii.) The equivalences $\beta''\beta' F + (\beta'' F' + F''(\alpha' \times I))(\alpha \times I) \simeq \beta''\beta' F + \beta'' F'(\alpha \times I) + F''(\alpha'\alpha \times I) \simeq \beta''(\beta' F + F'(\alpha \times I)) + F''(\alpha'\alpha \times I)$ ensure that $([\alpha'', \beta'', F''][\alpha', \beta', F'])[\alpha, \beta, F] = [\alpha'', \beta'', F'']([\alpha', \beta', F'][\alpha, \beta, F])$.

iii.) The equivalences $1F + S_g(\alpha \times I) \simeq F$ and $\beta S_f + F(1 \times I) \simeq F$ ensure that $[\alpha, \beta, F][1, 1, S_f] = [\alpha, \beta, F] = [1, 1, S_g][\alpha, \beta, F]$. □

LEMMA 3.4. *A class $[\alpha, \beta, F]$ is invertible (with respect to the previous composition law) if and only if α and β are homotopy equivalences.*

PROOF. Given two homotopy equivalences α, β such that $\beta f \sim f'\alpha$, let α', β' be their homotopy inverses with $A\colon \alpha'\alpha \sim 1$ and $B\colon \beta'\beta \sim 1$. For every $F\colon \beta f \sim f'\alpha$, we have a homotopy $H = -\beta' F + B(f \times I) - fA$ from $(\beta' f')\alpha$ to $(f\alpha')\alpha$ and thus there exists $F'\colon \beta' f' \sim f\alpha'$ such that $F'(\alpha \times I) \simeq H$ [5, p. 230, Lemma 4]. This means that $\beta' F + F'(\alpha \times I) + fA \simeq B(f \times I) + S_f$ and therefore $[\alpha', \beta', F'][\alpha, \beta, F] = [1, 1, S_f]$. At the same way one may prove the existence of a right inverse of $[\alpha, \beta, F]$, referring to [5, p. 230, Lemma 2]. □

We can now define $lh\mathrm{Top}^{\rightrightarrows}$ as the category where an object (f, g) is the pair of parallel maps $f, g\colon X \to Y$ and an arrow $[\alpha, \beta, F, G]\colon (f, g) \to (f', g')$ is the pair of classes of labelled squares $[\alpha, \beta, F]\colon f \to f', [\alpha, \beta, G]\colon g \to g'$.

The previous lemma establishes that an arrow $[\alpha, \beta, F, G]$ is an isomorphism in $lh\mathrm{Top}^{\rightrightarrows}$ if and only if α and β are homotopy equivalences.

Given $(\alpha, \beta, F, G) \in [\alpha, \beta, F, G]\colon (f, g) \to (f', g')$, where $f, g\colon X \to Y$, $f', g'\colon X' \to Y'$, consider the homotopy

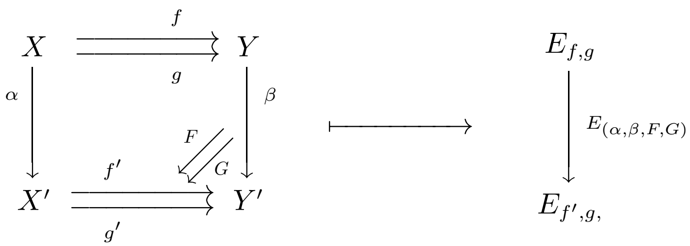
$$\Lambda = (-F(\overline{\varepsilon}_{0,1} \times I) + \beta\overline{\langle f, g\rangle}^{\flat}) + G(\overline{\varepsilon}_{0,1} \times I) : E_{f,g} \times I \to Y' .$$

We have $\varepsilon_{0,1}\Lambda^\sharp = \langle f', g'\rangle\alpha\overline{\varepsilon}_{0,1}$ and so, for the universal property of the pullback $E_{f',g'}$, we can assume the existence of a unique map

$$E_{(\alpha,\beta,F,G)}\colon E_{f,g} \to E_{f',g'}$$
$$E_{(\alpha,\beta,F,G)}(x,\lambda) = (\alpha\overline{\varepsilon}_{0,1}, \Lambda^\sharp)(x,\lambda) = (\alpha(x), (-F_x + \beta\lambda) + G_x)$$

such that $\overline{\varepsilon}_{0,1}E_{(\alpha,\beta,F,G)} = \alpha\overline{\varepsilon}_{0,1}$ and $\overline{\langle f', g'\rangle}E_{(\alpha,\beta,F,G)} = \Lambda^\sharp$.

In this way, we associate a map $E_{(\alpha,\beta,F,G)}$ to the diagram (α,β,F,G).



Consider the law E which assigns the space $E_{f,g}$ to the object $(f,g) \in lh\mathrm{Top}^{\rightrightarrows}$ and the homotopy class $[E_{(\alpha,\beta,F,G)}]$ to the arrow $[\alpha,\beta,F,G]$ of $lh\mathrm{Top}^{\rightrightarrows}$.

Proposition 3.5. *$E\colon lh\mathrm{Top}^{\rightrightarrows} \to h\mathrm{Top}$ is a functor.*

Proof. We are going to show that if $(\alpha',\beta',F',G') \in [\alpha,\beta,F,G]$, then the maps $E_{(\alpha',\beta',F',G')}$ and $E_{(\alpha,\beta,F,G)}$ are homotopic.

Let $A\colon \alpha \sim \alpha'$, $B\colon \beta \sim \beta'$ be two homotopies such that $F + f'A \simeq B(f \times I) + F'$ and $G + g'A \simeq B(g \times I) + G'$.

From the universal property of the pullback $E_{f',g'}$, there exists a homotopy $K\colon E_{(\alpha,\beta,F,G)} \sim E_{(\alpha',\beta',F',G')}$ if and only if there exist $K_1\colon \alpha\overline{\varepsilon}_{0,1} \sim \alpha'\overline{\varepsilon}_{0,1}$ and $K_2\colon \Lambda^\sharp \sim \Lambda'^\sharp$ such that $\langle f', g'\rangle K_1 = \varepsilon_{0,1}K_2$.

Since $A(\overline{\varepsilon}_{0,1} \times I)\colon E_{f,g} \times I \to X'$ is a homotopy between $\alpha\overline{\varepsilon}_{0,1}$ and $\alpha'\overline{\varepsilon}_{0,1}$, we have to give $K_2\colon E_{f,g} \times I \to Y'^I$ such that the adjoint $K_2^\flat\colon E_{f,g} \times I \times I \to Y'$ verifies:

$$K_2^\flat(x,\lambda,0,u) = \Lambda^\sharp(u) = (-F_x + \beta\lambda + G_x)(u)\,,$$

$$K_2^\flat(x,\lambda,1,u) = \Lambda'^\sharp(u) = (-F'_x + \beta'\lambda + G'_x)(u)\,,$$

$$K_2^\flat(x,\lambda,t,0) = f'A(\overline{\varepsilon}_{0,1} \times I)(x,\lambda,t) = f'A(x,t)\,,$$

$$K_2^\flat(x,\lambda,t,1) = g'A(\overline{\varepsilon}_{0,1} \times I)(x,\lambda,t) = g'A(x,t)\,.$$

The equivalence $-F(\overline{\varepsilon}_{0,1}\times I)+B(\overline{\varepsilon}_{0,1}\times I)\simeq f'A(\overline{\varepsilon}_{0,1}\times I)-F'(\overline{\varepsilon}_{0,1}\times I)$ and Lemma 3.2 give us the existence of a map $\mathcal{W}'\colon E_{f,g}\times I\times I\to Y'$ such that

$$\mathcal{W}'(x,\lambda,0,u)=-F(x,u)\,,\quad \mathcal{W}'(x,\lambda,t,1)=B(f\times I)(x,t)\,,$$
$$\mathcal{W}'(x,\lambda,t,0)=f'A(x,t)\,,\quad \mathcal{W}'(x,\lambda,1,u)=-F'(x,u)\,.$$

We have $\beta\langle\overline{f,g}\rangle^\flat+B(g\times I)(\overline{\varepsilon}_{0,1}\times I)\simeq B(f\times I)(\overline{\varepsilon}_{0,1}\times I)+\beta'\langle\overline{f,g}\rangle^\flat$ from Lemma 3.1 and so there exists a map $\mathcal{W}''\colon E_{f,g}\times I\times I\to Y'$ such that

$$\mathcal{W}''(x,\lambda,0,u)=\beta\lambda(u)\,,\qquad \mathcal{W}''(x,\lambda,t,1)=B(g\times I)(x,t)\,,$$
$$\mathcal{W}''(x,\lambda,t,0)=B(f\times I)(x,t)\,,\quad \mathcal{W}''(x,\lambda,1,u)=\beta'\lambda(u)\,.$$

The equivalence $G+g'A\simeq B(g\times I)+g'$ implies the existence of a map $\mathcal{W}'''\colon E_{f,g}\times I\times I\to Y'$ such that

$$\mathcal{W}'''(x,\lambda,0,u)=G(x,u)\,,\qquad \mathcal{W}'''(x,\lambda,t,1)=g'A\,,$$
$$\mathcal{W}'''(x,\lambda,t,0)=B(g\times I)(x,t)\,,\quad \mathcal{W}'''(x,\lambda,1,u)=G'(x,u)\,.$$

Finally we may define the required map $K_2^\flat\colon E_{f,g}\times I\times I\to Y'$ as

$$K_2^\flat(x,\lambda,t,u)=\begin{cases}\mathcal{W}'(x,\lambda,t,4u) & 0\le t\le 1/4\\ \mathcal{W}''(x,\lambda,t,4u-1) & 1/4\le t\le 1/2\\ \mathcal{W}'''(x,\lambda,t,2u-1) & 1/2\le t\le 1\,.\end{cases}$$

We can write now $E_{[\alpha,\beta,F,G]}$ for the homotopy class of the map $E_{(\alpha,\beta,F,G)}$.

Since $E_{(1,1,S,S)}(x,\lambda)=(x,c_{f(x)}+\lambda+c_{g(x)})$, we have $E_{[1,1,S,S]}=[1_E]$.

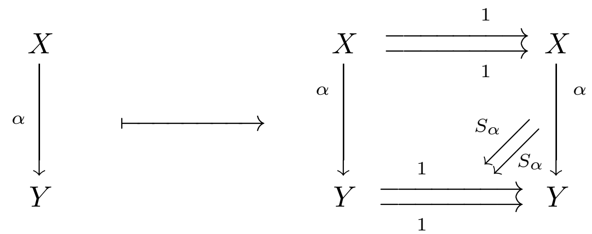
Let $[\alpha,\beta,F,G]$ and $[\gamma,\delta,H,K]$ be two composable arrows of $lh\mathrm{Top}^{\rightrightarrows}$.

$$\begin{aligned}
&E_{(\gamma,\delta,H,K)}E_{(\alpha,\beta,F,G)}(x,\lambda)=\\
&\quad=E_{(\gamma,\delta,H,K)}(\alpha(x),-F_x+\beta\lambda+G_x)=\\
&\quad=(\gamma\alpha(x),-H_{\alpha(x)}-\delta F_x+\delta\beta\lambda+\delta G_x+K_{\alpha(x)})=\\
&\quad=(\gamma\alpha(x),-(\delta F+H(\alpha\times I))_x+\delta\beta\lambda+(\delta G+K(\alpha\times I))_x)=\\
&\quad=E_{(\gamma\alpha,\delta\beta,\delta F+H(\alpha\times I),\delta G+K(\alpha\times I))}(x,\lambda)=\\
&\quad=E_{((\gamma,\delta,H,K)(\alpha,\beta,F,G))}(x,\lambda)
\end{aligned}$$

and so $E_{[\gamma,\delta,H,K]}E_{[\alpha,\beta,F,G]}=E_{[\gamma,\delta,H,K][\alpha,\beta,F,G]}$. ∎

Corollary 3.6. *Consider the maps $f, g: X \to Y, f', g': X' \to Y'$ and the homotopy equivalences $\alpha: X \to X', \beta: Y \to Y'$ such that $\beta f \sim f'\alpha, \beta g \sim g'\alpha$. The spaces $E_{f,g}$ and $E_{f',g'}$ have the same homotopy type and $R_{f,g} = R_{f',g'}$.*

Given a map $\alpha: X \to Y$, consider now the assignment



Notice that, for every homotopy $A: \alpha \sim \alpha'$, $S_\alpha + A \simeq A + S_\alpha$; then we can define the constant functor $\Delta: h\mathrm{Top} \to lh\mathrm{Top}^{\rightrightarrows}$ as $\Delta X = (1_X, 1_X)$ on the objects and $\Delta[\alpha] = [\alpha, \alpha, S_\alpha, S_\alpha]$ on the arrows.

Proposition 3.7. *Δ is left adjoint to E.*

Proof. The natural bijection $\phi: lh\mathrm{Top}^{\rightrightarrows}(\Delta W, (f, g)) \to h\mathrm{Top}(W, E_{f,g})$ is defined as $\phi[\alpha, \beta, F, G] = [h]$ with $h(w) = (\alpha(w), -F_w + G_w)$ (we may use a technique like the one used in the proof 3.5 to check that ϕ is well-defined).

The inverse ψ of ϕ is defined by $\psi[k] = [\overline{\varepsilon}_{0,1}k, f\overline{\varepsilon}_{0,1}k, S_{f\overline{\varepsilon}_{0,1}k}, (\langle\overline{f, g}\rangle k)^\flat]$ for every homotopy class $[k] : W \to E_{f,g}$ (observe that for every homotopy $K : k \sim k'$, the equivalences $S_{f\overline{\varepsilon}_{0,1}k} + f\overline{\varepsilon}_{0,1}K \simeq f\overline{\varepsilon}_{0,1}K + S_{f\overline{\varepsilon}_{0,1}k'}$ and $(\langle\overline{f, g}\rangle k)^\flat + g\overline{\varepsilon}_{0,1}K \simeq f\overline{\varepsilon}_{0,1}K + (\langle\overline{f, g}\rangle k')^\flat$ guarantees that ϕ is well-defined). ∎

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