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Dirac operators, heat kernels and microlocal analysis Part II: Analytic surgery(*)

R. MAZZEO – P. PIAZZA

Riassunto: *Sia X una varietà Riemanniana chiusa e sia $H \hookrightarrow X$ una ipersuperficie immersa. Sia $X = X_+ \cup_H X_-$ una decomposizione di X in due varietà con bordo, con $X_+ \cap X_- = H$. In questo articolo vengono presentate formule di decomposizione per alcuni invarianti geometrici e spettrali associati a $\eth_X$, un operatore di tipo Dirac su X. Vengono considerati in dettaglio l'indice di $\eth_X$, il fibrato indice ed il fibrato determinante associati ad una famiglia di tali operatori, l'invariante eta e la torsione analitica. Insieme ai risultati vengono illustrate le differenti tecniche impiegate per dimostarli.*

Abstract: *Let X be a closed Riemannian manifold and let $H \hookrightarrow X$ be an embedded hypersurface. Let $X = X_+ \cup_H X_-$ be a decomposition of X into two manifolds with boundary, with $X_+ \cap X_- = H$. In this expository article, surgery – or gluing – formulæ for several geometric and spectral invariants associated to a Dirac-type operator $\eth_X$ on X are presented. Considered in detail are: the index of $\eth_X$, the index bundle and the determinant bundle associated to a family of such operators, the eta invariant and the analytic torsion. In each case the precise form of the surgery theorems, as well as the different techniques used to prove them, are surveyed.*

(*) For Part I see ref. [52].

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1 – Introduction

The behaviour of global invariants for Dirac operators and Laplacians with respect to decompositions of their underlying compact Riemannian manifolds has become a topic of much interest over the past several years. We are thinking here of *geometric* invariants such as index and determinant bundles, as well as *spectral* invariants such as the eta invariant and analytic torsion. So-called 'gluing theorems' for these invariants provide new insights into their nature and have facilitated their use in other areas. One technique to study these problems was developed by the first author and Melrose [34], and McDonald [35], and is called *analytic surgery*. In this paper we give a brief introduction to this method and to a few of the problems for which it has proved useful, and also to survey a few other methods developed by other authors to study gluing problems.

In the most general terms, suppose that we are given a decomposition of the compact manifold X into two pieces, $X = X_+ \cup X_-$, where $X_\pm$ are submanifolds with boundary. Any geometric datum, such as a Riemannian metric g, a bundle E, or a spin structure and its associated Dirac operator $\eth$, restricts to give the corresponding structures on each of these pieces. In the next section we shall give precise definitions of some of the global invariants in which we are interested; for the sake of being concrete, and referring to that section for its definition, let us consider the eta invariant of a Dirac operator, $\eta(\eth)$. Setting aside, for the moment, the issue of boundary conditions, the simplest formulation of one of the problems we wish to discuss is whether there is a reasonable formula for $\eta(\eth_X)$ in terms of $\eta(\eth_{X_\pm})$; here, for any manifold Z, possibly with boundary, $\eth_Z$ denotes its Dirac operator relative to some fixed spin structure and metric. Amongst the various considerations we shall need to address, even just to formulate a reasonable conjecture more precisely, is the issue of boundary conditions, and also such matters as the dependence of the eta invariant on the underlying metric. Upon doing this, it will become apparent that it is very natural, or at least very convenient, to study families of degenerating metrics, or families of boundary conditions, and that the defect between $\eta(\eth_X)$ and $\eta(\eth_{X_\pm})$ is reasonably gauged by some measure of the variation in these families. The problem then will be to express this defect in some explicit way.

The structure of this paper will be somewhat informal, inasmuch as we consider these various problems in successively greater degrees of precision. In the rest of this introduction we formulate the surgery problem somewhat more carefully; the reader should note that different authors describe it in seemingly quite different ways, depending on their precise contexts and the applications they have in mind. We are trying here to present these approaches from a more uniform perspective. After this 'first-level' formulation, we proceed to discuss two competing points of view related to the issue of boundary conditions: namely, is it more natural to consider various geometric structures on a manifold with boundary Z as smooth up to the boundary (and possibly having some product structure near the boundary $Y = \partial Z$), or else as defined relative to an infinite cylindrical geometry near the ends. As was recognized already in [1], these two points of view are essentially equivalent, but choosing one or the other as primary tends to inform our intuitions in different ways. (Of course, there are many other geometries on a manifold with boundary, but these two have, up until now at least, played the most prominent rôles in the sorts of problems we consider.) We include here a brief overview of the calculus of b-pseudodifferential operators, as a preamble to the surgery calculus discussed later. We conclude the introduction by discussing the three principle methods used to study gluing theorems: those developed by Bunke, Vishik, and that of the first author, Melrose and Hassell.

In the remainder of the paper we shall, as promised, give more careful explanations of many of these issues. In §2, we discuss some of the different settings and invariants for which each of these methods has proved useful, or at least has been applied. In the three succeeding sections we give more careful discussions of these three methods, concentrating, it must be admitted, on the final one. Unfortunately, we do not have the time or space to go too deeply into any of the analytic subtleties in any of these approaches, but instead wish to present them side by side, indicating some of their relative strengths and weaknesses in hopes that this will be useful for future applications. In the final sections we give a more extended discussion of two further applications of the surgery calculus: the first is to the signature formula on manifolds with corners as in [26], while the second is to gluing formulæ for determinant bundles as in [54].

1.1 – The surgery problem

We start, as above, with the decomposition

$$X = X_+ \cup X_-, \qquad \text{where} \qquad X_+ \cap X_- = H \tag{1}$$

is a smooth, oriented hypersurface in X and the pieces $X_\pm$ are smooth manifolds with boundary. (We are implicitly assuming that H disconnects X; this is not necessary, but we consider only this case so as to minimize notation.) A Riemannian metric g on X induces metrics $g_\pm$ on $X_\pm$, and if Δ_Z denotes the (scalar) Laplacians on any one of these manifolds, $Z = X, X_\pm$, then a primitive form of the analytic surgery problem is to determine the relationship between $\mathrm{spec}\,(\Delta_X)$ and $\mathrm{spec}\,(\Delta_{X_\pm})$. (To define the latter quantities, we use, for example, Dirichlet conditions on H.) While precise relationships between individual eigenvalues are generally impossible to establish, it is easier to find relationships between some aggregate invariants of these spectra, such as the determinants $\det(\Delta)$, or even between their resolvents $(\Delta - \lambda)^{-1}$ or heat kernels, $\exp(-t\Delta)$.

At a slightly higher level of complexity, suppose that $\eth_X$ is the Dirac operator on X with respect to some fixed spin structure and the metric g, or even simply a generalized Dirac-type operator (which does not require a spin structure *per se*). The issue of boundary conditions for the restricted Dirac operators $\eth_{X_\pm}$ on $X_\pm$ is now more subtle, and it is well-known that one must use global boundary conditions, of the sort introduced by Atiyah, Patodi and Singer, to obtain an elliptic boundary problem. We discuss this in the next subsection. At any rate, having done this, once again we ask for relationships between the spectra of these operators or between the resolvents or heat kernels associated to their squares, $\eth^2$. The most common global spectral invariants in this context are the eta invariant $\eta(\eth)$, which is of particular interest only when $\dim X$ is odd, and the analytic torsion, $\tau(X)$. Again referring to the eta invariant for concreteness, it is not quite true in general that the eta invariant for $\eth_X$ is simply the sum of the eta invariants for $\eth_{X_\pm}$. So, as indicated earlier, the problem reduces to finding a good formula for the defect between these two expressions (just as the eta invariant is the defect between the two sides of the index formula). From this point of view, the defect arises because the boundary conditions which the $\eth_{X_\pm}$ inherit by restriction from $\eth_X$ do not match the natural global APS boundary conditions on

$X_\pm$. One may change perspective, though, and consider instead the eta invariant of $\eth_X$ relative to a family of metrics g_ε on X which degenerate along the hypersurface H. Denoting this family of operators by $\eth_{X,\varepsilon}$, then $\eta(\eth_{X,\varepsilon})$ depends on this family of metrics, but in an understandable way, at least for $\varepsilon > 0$ (here $\varepsilon = 0$ corresponds to some sort of degenerate limit). Now the problem becomes to determine the defect between the eta invariant of the limiting operator and the limit of the eta invariants. One may pose a similar problem for the log of the analytic torsion, $\log \tau(X)$.

The types of metric degenerations we shall discuss, and shall call *surgery degenerations*, arise when g_ε elongates transversally to H, but stays bounded (and converges smoothly to a limit) away from H in such a way that in the limit as $\varepsilon \to 0$, the interiors of the components $X_\pm$ inherit complete metrics with asymptotically cylindrical end structures. It may indeed be reasonable, and even better in some circumstances, to study other sorts of degenerations. For example, there is an enormous literature concerning degenerations of compact Riemann surfaces endowed with their hyperbolic metrics into limits with hyperbolic cusp ends — indeed, a dense set of points on the boundary of Teichmüller space of a surface may be 'reached' in this way — and Arakelov degenerations have also received considerable attention, cf. [62], [24]. Whether these other geometries are more favourable for some of our spectral questions is not known.

Beyond the questions concerning these numerical invariants are some others, particularly when one studies families of degenerating metrics and their associated Laplacians or Dirac operators. For the cylindrical degenerations we shall study, the spectrum of Δ_{g_ε} or $\eth_{g_\varepsilon}$ is discrete when $\varepsilon > 0$, but is continuous when $\varepsilon = 0$ (with possibly some additional discrete spectrum). It is natural to enquire how the transition between these two states takes place; in particular, how accurately may one describe the convergence of the discrete spectrum to continuous spectrum. This sort of question seems relevant in light of the extensive recent work on Novikov-Shubin invariants, which are some sort of measure of the 'germ' of continuous spectrum at 0 for the Hodge Laplacian on differential forms on universal covers of compact manifolds. This sort of geometry is quite different from cylindrical end geometry, but it is clear that there is much to be learned about the fine structure of the continuous spectrum of geometric operators on complete manifolds.

1.2 – APS vs. L^2 boundary conditions

Obviously there is a substantial geometric difference between complete metrics (with cylindrical ends) on $X_\pm$ and the restrictions of the metric g to these components, and just as obviously there are substantial analytic differences between the Laplacians or Dirac operators for these metrics. Whether to approach the surgery problem via restriction or degeneration of metrics is a moot point: as we shall see, each point of view has its strengths and weaknesses. But at heart is the purely qualitative and subjective question concerning which classes of functions or metrics or differential operators on a compact manifold with boundary one should consider to be the most natural.

At first, it seems odd to say anything other than that the classes of objects, e.g. functions, metrics, etc., which are smooth up to the boundary in the usual sense are the most natural ones. However, there is a good argument to be made that this is not necessarily the case. At the very least, the class of metrics with asymptotically cylindrical ends, the geometric elliptic operators corresponding to these metrics, and finally, the class of b-pseudodifferential operators which generalize these, have proven to be essential in a number of recent geometric investigations. Amongst these we wish to mention the recent 'direct' proof of the index theorem of Atiyah, Patodi and Singer on manifolds with boundary obtained by Melrose [36]. Although in many ways equivalent to the original approach to this theorem, its slightly different perspective and the use of b-pseudodifferential operators led the way to previously unknown results, such as the proper generalization of the index theorem for families of Dirac operators on manifolds with boundary by the second author and Melrose [44], [45], following partial results by Bismut and Cheeger [5] using an older approach. The predecessor to this paper [52] surveys these results. In addition, the long exact sequence in analytic K-homology for manifolds with boundary, or even with corners, had proved somewhat difficult to even formulate correctly in more traditional terms because of the necessity for keeping track of boundary conditions, but when recast in the language of b-pseudodifferential operators, it became much more transparent and amenable to proof, [43]. There are other, more recent, index-theoretic applications of the b-calculus, such as the higher Atiyah-Patodi-Singer index theorem on Galois coverings by Leichtnam and the second author, see [30], [31].

Beyond these essentially index-theoretic applications, manifolds with cylindrical ends, or degenerations to them, have also played important rôles in many other sorts of problems, and by many other authors; we mention only the important recent work [60], [46], [28], [47] in Donaldson theory and Seiberg-Witten theory.

The two points of view are closely related, and it is by playing them off against one another that one may obtain the best insights. To illustrate this, we describe in detail a very elementary fact, requisite to much of what follows, namely, the equivalence between the global Atiyah-Patodi-Singer (henceforth APS) boundary conditions for the Dirac-type operator $\eth$ on manifolds with boundary Z, assuming that all structures are of product-type near the ends, and the natural L^2 boundary condition on the prolongation $\widehat{Z}$ of Z to a manifold with infinite cylindrical ends. This equivalence was already noted and used in the original paper [1].

We begin with a Dirac-type operator $\eth$ over Z. To say that $\eth$ is of Dirac type means that it acts between sections of two different bundles E and F, and that there is a parallel bundle map

$$\gamma : \mathrm{Cl}\,(TZ) \longrightarrow \mathrm{End}\,(E, F)$$

which is a fibrewise homomorphism from the Clifford bundle over Z to the bundle of endomorphisms from E to F, in terms of which, locally,

$$\eth = \sum_{j=1}^{n} \gamma(e_j)\nabla_{e_j} + R,$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal frame and $R \in \mathcal{C}^\infty(Z; \mathrm{End}\,(E, F))$. Requirements of self-adjointness impose compatibility conditions on γ and R.

Next, suppose that t is a smooth defining function for $Y = \partial Z$, so that t vanishes simply on Y and is everywhere positive in the interior of Z. A metric g which is a product near Y takes the form

$$g = dt^2 + h$$

where h is a smooth metric on Y which is independent of t for $t \leq \varepsilon$. The operator $\eth$ is product type near Y if in some collar neighbourhood of the boundary, the bundles E and F are lifted from Y,

$$E = \pi^* E_Y, \qquad F = \pi^* F_Y, \qquad \text{where} \quad \pi : Y \times [0, t_0) \longrightarrow Y,$$

and in terms of these partial trivializations, $\eth$ takes the form

$$\eth = \gamma(\partial_t)\,(\partial_t + A) + R.$$

Here A is some t-independent first order elliptic operator on Y, acting on sections of E and R is is also t-independent for $t \leq t_0$. In the cases of interest to us, A is self-adjoint, and we shall also assume that $R \equiv 0$ for simplicity. More detailed descriptions of generalized Dirac-type operators are given in [36] and [49].

Since A is self-adjoint, elliptic and first order, its spectrum is a discrete sequence of real numbers $\{\lambda_j\}$ which is unbounded both above and below. The corresponding eigensections of E will be denoted ϕ_j. There is no natural local elliptic boundary condition for this operator. The 'correct' global boundary condition was one of the very important discoveries in [1]. To state it, we first define the orthogonal projection

$$\Pi_0^+ : L^2(Y; E) \longrightarrow L^2(Y; E)$$

onto the sum of eigenspaces for A with nonnegative eigenvalues. Thus $\Pi_0^+(\phi_j) = 0$ whenever $\lambda_j < 0$ and $\Pi_0^+(\phi_j) = \phi_j$ whenever $\lambda_j \geq 0$. The APS boundary conditions involve letting the operator $\eth$ act on the domain

$$\{u \in \mathcal{C}^\infty(X; E)\ ;\ \Pi_0^+(u_Y) = 0\}. \tag{2}$$

This boundary projection has a classical analogue: if X is the disc in $\mathbb{C}$ and $\eth$ is the Cauchy-Riemann operator, then it is elementary that the restriction of holomorphic functions to S^1 are precisely those with only nonnegative Fourier coefficients. (Although neither the metric nor the operator are of product type here, it is not hard to transform them to be of this form.)

To explain this boundary projection better, we consider elements u of the nullspace of $\eth$. Solutions of $\eth u = 0$ may be analyzed near Y by introducing the eigendecomposition

$$u(t, y) = \sum_j u_j(t)\phi_j(y), \quad \text{so that} \quad \eth u(t, y) = \gamma(\partial_t) \sum_j \Big(u_j'(t) + \lambda_j u_j(t)\Big)\, \phi_j(y),$$

valid in the collar neighbourhood $\mathcal{U} = [0, t_0) \times Y$. Thus if $\eth u = 0$, then $u_j(t) = a_j e^{-\lambda_j t}$ for some constants a_j and for all j. Extend the variable t, the bundles E and F and the operator $\eth$ to the manifold

$$\widehat{Z} = Z \cup_Y (Y \times (-\infty, 0]),$$

obtained by adjoining the half-cylinder $\mathbb{R}^- \times Y$ to Z along the common boundary Y. Then the solution u automatically extends to a solution of this equation on $\widehat{Z}$. More importantly, $\Pi^+(u(0,\cdot)) = 0$ if and only if this extension decays exponentially. In particular, elements of the nullspace of the elliptic boundary problem $(\eth, \Pi_0^+)$ on Z are in one-to-one correspondence with the L^2 nullspace of $\eth$ on $\widehat{Z}$.

The associated inhomogeneous elliptic boundary problem is

$$\eth u = f \quad \text{in } X, \qquad \Pi_0^+(u|_Y) = \phi,$$

for f and ϕ in some appropriate spaces of sections. Assuming that $\eth$ is symmetric, the adjoint boundary problem is given by the pair $(\eth, \Pi^+)$, where Π^+ is the spectral projection onto the *positive* part of the spectrum of A. Notice that $\operatorname{ran}(\Pi_0^+) \ominus \operatorname{ran}(\Pi^+) = \ker A$. Elements of the nullspace of $(\eth, \Pi^+)$ are in one-to-one correspondence with the *extended* L^2 nullspace of $\eth$ on $\widehat{Z}$, which contains all temperate solutions of $\eth u = 0$. To obtain a self-adjoint boundary problem, we see that we must restrict the domain for this operator to be intermediate between the domains for the two boundary problems $(\eth, \Pi_0^+)$ and $(\eth, \Pi^+)$. It turns out that the self-adjoint extensions of $(\eth, \Pi^+)$ are in one-to-one correspondence with the Lagrangian subspaces of $\ker A$. The symplectic structure with respect to which these subspaces are Lagrangian is the one induced on $\ker A$ from the L^2 inner product and the almost complex structure induced from Clifford multiplication by ∂_t. (We are using that $\gamma(\partial_t)$ anticommutes with A, but in particular preserves $\ker A$, and that the Clifford relations imply that $\gamma(\partial_t)^2 = -I$.) If $\Lambda \subset \ker A$ is any Lagrangian subspace, then we define the augmented projection Π_Λ^+ by demanding that $\operatorname{ran}(\Pi_\Lambda^+) \ominus \operatorname{ran}(\Pi^+) = \Lambda$. The corresponding self-adjoint elliptic boundary problem is then given by the pair $(\eth, \Pi_\Lambda^+)$.

To conclude this discussion, we observe that there is a natural Lagrangian subspace $\Lambda_{\text{sc}} \subset \ker A$ which provides the connection between the ‘finite’ elliptic boundary problem and the operator on the manifold $\widehat{Z}$. It is the subspace of asymptotic limits of solutions of $\eth u = 0$:

$$\Lambda_{\mathrm{sc}} \equiv \{ \lim_{t \to -\infty} u(t,y) : u \text{ a bounded solution of } \eth u = 0 \text{ defined on all of } \widehat{Z} \},$$

and is called the *scattering Lagrangian* associated to the operator. The fact that the solution u must be globally defined is very important in this definition. It is not immediately clear why this Λ should even have the correct dimension, let alone be Lagrangian, but these follow from Green’s formula and adjointness considerations, cf. [49], [36].

1.3 – The calculus of b-pseudodifferential operators

The simple observations of the last subsection indicate that it is at least as fruitful to work in the category of manifolds with cylindrical ends as of manifolds with boundary, and thus we have reconnected with our earlier question about the most natural classes of functions, operators, etc., on a manifold with boundary. The preceding discussion points out one advantage to studying geometric elliptic operators on complete manifolds: there is no need to choose boundary conditions explicitly, because the L^2 requirement imposes a natural set of such conditions.

There are other advantages too. For example, one often underappreciated fact is that there is a lack of naturality inherent in the usual spaces of smooth functions and pseudodifferential operators on a manifold with boundary Z: general pseudodifferential operator do not map the space $\mathcal{C}^\infty(Z)$ to itself. This prompted Boutet de Monvel’s introduction of the transmission condition for symbols, and his pseudodifferential calculus adapted to boundary problems in the mid 1970’s [8]. Several years later a quite different approach was initiated by Melrose, resulting in the so-called b-calculus, or calculus of b-pseudodifferential operators on a manifold with boundary Z. In fact, this b-calculus is the first step toward a rather general microlocal approach for studying a hierarchy of spaces of degenerate differential pseudodifferential operators. The surgery calculus we discuss later is one amongst many in this hierarchy, and as can be seen from the geometric and analytic problems motivating it, is some sort of extension of the b-calculus. Because of this relationship, we include a very brief introduction to the b-calculus here. This will be continued, and the surgery calculus itself will be discussed, later in the paper.

In defining this 'calculus' (by which we mean a set of pseudodifferential operators which are essentially closed under composition, up to some elementary and computable obstructions having to do with integrability of certain functions) it is customary to start by introducing the space of b-vector fields on Z (although one might easily regard one of the other objects we introduce below as the 'primary' object of the theory). This class of vector fields is defined by

$$\mathcal{V}_b(Z) = \{V \in \mathcal{C}^\infty(Z;TZ) : V \quad \text{tangent to} \quad \partial Z = Y\},$$

and this condition is obviously closed under Lie bracket, so that $\mathcal{V}_b(Z)$ is a Lie algebra. In terms of a smooth boundary defining function x, and any choice of local coordinates y on Y, $\mathcal{V}_b$ is generated over $\mathcal{C}^\infty(Z)$ by

$$x\partial_x, \partial_{y_1}, \dots, \partial_{y_k}, \quad k = \dim Y.$$

Alternately, we can also define

$$\mathcal{V}_b = \{V \in \mathcal{C}^\infty(Z;TZ) : Vx = xf \quad \text{for some } f \in \mathcal{C}^\infty(Z)\}.$$

Next, a metric g is said to be a b-metric if $g(V,W) \in \mathcal{C}^\infty(Z)$, $g(V,V) \geq 0$ for every $V, W \in \mathcal{V}_b$. In terms of the same local coordinates near Y, any such g is a positive smooth symmetric two-tensor in the 1-forms dx/x and dy^j. A slightly more tractable, and just as useful subclass of these are the *exact* b-metrics. g is called exact if there exists a boundary defining function $x \in \mathcal{C}^\infty(Z)$ and some smooth (in the ordinary sense) symmetric two-tensor h such that

$$g = \frac{dx^2}{x^2} + h.$$

If we introduce $t = -\log x$, and if h is independent of x in some neighbourhood of Y, then an exact b-metric is nothing more than a metric with an infinite product cylindrical end, and a general exact b-metric decreases at an exponential rate to a product cylindrical metric in these coordinates.

From $\mathcal{V}_b(Z)$ we can define the ring of b-differential operators $\mathrm{Diff}_b^*(Z)$: this contains all operators which may be written as locally finite sums of products of elements of $\mathcal{V}_b$. Thus, in the same local coordinates,

$$\mathrm{Diff}_b^m(Z) \ni L \Longrightarrow L = \sum_{j+|\alpha| \leq m} a_{j,\alpha}(x,y)(x\partial_x)^j \partial_y^\alpha.$$

Examples of such operators include any geometric operator, such as the Laplacian on differential forms or Dirac operator, associated to an exact b-metric.

Although any $L \in \mathrm{Diff}_b^*(Z)$ is degenerate in the ordinary sense, it is still possible to define a meaningful notion of ellipticity: a b-operator L is said to be elliptic if it may be represented locally as an elliptic combination of the basic spanning set of b-vector fields listed above. One is then led to ask the question as to whether such operators have any 'right' to be called elliptic, i.e. whether they enjoy any of the properties familiar from elliptic theory on compact manifolds. Specifically, are these operators Fredholm on any natural function spaces, and what are the regularity properties of solutions of $Lu = 0$ or $Lu = f$? Note that because elliptic b-operators are elliptic in the ordinary sense in the interior of Z, these questions really involve only 'local' behaviour at ∂Z.

To investigate these questions, it is natural to use pseudodifferential methods, and the heart of this technique is to introduce a class of pseudodifferential operators $\Psi_b^*(Z)$ which contains $\mathrm{Diff}_b^*(Z)$, and which is also hopefully large enough to include inverses, or at least good parametrices, for the elliptic differential b-operators. As a clue to how one might define these pseudodifferential operators, observe that any element of $\mathcal{V}_b(Z)$, hence any b-differential operator, is approximately invariant under dilations in the variable x (which correspond to translations in the variable t). Thus one might hope to characterize elements of $\Psi_b^*(Z)$ by this same property, and indeed this is the case. Actually, it is easiest to characterize these operators by geometric properties of their Schwartz kernels, which are distributions on the product $Z \times Z = Z^2$. Because elements of $\mathrm{Diff}_b^*(Z)$ are degenerate at ∂Z, we expect pseudodifferential operators which represent inverses for the elliptic elements to have some sort of singularity at $(\partial Z)^2$. The main idea is that these singularities can be characterized geometrically: instead of regarding the Schwartz kernel of an element $B \in \Psi_b^*(Z)$ as a more singular distribution on the relatively simple space Z^2, we instead regard it as a simpler distribution on a geometrically more complicated space Z_b^2, the b-stretched product of Z with itself. This new space is obtained from Z^2 by blowing up the corner $(\partial Z)^2$; said differently, introduce polar coordinates around this corner and include as part of the blow-up the new face where the polar distance variable $r = 0$. Z_b^2 is a manifold with corners. It has three codimension one boundary faces, $Z \times \partial Z$, $\partial Z \times Z$, and the new one created from the blow-up, and away from this final hypersurface it is diffeomorphic to Z^2. The Schwartz kernel of B is required to be smooth in the interior of Z_b^2 away from the diagonal, where it is to have an ordinary pseudodifferential singularity, and at each of the codimension one boundary hypersurfaces it is required to have complete polyhomogeneous (i.e. classical) expansions. In fact, it is even required to be smooth up to the front face. Notice, however, that back on the original manifold Z^2, this means that it is only smooth in polar coordinates around the corner, away from the diagonal. When $B \in \mathrm{Diff}_b^*(Z)$, then its Schwartz kernel lifts to a δ-section supported along the lifted diagonal of Z_b^2.

Complete and accessible discussions of this space of operators are to be found in [36], and [33], to which we refer the interested reader for more details.

To return to the original objective, though, once the space of b-pseudodifferential operators on Z, $\Psi_b^*(Z)$, has been defined, and certain basic facts about it, such as its closure under composition and a satisfactory symbol calculus, have been established, then one may proceed with the investigation of the elliptic differential b-operators. To phrase the main results, one lets these operators act not just on the ordinary Sobolev spaces $H^s(Z)$, but rather on weighted b-Sobolev spaces $x^\delta H_b^s(Z)$. The subscript b refers to the fact that the differentiations involved in defining these spaces should be with respect to the elements of $\mathcal{V}_b$, while the factor x^δ allows for changing the rates of growth or decay at ∂Z. The basic result is that for all but a discrete set of weight parameters δ, an elliptic b-operator L is Fredholm on $x^\delta H_b^s(Z)$. Furthermore, an arbitrary solution of $Lu = 0$ admits a complete polyhomogeneous expansion in powers of x (and possibly $\log x$) as $x \to 0$. This polyhomogeneity is the natural replacement for the special case of smoothness up to the boundary (which is what occurs when all powers in the expansion are nonnegative integers and no logarithmic factors occur). These results are all consequences of the fact that one has a very precise description of a good parametrix for L. In fact, that is really the point of the theory. After the not completely insignificant effort involved in defining the calculus and constructing the parametrix, we then have a more or less complete geometric description of the Schwartz kernel of the generalized inverse for L on any of the admissible weighted Sobolev spaces. From this it is possible to simply 'read off' any more refined mapping or regularity properties about L one might wish to know.

This overview of the b-calculus is intended to motivate the similar, but unfortunately more elaborate, description of the surgery calculus later.

1.4 – Three different approaches to the problem

In this final section of the introduction we briefly introduce three different methods which have been developed to study the surgery problem. These were all developed roughly simultaneously and independently, but each was directed toward, and achieved, somewhat different goals. What we call the first approach was developed by Bunke [11], the second by Vishik [61], and later used by Brüning and Lesch [10], while the final one was contained in work of the first author with Melrose [34], and then also with Hassell [25]. To avoid being overly self-referential, this last approach will be referred to as that of MM/HMM. In later sections of this paper, we amplify the descriptions of these approaches — rather cursorily for the first two and in more detail for the third. For the most part, we shall only discuss Dirac-type operators because they have provided the main setting for applications.

As we have seen, the main issue is to somehow 'disconnect' the operator $\eth_{X_+}$ from $\eth_{X_-}$, and this may be done either by use of boundary conditions or by literally disconnecting the two halves geometrically by placing them at infinite distance from one another. Bunke's approach is the least intricate, technically, and essentially uses both of these types of considerations. Vishik's ideas involve a variation of boundary conditions along H, while those of MM/HMM rely on the idea of geometric separation.

The goals of these papers are also quite different. Bunke's intent is to find a gluing formula for the eta invariant. Vishik was concerned with gluing formulæ for determinants of elliptic operators, and particularly for the analytic torsion. In the somewhat more tractable version of his ideas developed by Brüning and Lesch, the goal is to find another proof of the gluing formula for the eta invariant. The techniques of MM/HMM are directed toward proving uniformity of the resolvent and heat kernel associated to $\eth^2$ in the 'analytic surgery limit' as the manifold X stretches to infinite length along the hypersurface H. Gluing formulæ for the eta invariant, and analytic torsion [23], are then consequences of this uniformity, but far more detailed information is obtained along the way. The expense, of course, is that the development of this approach is by its nature the most technically intricate of the three.

We now go into only slightly more detail. Bunke's setup involves considering two different manifolds. The first is the disjoint union $X_+ \sqcup X_-$, each endowed with long (but finite) cylindrical ends, while the second is the disjoint union of X, endowed with a long cylindrical section around H, and a long cylindrical piece $[-T, T] \times H$. The goal is to show that there is some abstract unitary equivalence between the Dirac operators on these two (sets of) manifolds; since they are unitary invariants, the eta invariants for these manifolds must also coincide. Three of the four components are the various terms one expects in the gluing formula for the eta invariant, and the fourth represents the defect term, and it may be computed 'explicitly'. One subtlety here is in determining how the different boundary conditions at the various ends arise in order that the unitary isomorphism be valid.

Vishik's setup, on the other hand, involves the consideration of a family of boundary conditions along the hypersurface H. Each corresponds to some self-adjoint elliptic boundary problem. At one extreme, this problem corresponds to the operator $\eth$ on the closed manifold X, where the hypersurface H becomes 'invisible'; these are the transmission boundary conditions. At the other extreme, the boundary conditions are the natural APS ones on each half. The analytic torsion, or eta invariant, may be computed for each operator in this family, and the problem then consists of computing the variation of these invariants with respect to the parameter. The total variation with respect to this parameter represents the defect.

Finally, as indicated earlier, in MM/HMM the goal is to develop a 'surgery calculus' $\Psi_s^*(X)$, that is, a calculus of pseudodifferential operators on X, depending on a parameter ε, and which incorporates the sorts of degeneracies seen in the family of Dirac operators or Laplacians with respect to metrics undergoing degeneration to infinite cylindrical ends. Thus, for $\varepsilon > 0$, the surgery calculus restricts to the ordinary pseudodifferential calculus, while at $\varepsilon = 0$ it somehow induces the b-calculus. The point is to show how the transition between these quite different calculi takes place. Once this calculus is defined, and its basic analytic properties established, such as a symbol calculus, closure under composition, etc., then one may use it to construct parametrices for $(\eth^2_{X,g_\varepsilon} - \lambda)$, for example. As with the b-calculus, if one is able to describe the behaviour of this resolvent, or of the heat kernel, uniformly with respect to ε, and explicitly, it is then straightforward to examine the behaviour of these auxiliary numerical invariants. One also obtains more detailed information, such as the way in which the discrete spectrum accumulates into continuous spectrum.

2 – Some applications of the surgery formula – an overview

We now describe in somewhat greater detail four different types of objects, for the study of which some form of the analytic surgery technique has proved useful. These are index bundles, the eta invariant, analytic torsion and determinant bundles.

In each of the following settings we shall, again for simplicity, consider only the case of a Dirac-type operator $\eth$, acting between sections of the bundles E and F over the manifold X. We shall describe at least the general form of the gluing theorems in each context, leaving the more precise statements until later.

2.1 – Index bundles

First we consider the numerical index. We assume that X is *even*-dimensional and, for simplicity, spin. We denote by $\not{S} = \not{S}^+ \oplus \not{S}^-$ the spin bundle and its splitting into the plus and minus spin bundles. The Dirac operator $\eth$ is odd with respect to the natural $\mathbb{Z}_2$-grading, and so takes the form

$$\begin{pmatrix} 0 & \eth^- \\ \eth^+ & 0 \end{pmatrix} \qquad \eth^- = (\eth^+)^*$$

with $\eth^\pm : \mathcal{C}^\infty(X, \not{S}^\pm) \to \mathcal{C}^\infty(X, \not{S}^\mp)$. Since the manifold X is closed and compact the Dirac operator is Fredholm on any Sobolev space $\eth^+ : H^m(X, \not{S}^+) \to H^{m-1}(X, \not{S}^-)$ with index $\mathrm{ind}\,(\eth) = \dim(\ker(\eth^+)) - \dim(\ker(\eth^-))$ which is independent of m by elliptic regularity. In this even-dimensional setting, our primary interest is in $\eth^+$ and not in the full self-adjoint operator $\eth$.

Suppose now that $X = X_+ \cup_H X_-$; up to a perturbation not affecting the index we can assume that the metric near the disconnecting hypersurface H is of product type. This means that the Dirac operator on each piece $X_\pm$ takes the product form introduced in the previous section, with t now denoting a defining function for H. Notice that the vector field ∂_t will be normal to H, the common boundary of $X_\pm$, but inward pointing for one manifold, say X_+, and outward pointing for the other. As in §1.2, we denote by Π_0^+ the augmented APS spectral projection for the boundary operator $\eth_H$. Because of the discrepancy in the orientation of the normals it is easy to check directly that the two APS boundary value problems can be written as $(\eth^+_{X_+}, \Pi_0^+)$ and $(\eth^+_{X_-}, \mathrm{Id} - \Pi^+)$. Applying the Atiyah-Singer index theorem [2] to $\eth_X$ and the Atiyah-Patodi-Singer index theorem [1] to the two boundary value problems and observing moreover that the two eta invariants cancel because of the opposite orientation of the normals, we obtain the following surgery formula for the index

$$\operatorname{ind}(\eth_X) = \operatorname{ind}(\eth_{X_+}, \Pi_0^+) + \operatorname{ind}(\eth_{X_-}, \mathrm{Id} - \Pi^+) + \dim(\operatorname{null} \eth_H). \tag{3}$$

Notice that $\dim(\operatorname{null} \eth_H) = \dim(\Pi_0^+ - \Pi^+)$. Suppose now that X is the typical fibre of a fibration of compact manifolds: $\phi : M \to B$. Following what is now standard notation, we denote the family of metrics on the fibres by $g_{M/B}$; we also assume that the fibres carry smoothly varying spin structures. We denote by M^z the fiber over z; thus $M^z \equiv \phi^{-1}(z) \cong X$. For each $z \in B$ we can consider the Dirac operator $(\eth_M)_z$ naturally defined by the spin structure of M^z. We obtain a family $\eth_M = ((\eth_M)_z)_{z \in B}$ of Dirac operators.

Let H be a disconnecting hypersurface of the fibration $M \to B$ and assume that H also fibres over B: $\phi|_H : H \to B$. Thus $M = M_+ \cup_H M_-$ and each fiber M^z is the union along H^z of two manifolds with boundary: $M_+^z \cup_{H^z} M_-^z$.

We obtain in this way four families of Dirac operators: $\eth_M$, $\eth_{M_+}$, $\eth_{M_-}$ and $\eth_H$. Since the family $\eth_M$ is defined on a closed manifold, the familiar construction of an *index bundle*, as in [3], provides us with an index class $\operatorname{Ind}(\eth_M) \in K^0(B)$. The problem is to formulate and prove the analogue of (3). If the vector spaces $\ker(\eth_H)_z$ are of *constant rank* as z varies in B, then the APS boundary value problems for M_+^z and M_-^z define, as z

varies, two *continuous* families of boundary value problem

$$(\eth_{M_+}, \Pi_0^+) = (\eth_{M_+^z}, (\Pi_0^+)_z)_{z\in B} \qquad (\eth_{M_-}, \mathrm{Id}-\Pi^+) = (\eth_{M_-^z}, \mathrm{Id}-(\Pi^+)_z)_{z\in B}.$$

Notice that because of the assumption of constant rank, $\mathrm{Ker}\,(\eth_H) = \cup_{z\in B}\ker(\eth_H)_z$ is a *continuous* (in fact smooth) vector bundle.

However if the constant rank assumption is not satisfied we know that the two APS families will not be continuous. In this case, as explained at length in Part I of this survey [52] we need to fix a *spectral section* P for the family $\eth_H$ (see [43] [44]). Thus $P = (P_z)_{z\in B}$ is a smooth family of pseudodifferential operators of order zero which are self-adjoint projections and with the additional property that there exists a positive constant $R \in \mathbb{R}$ such that

$$(\eth_{H^z})u = \lambda u \Rightarrow \begin{cases} P_z u = u & \text{if } \lambda > R \\ P_z u = 0 & \text{if } \lambda < -R \end{cases} \tag{4}$$

Since $\eth_H$ is by construction a boundary family we know from [43] that there exist an infinite number of spectral sections; moreover two spectral sections, P_1, P_2, give rise to a difference element $[P_1 - P_2] \in K^0(B)$ (and in fact it can be proved that these differences exhaust all of $K^0(B)$). A spectral section P for $\eth_H$ fixes a *smooth* family of generalized APS boundary value problem $(\eth_{M_+}, P)$ and thus an index class $\mathrm{Ind}\,(\eth_{M_+}, P)$. Simply define $(\eth_{M_+}, P)_z$ as the operator $\eth_{M_+^z}$ with domain

$$\{u \in L^2(M_+^z, E_z); (\eth_{M_+^z})u \in L^2(M_+^z, E_z), P(u|_{\partial M_+^z}) = 0\}.$$

We also obtain an index class for $\eth_{M_-}$ by considering the family of boundary value problems $(\eth_{M_-}, \mathrm{Id}-P)$ (recall that the normal to M_- is oriented in the outward direction).

We can now state the decomposition formula for index bundles ([18]).

THEOREM 1 (Dai-Zhang). *Let P_1 and P_2 be spectral sections for $\eth_H$. Then the following formula holds*

$$\mathrm{Ind}\,(\eth_M) = \mathrm{Ind}\,(\eth_{M_+}, P_1) + \mathrm{Ind}\,(\eth_{M_-}, \mathrm{Id} - P_2) + [P_1 - P_2] \quad \text{in} \quad K^0(B)$$

The corresponding formula for the Chern characters follows directly from the family index formula of Melrose-Piazza. Notice that if in particular $\operatorname{Ker}(\eth_H)$ is a smooth vector bundle then we can choose $P_1 = \Pi_0^+$, $P_2 = \Pi^+$; then $[\operatorname{null}(\eth_H)] = [P_1 - P_2]$ and we obtain the precise analogue of (3).

The proof of the surgery formula for the index bundle, as given by Dai-Zhang, follows Bunke's method. One can also prove it with the surgery calculus of MM, as in [54].

2.2 – The eta invariant

The eta invariant $\eta(\eth)$ is a spectral invariant which was discovered originally in the context of the APS index theorem as the boundary correction term. Because indices are really even-dimensional phenomena, eta invariants are therefore of most interest when the dimension of X is odd – in fact, there are a number of difficult analytic subtleties in even defining the eta invariant in even dimensions. It was pointed out by Singer [58] that the eta invariant of Dirac operators in odd dimensions actually shares many properties with the index of Dirac operators in even dimensions: it could even be regarded as the odd-dimensional analogue of the index. This has been made more precise and generalized considerably by Melrose [40]. Other aspects of Singer's assertions have been addressed and proved by Wojciechowski [63], [64].

In any case, the basic definition is given as follows. Since $\eth$ is elliptic, and is self-adjoint when $n \equiv \dim X$ is odd, its spectrum is discrete, and we denote it by $\{\lambda_j\}$. This sequence is unbounded in both directions. The Weyl asymptotics show that the eta function

$$\eta(s) \equiv \sum \frac{\operatorname{sgn} \lambda_j}{|\lambda_j|^s}$$

is defined and holomorphic in the half-plane $\operatorname{Re} s > n$. From the usual analysis of the short-time asymptotics of the heat kernel associated to $\eth^2$, this function extends meromorphically over the complex plane. This extension is actually regular at $s = 0$ (and this is precisely the point that becomes much more subtle for non-Dirac operators and in even dimensions). This value is defined to be the eta invariant $\eta(\eth)$.

It is usually easier to work with a different expression for this invariant. First note that for $\mathrm{Re}\, s$ sufficiently large,

$$\eta(s) \equiv \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty t^{\frac{s-1}{2}} \mathrm{Tr}\left(\eth e^{-t\eth^2}\right) dt.$$

The integral continues to converge up to $s = 0$. This is not obvious, but follows from Getzler's rescaling technique, which is explained in [4] and [36]. The factor in front is regular at $s = 0$ too, and so

$$\eta(\eth) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \mathrm{Tr}\left(\eth e^{-t\eth^2}\right) dt.$$

Since we are also interested in the eta invariant on the manifolds with boundary $X_\pm$, we must also discuss how to define these. Recall the two different sorts of metrics we have been considering, those which make these manifolds compact with boundary, and are of product type near the boundaries, and those which are complete with infinite cylindrical ends. In the former case, following our earlier discussion, once we have introduced Lagrangian subspaces $\Lambda_\pm \subset \ker \eth_H$, we obtain self-adjoint elliptic boundary problems $(\eth_{X_+}, \Pi^+_{\Lambda_+})$ and $(\eth_{X_-}, \Pi^+_{\Lambda_-})$. These operators have discrete spectrum, and at least formally the preceding definitions make sense. The details of making these plausible definitions rigorous has been carried by MÜLLER [49]. The invariants we obtain will be denoted $\eta(\eth_{X_\pm}, \Lambda_\pm)$ for simplicity.

We may now state one form of the surgery problem for eta invariants explicitly: find a tractable expression for the defect

$$\delta(\Lambda_+, \Lambda_-) \equiv \eta(\eth_X) - \eta(\eth_{X_+}, \Lambda_+) - \eta(\eth_{X_-}, \Lambda_-).$$

Notice that we have written this defect as a function of the two Lagrangians $\Lambda_\pm$. That these should be the essential variables on which it depends requires some work. Furthermore, it is also of substantial interest to see whether there is some choice of Lagrangians for which this formula becomes particularly simple or natural.

The other geometric setup is when $X_\pm$ are endowed with exact b-metrics, and the degeneration from X to these complete metrics is a surgery degeneration as defined more precisely later in this paper. The biggest obstacle now is that the eta invariant $\eta(\eth)$ ostensibly requires the operator $\eth$ to have discrete spectrum, while in this case we know that the Dirac operator has continuous spectrum. Thus it is not even clear how to define the eta function $\eta(s)$. Starting from the definition in terms of the heat kernel we see the obstacle in a different way. The heat kernel for $\eth^2$ on either X_+ or X_- is a smooth function, but it does not decay along the cylindrical ends, and hence is not integrable. The way out of this impasse is to use the regularized b-trace defined by MELROSE [36]. This b-trace is an extension of the ordinary trace to the ring of smoothing b-pseudodifferential operators. For any such operator R, the pointwise trace along the diagonal of the Schwartz kernel of R, (i.e. just its restriction to the diagonal, if R acts on functions) has an asymptotic expansion in terms of nonnegative powers of the boundary defining function x. It therefore makes sense to define

$$(5) \qquad {}^b\mathrm{Tr}\,(R) = \lim_{\varepsilon\to 0}\left(\int_{x\geq\varepsilon} K_R(x,y)\,\frac{dxdy}{x} + \log\varepsilon\cdot\int_{x=0} K_R(0,y)\,dy\right)$$

Here $K_R(x,y)$ is the (pointwise trace of the) Schwartz kernel of R on the diagonal, using some local coordinate system y on the boundary. Granting the naturality of this definition, it is then reasonable to define the regularized b-eta invariant, ${}^b\eta(\eth)$, via the same heat-kernel formula, but substituting the b-trace for the ordinary trace.

Although this definition may seem ad hoc, it follows from work of MÜLLER [49] that on a manifold $\widehat{Z}$ with infinite (product) cylindrical ends, if Λ_{sc} is the scattering Lagrangian, i.e. the subspace of $\ker\eth_Y$ obtained as the set of asymptotic limits of $\eth u = 0$ on $\widehat{Z}$, and if Z_T represents the truncation of $\widehat{Z}$ to any compact piece, where $t\geq T$ for some sufficiently negative T, then

$$\eta(\eth_{Z_T},\Pi^+_{\Lambda_{\mathrm{sc}}}) = {}^b\eta(\eth_{\widehat{Z}}).$$

In particular, the term on the left does not depend on the length of the cylindrical ends. This serves as ample evidence that the b-eta invariant is a natural object.

The easiest case to calculate this defect is when $\ker\eth_H = \{0\}$. Then necessarily both Λ_+ and Λ_- are also trivial, and so it is hardly surprising (and is consonant with our notation) that the defect vanishes. In this setting of boundary problems on finite manifolds this was proved by BUNKE [11], while in the setting of degeneration to exact b-metrics it was proved in [34].

It is much more interesting, of course, to see what happens when $\ker \eth_H$ is nontrivial. Then $\delta(\Lambda_+, \Lambda_-)$ does not necessarily vanish. Rather nicely, it turns out that the 'best' Lagrangians with respect to which to compute this defect are the ones induced from the asymptotic limits of bounded solutions on the cylindrical extensions of $X_\pm$. The importance of these Lagrangians even for the boundary problem on the compact manifold was first proved by MÜLLER [49]. There are several nice formulæ for the defect in this case. The one obtained by Bunke is in the form of an averaged Maslov invariant,

$$\delta(\Lambda_+, \Lambda_-) = \int_G \mu(g\Lambda, \Lambda_+, \Lambda_-)\, dh.$$

Here G is the Lagrangian Grassmanian of $\ker \eth_H$, dh is normalized Haar measure on it, μ is the Maslov invariant, which is a function of three separate Lagrangians, and Λ is an arbitrary third Lagrangian. The formula found for the defect in this case in [25] is more elementary and purely linear algebraic. Again when $\Lambda_\pm$ are the scattering Lagrangians,

$$\delta(\Lambda_+, \Lambda_-) = \frac{i}{2\pi} \log \operatorname{Sdet} (I - S^r_+ S^r_-). \tag{6}$$

Here Sdet is the superdeterminant of a $\mathbb{Z}_2$-graded diagonal operator,

$$\operatorname{Sdet} \begin{pmatrix} A^0 & 0 \\ 0 & A^1 \end{pmatrix} \equiv \det A^0 \cdot (\det A^1)^{-1}$$

and $S^r_\pm$ are differences of certain projection operators associated to the scattering Lagrangian subspaces $\Lambda_{\rm sc}$ in $\ker(\eth_H)$. We refer to [25] for further explication.

A similar, but less tidy, finite-dimensional linear algebraic formula for this term was discovered (earlier) by LESCH and WOJCIECHOWSKI [32] in their study of the closely related problem of the dependence of the eta invariant on cylinders with boundary conditions given by arbitrary Lagrangian subspaces. It arises, as does the expression above, from one other useful way to think of this defect, namely as the eta invariant of an associated one-dimensional problem. The Dirac-type operator $\gamma\partial_s$ acts on the space of sections of the trivial bundle $I \times \ker(\eth_H)$ over $I = [-1, 1]$, where γ denotes Clifford multiplication by the unit normal to H. Imposing the boundary conditions associated with the Lagrangian subspaces $\Lambda_\pm$ at $s = \pm 1$, we obtain a self-adjoint operator with discrete spectrum. Then it is also true that the defect is simply the eta invariant of this operator,

$$\delta(\Lambda_+, \Lambda_-) = \eta(\gamma\partial_s, \Lambda_+, \Lambda_-). \tag{7}$$

The seemingly more explicit expression (6) is deduced from this one.

2.3 – Analytic torsion

The analytic torsion was first defined in the seminal paper of Ray and Singer [56]. In order to introduce it we first need to recall the definition of zeta function $\zeta_P(s)$ associated to a (second order) self-adjoint elliptic differential operator P on a closed manifold Z [57]. This zeta function is defined in manner similar to, but simpler than, the eta function. Thus, if $\mathrm{spec}(P) = \{\lambda_j\}$, now a sequence of real numbers tending only to infinity, then

$$\zeta_P(s) \equiv \sum_{j=0}^{\infty} \lambda_j^{-s}.$$

We are assuming here that all eigenvalues are positive. If there are finitely many nonpositive ones we simply omit them from this sum. The definition that Ray and Singer gave to the determinant is

$$\det{}' P = e^{-\zeta_P'(0)}.$$

The notation $\det'$ is meant to indicate that the nonpositive eigenvalues (in particular, the zero eigenvalue) have been omitted.

This determinant has emerged as one of the central objects of study in spectral geometry (and also plays a prominent rôle in string theory). In particular, it was a crucial ingredient in the proof of compactness of isospectral sets by Osgood, Phillips and Sarnak, and many new results about it have been obtained and applied by many authors over the past decade. We mention only the work of S.-Y. Chang, Yang, Gursky, Okikiolu, in addition to that of Vishik.

Just like the eta invariant, $\det' P$ depends on the Riemannian metric g: its variation with respect to a family of metrics is actually computable as an integral of local quantities. Ray and Singer observed, though, that if one computes the determinants $\det'(\Delta_p)$ of the Hodge-Laplacian on p-forms, and takes a certain weighted sum of these expressions, then the resulting object has much more invariance. This weighted sum is the analytic torsion $T(Z, g)$ defined by

$$\log T(Z, g) = \sum_{p=0}^{n} (-1)^p p \det{}'(\Delta_p).$$

This expression may also be defined when the Hodge-Laplacian is twisted by some flat bundle E, and this yields a number $T(Z, E, g)$. Actually, this expression is independent of the metric only when the twisted de Rham complex is acyclic, i.e. has all cohomology groups vanishing. Notice that this never occurs in the untwisted case. In the general case there is an explicit factor which contains all metric dependence.

To be somewhat more precise fix a set of bases $\{\mu\} = \{\mu_j^{(i)}\}$ for the cohomology spaces $H^i(Z)$. Next, using the Hodge theorem, let $\{\omega_j^{(i)}\}$ be a basis of harmonic forms for each of these spaces which is orthonormal with respect to the L^2-inner product induced by g. Now let $\Lambda(g, \{\mu\})$ be the determinant of the change of basis matrix. Following [56] we define $T(M, \{\mu\})$ as the product $T(Z, \{\mu\}) \equiv T(Z, g) \cdot \Lambda(g, \{\mu\})$. It is this quantity which is *independent* of the metric g. Because of this somewhat surprising invariance, and for other (more compelling) reasons, Ray and Singer made their famous conjecture that the analytic torsion $T(Z, \{\mu\})$ agrees with the Reidemeister torsion $\tau(Z, \{\mu\})$, a PL invariant of the manifold Z defined by Reidemeister and Franz in the 1930's. This conjecture was proved in the late 1970's independently by Cheeger [13] and Müller [48].

By now there are numerous proofs of this Cheeger-Müller theorem. Many of these rely on ideas related to surgery degeneration, including Cheeger's original proof [13], and also the fairly recent proof by Burghelea, Friedlander and Kappeler, which is based on Witten's deformation method [9]. Müller's proof [48] was of a somewhat different nature. The proofs we highlight here are those by Hassell [23], who applied the surgery calculus of MM/HMM to obtain a gluing formula for the analytic torsion, and hence could then follow Cheeger's strategy to prove its equivalence with Reidemeister torsion, and VISHIK's gluing formulæ for determinants [61], the proofs of which employ his method of changing boundary conditions, as described below in the slightly different context of gluing formulæ for eta invariants (developed by BRÜNING and LESCH [10])).

It is obviously not surprising that the results and methods for obtaining gluing formulæ for determinants are much the same as for eta invariants. Both are obtained from integrating the appropriate heat kernel, or combinations thereof. At any rate, the studies of these two quantities are intimately interrelated. For reasons of space and time, we shall concentrate almost exclusively on results obtained for the eta invariant in the remainder of the paper.

2.4 – Determinant bundles

Let E be a finite dimensional vector space and suppose that $T : E \to E$ is linear. Then T induces in a natural way a map $\det(T) : \Lambda^{\max}(E) \to \Lambda^{\max}(E)$ and hence an element $\det(T) \in (\Lambda^{\max}(E))^* \otimes \Lambda^{\max}(E)$; the numerical determinant of T is simply obtained by fixing a basis of E, or at least a nonzero element of $\Lambda^{\max}(E)$. If $T \in \mathrm{Hom}\,(E, F)$, with $\dim E = \dim F$, then $\det(T)$ is again well defined as an element of $(\Lambda^{\max}(E))^* \otimes \Lambda^{\max}(F)$.

The natural exact sequence of vector spaces

$$0 \to \ker T \to E \to F \to \operatorname{coker} T \to 0$$

induces a natural isomorphism

$$(\Lambda^{\max}(E))^* \otimes \Lambda^{\max}(F) \cong (\Lambda^{\max} \ker T)^* \otimes (\Lambda^{\max} \operatorname{coker} T).$$

Suppose now that the vector spaces E, F and the linear map T depend smoothly on a parameter $z \in B$. In other words suppose that $T \in \mathcal{C}^\infty(B, \mathrm{Hom}\,(E, F))$. Applying the preceeding remarks we obtain a smooth section $\det T \in \mathcal{C}^\infty(B, \mathcal{L})$ of the *determinant line bundle* $\mathcal{L}$ with fiber at z equal to $\mathcal{L}_z = (\Lambda^{\max}(E_z))^* \otimes \Lambda^{\max}(F_z)$.

Notice again that for each fixed $z \in B$

$$\mathcal{L}_z \cong (\Lambda^{\max} \ker T_z)^* \otimes (\Lambda^{\max} \operatorname{coker} T_z). \tag{8}$$

These elementary remarks show that the determinant of a family of linear maps is only defined as a *section* of a complex line bundle. Of course if would be desirable to have a determinant *function* DET : $B \to \mathbb{C}$ assigning a number to each linear map T_z. If $\mathcal{L}$ is trivial this can certainly be done by fixing a trivializing section $\tau \in \mathcal{C}^\infty(B, \mathcal{L})$ and comparing $\det T$ and τ, viz.

$$(\det T)(z) = \mathrm{DET}_\tau(T_z)\tau(z).$$

The determinant function DET_τ so obtained depends of course on the trivializing section τ and it is natural to ask whether it possible to agree on a *canonical* choice. This is particularly important in physics. One way to proceed would be to assume that the determinant line bundle is equipped with a metric and compatible connection. Using a nontrivial covariant constant section $\overline{\tau}$ to trivialize this bundle would fix the determinant function up to a global phase $C \in U(1)$. The local and global obstructions to the existence of $\overline{\tau}$ are given by the curvature and holonomy of the given connection, respectively. In the usual physics parlance these are called the local and global anomalies.

Again partly motivated by physics, the problem arises as to whether these ideas can be extended to the infinite dimensional context where the operator T is replaced by a family of Dirac operators. Thus let $\eth = (\eth_z)_{z \in B}$ be a smooth family of Dirac operators, associated as in §2.1 to a smooth fibration of closed compact manifolds $\phi : M \to B$ with even dimensional fibres. Since each $\eth_z$ is Fredholm, it makes sense to define the complex line

$$\Lambda^{\max} \ker(\eth_z)^* \otimes \Lambda^{\max} \mathrm{coker}\,(\eth_z).$$

However, since the kernel and cokernel of $\eth_z$ are not constant in z, these complex lines do *not* vary smoothly with the parameter z. The first step then is to show that there exists a smooth complex line bundle $\mathcal{L}(\eth)$ over B with the property that its fiber over $z \in B$ is naturally identified with $\Lambda^{\max} \ker(\eth_z)^* \otimes \Lambda^{\max} \mathrm{coker}\,(\eth_z)$. The second step is to introduce a metric and compatible connection on $\mathcal{L}(\eth)$ in some natural way, and then to compute in geometric terms the curvature and the holonomy, i.e. the local and global anomaly.

This program was accomplished by Quillen in his seminal paper [55] in the special case of $\overline{\partial}$-operators on Riemann surfaces acting on a vector bundle E and with parameter space $\mathcal{A}$ equal to the moduli space of holomorphic structures on E. Since $\mathcal{A}$ is simply connected, only information about the curvature of the so-called Quillen metric is required to determine whether the determinant line bundle may be trivialized by parallel transport. These results of Quillen were extended to the general case by BISMUT and FREED in two papers [7] (see also [15], [65], [21], [17]). We now illustrate a few of the main ideas behind these works.

First we define the determinant line bundle $\mathcal{L}(\eth)$ associated to the family $\eth = (\eth_z)_{z\in B}$. We only treat the Dirac case here, but this construction can be applied to any family of Fredholm operators.

Since the fibers of $\phi : M \to B$ are even dimensional, each Dirac operator may be written as

$$\begin{pmatrix} 0 & \eth_z^- \\ \eth_z^+ & 0 \end{pmatrix} \qquad \eth_z^- = (\eth_z^+)^*$$

with $\eth_z^\pm : \mathcal{C}^\infty(M^z, \not{S}_z^\pm) \to \mathcal{C}^\infty(M^z, \not{S}_z^\mp)$. If $E = E^+ \oplus E^-$ is a $\mathbb{Z}_2$ graded vector space we use the notation $\det(E)$ for the complex line $\Lambda^{\max}(E^+)^* \otimes \Lambda^{\max}(E^-)$.

Clearly if $(\ker(\eth_z^\pm))_{z\in B}$ form two smooth vector bundles, $\operatorname{Ker}(\eth^+)$, $\operatorname{Ker}(\eth^-)$, then the line bundle

$$\mathcal{L}(\eth) = \Lambda^{\max}\operatorname{Ker}(\eth^+) \otimes \Lambda^{\max}\operatorname{Ker}(\eth^-) = \det(\operatorname{Ker}\eth)$$

is globally well defined. Notice that if $\Delta_z^\pm = \eth_z^\mp \eth_z^\pm$ then it is also true that $\mathcal{L}(\eth) = \det(\operatorname{Ker}\Delta)$. In general consider the set $U_\lambda = \{z \in B; \lambda \notin \operatorname{spec}(\Delta_z)\}$. Since the spectrum of each Laplacian is discrete, this is either a non-empty open set, or else the empty set. Since the latter may happen for at most a countable set of values of λ, we may cover B by a finite collection of such sets U_{λ_k}. Let $\Pi^\pm_{[0,\lambda)}(z)$ be the spectral projection associated to the interval $[0, \lambda)$ for the Laplacian $\Delta_z^\pm$. Consider

$$H^\pm_{[0,\lambda)}(z) = \operatorname{Im} P^\pm_{[0,\lambda)}(z).$$

This is simply the direct sum of the eigenspaces of $\Delta_z^\pm$ associated to the eigenvalues in $[0, \lambda)$. As z varies in U_λ these vector spaces define a $\mathbb{Z}_2$-graded smooth vector bundle $H_{[0,\lambda)} = H^+_{[0,\lambda)} \oplus H^-_{[0,\lambda)}$. We *define* $\mathcal{L}(\eth)$

restricted to U_λ as $\det(H_{[0,\lambda)})$. This is a smooth complex line bundle over U_λ and moreover for each fixed $z \in U_\lambda$ there is a natural isomorphism

$$(\mathcal{L}(\eth))_z \equiv \det(H_{[0,\infty)}(z)) \cong (\Lambda^{\max}(\ker \eth_z^+))^* \otimes \Lambda^{\max}(\ker \eth_z^-).$$

coming from the exact sequence

$$0 \to \ker \Delta_z^+ \to H^+_{[0,\lambda)}(z) \to H^-_{[0,\lambda)}(z) \to \ker \Delta_z^- \to 0.$$

Now if $\mu > \lambda$, then on $U_\lambda \cap U_\mu$ we have $H_{[0,\mu)} = H_{[0,\lambda)} \oplus H_{[\lambda,\mu)}$ and thus $\det(H_{[0,\mu)}) \cong \det(H_{[0,\lambda)}) \otimes \det(H_{[\lambda,\mu)})$. Moreover the restriction of $\eth_z^+$ to $H^+_{[\lambda,\mu)}(z)$ is an isomorphism for each $z \in U_\lambda \cap U_\mu$; this means that we can identify $\det(H_{[0,\mu)})$ and $\det(H_{[0,\lambda)})$ over $U_\lambda \cap U_\mu$ using the non-vanishing section $\det((\eth^+)_{[\lambda,\mu)})$. The resulting line bundle, which is now defined over all of B is, by definition, the determinant line bundle $\mathcal{L}(\eth)$ defined by the family $\eth$. By construction there is a natural isomorphism

$$(\mathcal{L}(\eth))_z \equiv (\Lambda^{\max} \ker(\eth_z^+))^* \otimes (\Lambda^{\max} \text{coker}\,(\eth_z^-))$$

for each fixed $z \in B$ (as expected). If the family $\eth$ has index zero we have $\dim(H^+_{[0,\lambda)}) = \dim(H^-_{[0,\lambda)})$ for each $\lambda > 0$ and it makes sense to speak about $\det((\eth^+)_{[0,\lambda)}$ as a section of $\det(H_{[0,\lambda)})$. These sections patch togeher (simply because $\det(\eth^+_{[0,\mu)}) = \det(\eth^+_{[0,\lambda)}) \otimes \det(\eth^+_{[\lambda,\mu)}))$ and we obtain a smooth section $\det(\eth^+) \in \mathcal{C}^\infty(B, \mathcal{L})$. This is the analogue of the section $\det(T)$ considered at the beginning of this section in the finite dimensional case.

The definition of the determinant bundle involves only the *small* eigenvalues of the operators $\Delta_z^\pm$. This is not the case for the natural metric and metric-compatible connection, introduced by Quillen and Bismut and Freed respectively, which involve instead the full spectrum of $\Delta_z^\pm$.

To define the Quillen metric $\|\cdot\|_Q$ first observe that each $H_{[0,\lambda)}$, and thus each $\det(H_{[0,\lambda)})$, inherits a natural metric coming from the L^2-metric of $\mathcal{C}^\infty(M^z, \not{S}_z)$. The problem with this L^2-metric, which we denote by $|\cdot|_\lambda$, is that it is *not* well defined: there is discrepancy between $|\cdot|_\lambda$ and $|\cdot|_\mu$ equal to the product of the eigenvalues of Δ_z^+ in the interval (λ, μ). Denote by $\zeta(s, \Delta_z^+, \lambda)$ the zeta function for the operator $P^+_{(\lambda,\infty)}(z)\Delta_z^+$.

This gives a $\mathcal{C}^\infty$ function $U_\lambda \ni z \to \zeta'(0, \Delta_z^+, \lambda)$ and it is possible to see that the metrics

$$\| \cdot \|_Q = e^{-\zeta'(0,\Delta^+,\lambda)/2} | \cdot |_\lambda \tag{9}$$

patch together to define a *global* metric on $\mathcal{L}(\eth)$. This is the Quillen metric; it involves the heat kernel of Δ_z^+ for all times. It gives another use for the determinant of a Laplacian as defined in §2.3; in fact for the section $\det(\eth^+) \in \mathcal{C}^\infty(B, \mathcal{L})$ (which vanishes precisely when the operator $\eth_z$ is not invertible),

$$\| \det(\eth^+) \|_Q^2 = \det(\eth^- \eth^+).$$

The Bismut-Freed connection is somewhat more complicated to describe and we shall not enter into the details here. Just like the Quillen metric, it is defined on each $\det(H_{[0,\lambda)})$ and then shown to be independent of choices. On each $U_{[0,\lambda)}$ the Bismut-Freed connection, henceforth denoted by $\nabla^{\mathcal{L}}$, is the sum of two pieces

$$\nabla^{\mathcal{L}}|_{U_\lambda} = \nabla^\lambda + \beta^+(\lambda). \tag{10}$$

The first summand ∇^λ is a connection which comes ultimately from the metric but is not globally defined; the second piece is a 1-form $\beta^+(\lambda)$ which is given by a t-integral over $\mathbb{R}^+$ involving $\eth^\pm$ and the heat-kernel $\exp(-t\Delta^\pm)$. This term should be thought of as a sort of eta invariant needed to make the various definitions ∇^λ coherent. Bismut and Freed prove that this connection is compatible with the Quillen metric. Notice that once again, even from this vague description, it is clear that we require the heat-kernel for all times.

Given a family of Dirac operators $\eth = (\eth_z)_{z\in B}$ as above we now have a determinant line bundle $\mathcal{L}(\eth)$, with a natural metric $\| \cdot \|_Q$ and metric compatible connection $\nabla^{\mathcal{L}}$. One of the main contributions of Bismut-Freed is the explicit computatation of the curvature and holonomy of $\nabla^{\mathcal{L}}$; in other words they give geometric formulae for the local and global anomaly. We refer to their papers for a statement of the precise results.

Our main concern here is of a different nature. Suppose as in §2.1 that the fibration $M \to B$ defining the Dirac family is the union along a fibering hypersurface H of two fibrations with boundary $M = M_+ \cup_H M_-$. We have now four Dirac families, $\eth_M$, $\eth_{M_\pm}$ and $\eth_H$. If we fix a spectral section P for $\eth_H$ then we obtain two families of Fredholm operators, as in

§2.1 and thus two determinant bundles $\mathcal{L}(\eth_{M_+}, P)$, $\mathcal{L}(\eth_{M_-}, \mathrm{Id} - P)$. The questions we shall address later in the paper are

- Q1. Is there a natural isomorphism $\mathcal{L}(\eth) \longrightarrow \mathcal{L}(\eth_{M_+}, P) \otimes \mathcal{L}(\eth_{M_-}, \mathrm{Id} - P)$?
- Q2. Is it possible to define Quillen metrics and Bismut-Freed connections on these two line bundles, $\mathcal{L}(\eth_{M_+}, P)$, $\mathcal{L}(\eth_{M_-}, \mathrm{Id}-P)$, and prove surgery formulæ for the corresponding curvature and holonomy?

3 – A closer look at the methods of Bunke and Vishik

Although the remainder of this paper is devoted mainly to a more detailed discussion of the surgery calculus of MM/HMM along with a few of its applications, we wish to describe the other two principal methods, those of Bunke and Vishik, in at least a bit more detail than we have up until now.

3.1 – Bunke's unitary equivalence

The method developed by Bunke [11] was directed specifically at finding a gluing formula for the eta invariant. Continuing our discussion from §1.4, Bunke considers a compact manifold X split along a hypersurface H as usual, and with a metric g containing an exactly cylindrical piece around H. On the manifolds with boundary, $X_\pm$, the Dirac operators are endowed with augmented APS boundary conditions associated to a choice of Lagrangian subspaces $\Lambda_\pm \subset \ker(\eth_H)$. The issue is to find a good expression for the defect

$$\delta = \delta(\Lambda_+, \Lambda_-) \equiv \eta(\eth_X) - \eta(\eth_{X_+}, \Lambda_+) - \eta(\eth_{X_-}, \Lambda_-).$$

In the easier case, when $\eth_H$ is invertible, Bunke shows that the reduction mod $\mathbb{Z}$ of the defect vanishes. He goes on to obtain a formula for the (no longer reduced) defect in the general case:

$$(11)\quad \delta(\Lambda_+, \Lambda_-) = m(\Lambda_+, \Lambda_-) - 2I(P_+, P_-) + \dim\ker(D_+) - \dim\ker(D_-).$$

Here $m(\Lambda_+, \Lambda_-)$ is an invariant of the pair of Lagrangian subspaces which is given in a few different ways. The first is in terms of the averaged Maslov class, as explained in §2.2, while the second, the form in which it was originally found by Lesch and Wojciechowski [32], is as a sum of eigenvalues of sum matrix. This expression may be written much more simply and neatly as in (6), as discovered in [25]. To explain the other terms on the right, we must first explain his proof a bit more.

The core of the proof involves comparing two different Dirac operators. The first is the sum of Dirac operators $\eth_{X_\pm}$ on the disjoint sum of $X_\pm$, where these components are now assumed to have finite product cylindrical ends. The other is for the sum of Dirac operators on the disjoint union of the manifold X, assumed to contain a long cylindrical piece around H, and the cylinder $C_R \equiv [-R, R] \times H$. On each of these pieces, the Lagrangians $\Lambda_\pm$ are used to augment the APS conditions at the appropriate boundary components. These sums of operators are called D_0 and D_+, respectively. The operator D_- is obtained from D_0 by applying a unitary map, defined using a partition of unity, which identifies the pieces of $X_+ \sqcup X_-$ with the equivalent pieces of $X \sqcup C_R$. This is done only so that the operators $D_\pm$ live on the same manifold.

The other terms on the right in (11) may now be explained. The dimensions of the kernels of $D_\pm$ are the obvious numbers. $P_\pm$ are the positive spectral projections for the operators $D_\pm$, and after Bunke shows that their difference $P_+ - P_-$ is compact, the relative index between them, $I(P_+, P_-)$, is well-defined.

In the nondegenerate case, only the first term on the right in (11) is necessarily trivial. However, if $\eth_X$ itself has only trivial nullspace (at least when the cylindrical piece is sufficiently long), then the other terms on the right in (11) also vanish.

This formula is obtained by comparing the heat kernels of the operators $D_\pm$. This is accomplished by comparing a particular regularization of the integrals required to define the eta functions. These regularizations are

$$R_\pm(s,t) \equiv \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty r^{\frac{s-1}{2}} D_\pm e^{-(t+r)D_\pm^2}\, dr.$$

The difference of eta invariants should arise as the limit as $t \to 0$ of the difference of traces of the $R_\pm$ at $s = 0$. Unfortunately, these operators are not continuous in the trace norm down to $t = 0$, which makes this procedure not entirely straightforward. Additional terms are added on to ensure that the limit exists, and these ultimately account for the various terms in the expression (11) for the defect.

We note again that the dependence of the invariant $m(\Lambda_+, \Lambda_-)$ on the Lagrangians $\Lambda_\pm$ was first considered by LESCH and WOJCIECHOWSKI [32], and they obtained one of the linear algebraic expressions for this number, but not its identification with the averaged Maslov class.

Amongst the advantages of this procedure are its relatively elementary nature, and the reasonably explicit identification of the integer part of the defect.

3.2 – Vishik's variation of boundary conditions

The second approach, by Vishik, was developed for the study of determinants and analytic torsion. In [61] VISHIK studies determinants of elliptic pseudodifferential operators in great generality and detail and gives, amongst other things, a new proof of the Cheeger-Müller theorem. Vishik's approach was recently adapted by BRÜNING and LESCH [10] to give another proof of the gluing formula for the eta invariant. Since this paper is somewhat more accessible than those of Vishik, and because we have chosen to concentrate on the surgery formula for the eta invariant specifically, we follow the discussion from this latter paper instead.

Instead of considering a family of metrics degenerating (or lengthening) transversally to H, the perspective is now the 'more classical' one, involving boundary conditions. The goal is to define a family of elliptic boundary problems $(\eth_X, \Pi_\theta)$ on the manifold $X_+ \sqcup X_-$. The boundary conditions are given by a family of orthogonal projections Π_θ, $|\theta| < \pi/2$, acting on the direct sum $L^2(H;E) \oplus L^2(H;E)$. The two copies of this Hilbert space arise because H needs to be thought of as the boundary of X_+ and of X_- separately. To define Π_θ we use the following notation. For a (sufficiently smooth) section u on $X_+ \sqcup X_-$, denote its restriction to $\partial X_\pm$ by $u(\pm 0)$. Also, let $\Pi^\pm_{X_\pm}$ and $\Pi^0_{X_\pm}$ denote the spectral projections onto the positive, negative and zero spectral subspaces of $\eth_H$, where H is considered alternately as the boundary of X_+ and X_-, respectively. Then

$$\begin{aligned}
\cos\theta\, \Pi^+_{X_+} u(+0) &= \sin\theta\, \Pi^+_{X_-} u(-0) \\
\sin\theta\, \Pi^-_{X_+} u(+0) &= \cos\theta\, \Pi^-_{X_-} u(-0) \\
\Pi^0_{X_+} u(+0) &= \Pi^0_{X_-}
\end{aligned}$$

defines the nullspace of the orthogonal projector on $L^2(H;E)^2$ which we term Π_θ.

This family of boundary conditions interpolates between two extremes. One extreme, at $\theta = \pi/4$, is the 'transmission condition': any section u which solves $\eth u = 0$ on $X_+ \sqcup X_-$ and also $\Pi_0 u = 0$ must extend smoothly across H, and thus corresponds to an element of $\ker(\eth_X)$; the other, when $\theta = 0$, corresponds to APS conditions independently on the two pieces $X_\pm$. As the parameter θ changes, the first projector 'rotates' to the other.

Having defined these boundary conditions, one obtains a family of self-adjoint problems, and for each operator in this family one considers the eta invariant, which we denote by $\eta(\eth_\theta)$. The main work in this proof is showing first that the eta invariant for this family of boundary problems is well-defined, i.e. that the eta function is regular at zero, and then computing the derivative of the eta invariant with respect to θ. After suitable normalizations, conjugating by unitary transformations, Brüning and Lesch show that this derivative vanishes. Unwinding the various normalizations leads to the same gluing formula.

While perhaps not quite as simple as Bunke's method, this method seems perhaps the simplest to adapt for the eventual study of gluing problems in more complicated situations. One such situation arises when the hypersurface H is the union of hypersurfaces with boundary H_j intersecting at a common codimension two submanifold, $Y = \partial H_j$ for all j. This generalization would be of particular importance if the signature formula of [26] is to be extended to manifolds with corners of arbitrary codimension, cf. §5.3 below.

4 – Pseudodifferential operators and the surgery problem

After the cursory treatments of the other methods in the previous section, we now turn to a description of some of the details of the surgery calculus of MM/HMM from [34], [25]. We follow these papers, as well as [36] closely. In in the first two subsections below we continue and extend the discussion of §1.3 on the b-calculus, and shall refer to the notation there without further comment.

4.1 – Degenerating metrics

Let $X = X_+ \cup_H X_-$ as in the previous sections. Assume, just for the time being, that the Riemannian metric on X is of product-type near H. Let $\mathcal{U}_H = (-T,T) \times H$ be a collar neighbourhood of H. Letting $T \to +\infty$ we obtain two manifolds, $\widehat{X}_+, \widehat{X}_-$, with infinite cylindrical ends. In general a manifold

$$\widehat{Z} = Z \cup_{\partial Z} (\partial Z \times (-\infty, 0])$$

with a cylindrical end and metric $dt^2 + h_{\partial Z}$ along the cylinder $((-\infty,0])_t \times \partial Z)$ can be compactified as a manifold with boundary $\overline{Z}$ with an exact b-metric, simply by making the change of variable $\log x = t$. In a neighbourhood of the boundary $\{x=0\}$, corresponding to $t=-\infty$, the metric takes the form

$$\frac{dx^2}{x^2} + h_{\partial Z}.$$

We can now give a natural analytic realization of the stretching procedure in this approach to the surgery problem. Let h be an arbitrary Riemannian metric on X and let $x \in \mathcal{C}^\infty(M)$ be a signed defining function for H; thus $H = \{x = 0\}$, $dx \neq 0$ on H, but $x < 0$ on X_- and $x > 0$ on X_+. Consider the one-parameter family of metrics

$$g_\varepsilon = \frac{dx^2}{x^2 + \varepsilon^2} + h. \tag{12}$$

For each $\varepsilon > 0$ the metric g_ε is non-degenerate on X. As $\varepsilon \to 0$ the metric g_ε develops a neck across H, the length of which is $L(\varepsilon) = (2\sinh^{-1}(\frac{1}{\varepsilon}) + O(1))$. The limiting metric, at $\varepsilon = 0$, is

$$g_0 = \frac{dx^2}{x^2} + h; \tag{13}$$

this is an *exact* b-metric on $X_+ \sqcup X_-$. We shall denote these two exact b-manifolds by $\overline{X}_+, \overline{X}_-$ and their disjoint union by $\overline{X} = \overline{X}_+ \sqcup \overline{X}_-$. In other words, g_0 endows the interior of $X_\pm$ with the structure of a Riemannian manifold with asymptotically (no longer necessarily product) cylindrical ends.

In summary, the family of metrics (12) models the degeneration of X, through the stretching of a tubular neighbourhhod of H, from a closed compact manifold to a manifold $\overline{X}$ which is the disjoint union of two manifolds with asymptotically cylindrical ends.

As already explained the invariants we are interested in are each associated (one way or another) to the heat kernel of the Dirac Laplacian $\eth^2$. Since the heat operator is defined through the resolvent $(\eth^2 - \lambda)^{-1}$ we conclude that the solution of the surgery problem for these invariants rests ultimately on a deep understanding of the uniform behaviour, down to $\varepsilon = 0$, of the resolvent associated to $\eth^2_{X,g_\varepsilon}$, the Dirac Laplacian defined by metric (12). Amongst the other consequences of this analysis will be a complete picture of the degeneration of the spectrum.

The problem breaks into two intimately related problems. The first is to describe the *limit picture*, i.e. the geometry and analysis corresponding to $\varepsilon = 0$. This is nothing more than the setting of the b-calculus which we have already introduced. The second is to find a geometric setting (i.e. an appropriately blown-up space) on which the relevant Schwartz kernels behave uniformly with respect to ε.

4.2 – The limit picture: more on the b-calculus

We have already discussed the b-calculus of pseudodifferential operators on manifolds with boundary in §1.3. We shall now continue this discussion with two goals. On the one hand we shall describe more precisely the analytic properties of the operators appearing in the limit of the surgery problem, while on the other hand parametrix construction for inverses of b-elliptic operators illustrates most of the main ideas in the more involved parametrix construction in the surgery calculus.

Getting down to business at last, consider the definition of the pseudodifferential calculus as given in §1.3. To motivate the introduction of the blown-up space there, we consider a simple example drawn from [36] This is the b-differential operator

$$(1-x)x\frac{d}{dx} + c \quad c \in \mathbb{R}$$

on the manifold with boundary $Z = [0,1]$. This operator is invertible acting on $x^\alpha(1-x)^\beta H^1_b([0,1])$ if $\alpha < -c, \beta > c$; the Schwartz kernel of its inverse is given quite explicitly by the distribution

$$K_c(x,x') = -\frac{x^c}{(1-x)^c}\frac{(1-x')^c}{(x')^c}H(x'-x), \tag{14}$$

where $H(\cdot)$ is the Heaviside function. This Schwartz kernel exhibits the usual diagonal singularities in the interior of $Z^2 = Z \times Z$, and has additional singularities on the boundary hypersurfaces $\mathrm{lb} \equiv (\partial Z \times Z) = \{x = 0\}$, $\mathrm{rb} \equiv (Z \times \partial Z) = \{x' = 0\}$, and on the part of the corner which intersects the diagonal (viz. $(\partial Z \times \partial Z) \cap \Delta = \{(0,0) \cup (1,1)\}$). The corner carries the most complicated singularities, resulting from the interaction of those coming from the diagonal Δ and those coming from the two boundary hypersurfaces. The key observation is that the corner singularities can be simplified, or *resolved*, by introducing (generalized) polar coordinates. For example near $(0,0)$ consider the singular change of coordinates

$$r = x + x', \quad \tau = \frac{x - x'}{x + x'}.$$

Then $2x = r(1+\tau), 2x' = r(1-\tau)$, and K_c may be written as the product of a $\mathcal{C}^\infty$ function and the distribution

$$\frac{(1+\tau)^c}{(1-\tau)^c} \times H(-\tau) \tag{15}$$

which is singular on three *non-intersecting* hypersurfaces, $\tau = 0$, $\tau = 1$ and $\tau = -1$. These new variables are not smooth at the corner relative to the original ones, but on the b-stretched product Z_b^2 of §1.3 they are both smooth and independent. Recall again that (assuming ∂Z is connected) Z_b^2 is the *blow-up* of Z^2 along $\partial Z \times \partial Z$, denoted $[Z^2; \partial Z \times \partial Z]$. If, as in the example above, ∂Z is not connected, then $Z_b^2 = [Z^2; (\partial Z \times \partial Z) \cap \Delta]$. As a set $[Z^2; \partial Z \times \partial Z]$ is obtained by replacing the corner $\partial Z \times \partial Z$ with its (inward pointing) spherical normal bundle $S_+N(\partial Z \times \partial Z)$:

$$[Z^2; \partial Z \times \partial Z] = Z^2 \backslash (\partial Z \times \partial Z) \sqcup S_+N(\partial Z \times \partial Z).$$

The b-stretched product comes with a natural surjective blow-down map

$$\beta_b^2 : [Z^2; \partial Z \times \partial Z] \longrightarrow Z^2$$

and is given the minimal $\mathcal{C}^\infty$ structure for which the lift of $\mathcal{C}^\infty(Z^2)$ and of the polar coordinates around the corner are smooth. There are four important submanifolds in Z_b^2; the lifted diagonal Δ_b, the left and right boundary faces, lb, rb, obtained by lifting the corresponding boundary hypersurfaces in Z^2 and finally the new boundary hypersurface created by the blow-up: $\mathrm{ff}(Z_b^2) = S_+N(\partial Z \times \partial Z)$. This is called the front face and by its very definition has the structure of a fibre bundle, with fibres diffeomorphic to the interval $[-1, 1]$.

The blow-up $[M; N]$ of a manifold with corners M along the the submanifold N may be defined in substantial generality, assuming only that N satisfies certain local triviality conditions. It formalizes the introduction of polar coordinates around N. We shall encounter other, more complicated, examples below.

The b-calculus is defined by specifying the singularities allowed in the Schwartz kernels of its elements. As the example illustrates, and this is really the main point, these singularities are best understood when they are resolved, i.e. lifted to Z_b^2.

In order to make the definition of §1.3 more precise recall first that if M is a manifold with boundary $\{x = 0\}$ and if $E \subset \mathbb{C} \times \mathbb{N}^+$ is a set of indices, then the space $\mathcal{A}^E_{\mathrm{phg}}(M)$ of polyhomogeneous conormal functions can be defined. It consists of functions which are smooth in the interior and have an asymptotic expansion of the type

$$\sum_{(z,k)\in E} a_{z,k} x^z (\log x)^k, \quad a_{z,k} \in \mathcal{C}^\infty(\partial M) \tag{16}$$

near the boundary. The index set E specifies which exponents are allowed in (16); the set $\mathbb{N} \times \{0\}$, which we denote by 0 for brevity when the context is clear, corresponds to functions smooth up to the boundary. A similar definition may be given when M is a manifold with corners; one simply requires expansions of this type at all boundary hypersurfaces and product-type expansions at the corners. Here it is necessary to specify an index family $\mathcal{E} = (E_1, \ldots, E_n)$, where E_j is an index set for the boundary hypersurface H_j, $j = 1, \ldots, n$.

Now consider the case $M = Z_b^2$ and fix an index family $\mathcal{E} = (E_{\mathrm{rb}}, 0, E_{\mathrm{lb}})$, where the boundary hypersurfaces are listed in the order left boundary, front face and right boundary. In particular, the index set associated to the front face $\mathrm{ff}(\mathrm{Z}_\mathrm{b}^2)$ is the one associated with smooth functions.

The b-calculus $\Psi_b^{*,\mathcal{E}}(Z)$ is the sum of two pieces:

$$\Psi_b^{*,\mathcal{E}}(Z) = \Psi_b^*(Z) + \widetilde{\Psi}_b^{-\infty,\mathcal{E}}(Z).$$

The first one, $\Psi_b^*(Z)$, is the *small* calculus: elements in $\Psi_b^*(Z)$ have Schwartz kernels on Z^2 lifting to Z_b^2 so as to have the usual interior singularities along Δ_b, vanish to infinite order at lb, rb and to be $\mathcal{C}^\infty$ up to the front face $\mathrm{ff}(\mathrm{Z}_\mathrm{b}^2)$. To say that a conormal singularity along Δ_b is smooth up to $\mathrm{ff}(\mathrm{Z}_\mathrm{b}^2)$ means that it extends smoothly across this face as a distribution conormal to the extended diagonal. For example, a b-differential operator has Schwartz kernel which is a smooth delta section along Δ_b, and hence $\mathrm{Diff}_b^*(Z) \subset \Psi_b^*(Z)$ as expected. The second summand contains the boundary terms, which are smooth in the interior and polyhomogeneous, with index family $\mathcal{E}$ at the boundary of Z_b^2:

$$\widetilde{\Psi}_b^{-\infty,\mathcal{E}}(Z) = \mathcal{A}_{\mathrm{phg}}^{\mathcal{E}}(Z_b^2).$$

Strictly speaking, it is also necessary to add to these two summands a third, containing 'very residual' parts of the calculus. Since these are not important for the present discussion, we shall not discuss them further.

The small b-calculus is an algebra. The full b-calculus itself is not, but only for the trivial reason that sometimes the boundary terms are not integrable. When they are, it is possible to give precise composition formulæ: when $A \in \Psi^{*,\mathcal{E}}(Z)$ and $B \in \Psi^{*,\mathcal{F}}(Z)$, the resulting operator $A \circ B$ will be an element of $\Psi^{*,\mathcal{G}}(Z)$, where the new index family $\mathcal{G}$ may be determined explicitly from $\mathcal{E}$ and $\mathcal{F}$. These composition formulæ are one cornerstone of the whole theory; in some sense, the main work in setting up one of these degenerate calculi is in proving such formulæ. They may either be proved directly, as in [36], which becomes less feasible in more complicated geometric situations, or else using general facts about pushforwards of polyhomogeneous conormal distributions with respect to so-called *b-fibrations* on manifolds with corners, cf. [38]. Closely related to these arguments are those used to establish the precise mapping properties for these operators, in particular their boundedness on weighted Sobolev spaces.

For an invertible elliptic b-pseudodifferential operator A, the inverse is an element of $\Psi_b^{*,\mathcal{E}}(Z)$ for some particular choice of the index set $\mathcal{E}$. This important result is, of course, the *raison d'être* for establishing the calculus. It is not immediately apparent why it should be necessary to enlarge the small calculus to include the polyhomogeneous boundary terms. We explain this issue now. Let $P \in \mathrm{Diff}_b^m(Z)$ be a b-elliptic operator. Using a suitably adapted symbol calculus, we may construct a parametrix $Q_\sigma \in \Psi_b^{-m}(Z)$ for P. This has the property that $P \circ Q_\sigma = \mathrm{Id} - R_\sigma$ with $R_\sigma \in \Psi_b^{-\infty}(Z)$. At this point it might seem that we are essentially done, but this is not the case because elements of $\Psi_b^{-\infty}(Z)$ are *not* compact on L^2. In fact the element $R_\sigma \in \Psi_b^{-\infty}(Z)$ is compact on L^2 if and only if the Schwartz kernel of R_σ restricted to the front face is equal to zero. In one direction this property should be clear. In fact, if $(K_{R_\sigma})|_{\mathrm{ff}} = 0$, then K_{R_σ} vanishes when restricted to any boundary face of ∂Z_b^2, and so its pushforward to Z^2 also vanishes on the entire boundary of Z^2. Compactness of operators with Schwartz kernels of this form, which are also smooth in the interior, follows from the Arzelà-Ascoli theorem.

To remedy this situation, we look for a correction term Q' with the property that

$$P \circ (Q_\sigma - Q') = \mathrm{Id} - (R_\sigma - R') \quad \text{with} \quad (K_{R_\sigma})|_{\mathrm{ff}} = (K_{R'})|_{\mathrm{ff}}. \tag{17}$$

Thus the operator Q' is intended to cancel the restriction to the front face of R_σ.

The restriction of the Schwartz kernel to the front frace is defined for any element in the small calculus; it defines a natural homomorphism

$$I : \Psi_b^*(Z) \longrightarrow \Psi_b^*(\overline{N^+(\partial Z)}).$$

with $\overline{N^+(\partial Z)} \cong [-1,1] \times \partial Z$ the compactified inward pointing normal bundle. This map is called either the normal or indicial homomorphism, and we use these two names interchangeably. (These two model operators exist for any of the degenerate calculi, but coincide only in the special case of the b-calculus.) It is defined by observing that the front face in Z_b^2 is canonically identified with the front face in the stretched product of $\overline{N^+(\partial Z)}$. Thus the restriction of the kernel to $\mathrm{ff}(\mathrm{Z_b^2})$ may be transferred to the other stretched product and using the dilation structure of $N^+(\partial Z)$ may be extended further to be homogeneous in the interior. This gives a kernel on $(\overline{N^+(\partial Z)})_b^2$, i.e. a b-pseudodifferential operator on $\overline{N^+(\partial Z)}$. In the special case of a b-differential operator P, locally given by

$$P = \sum_{j+|\alpha|\leq m} a_{j,\alpha}(x,y)(x\partial_x)^j \partial_y^\alpha$$

then it is not hard to check that this procedure leads to the indicial operator for P,

$$I(P) = \sum_{j+|\alpha|\leq m} a_{j,\alpha}(0,y)(x\partial_x)^j \partial_y^\alpha.$$

The normal homomorphism may be thought of as a noncommutative secondary boundary symbol.

To solve (17) then we need to find an operator Q', defined by a Schwartz kernel in Z_b^2, such that $I(P) \circ I(Q') = I(R_\sigma)$. Formally

$$I(Q') = I(P)^{-1} \circ I(R_\sigma) \tag{18}$$

fixes the Schwartz kernel of Q' near the front face, and this may then be extended to all of Z_b^2. Formula (18) shows that in order to construct a parametrix we need to *invert* the normal operator of $P \in \mathrm{Diff}_b^m(Z)$. It is because the inverse $I(P)^{-1}$ always involves polyhomogeneous boundary terms that we must always include the polyhomogeneous part of the general b-calculus. This may be seen already in the one-dimensional example above.

The invertibility of $I(P)$ is considered relative to weighted Sobolev spaces, and as in §1.3, the basic result is that except for a discrete set of values of the weight parameter $I(P)$ can be inverted; the inverse has polyhomogeneous expansions at lb and rb. Different weights give rise to different index sets in the expansion. In the special case $\eth_X \in \mathrm{Diff}_b^1$, the omitted set of weights coincides exactly to the spectrum of the boundary operator $\eth_{\partial X}$; the elements in the various index families, i.e. the exponents allowed in the polyhomogeneous expansions, are given explicitly in terms of $\mathrm{spec}_{L^2}(\eth_{\partial X})$. The same sort of result also holds for $\eth^2$.

In summary, we have indicated how, for each 'admissible' weight δ, i.e. one for which $I(P)^{-1}$ exists, this construction gives the Schwartz kernel of a right parametrix G_δ for $P \in \mathrm{Diff}_s^m(Z)$ acting on $x^\delta H_b^m$; G_δ itself is an element of $\Psi_b^{-m,\mathcal{E}(\delta)}$, where the index family $\mathcal{E}(\delta)$ can be explicitly described. The remainder term $R_\delta = G_\delta P - \mathrm{Id}$ is compact on $x^\delta L^2$. A left parametrix with similar properties is constructed similarly.

This parametrix construction may be applied to show that the actual resolvent $(\eth^2 - \lambda)^{-1} \in \Psi_b^{-m,\mathcal{E}_\lambda}$, for some explicitly given index family $\mathcal{E}_\lambda$.

4.3 – The surgery calculus

Having described the limit picture for the surgery problem at $\varepsilon = 0$ in the family of metrics (12), we now turn to a description of the uniform behaviour of the family of resolvents $(\eth^2_{X,g_\varepsilon} - \lambda)^{-1}$.

The basic idea is to incorporate the parameter ε into the geometric description of the Schwartz kernels. Thus consider the space $M = X \times [0, \varepsilon_0]$, with projection $\pi_\varepsilon : M \to [0, \varepsilon_0]$. The metric g_ε lifts to this space, and is nondegenerate along the fibres of π_ε. The vector field $\sqrt{x^2 + \varepsilon^2}\partial_x$ is of (essentially) unit length with respect to this metric, and thus appears in the definition of the Dirac operator $\eth_{X,g_\varepsilon}$ (henceforth denoted simply by $\eth_\varepsilon$). This vector field is not smooth on M – it has singularities along the submanifold $H \times \{0\} = \{x = 0, \varepsilon = 0\}$. As usual, we resolve these singularities by blowing up this submanifold.

We thus define the *single surgery space* M_s as the blow-up of M along $H \times \{0\}$:

$$M_s = [M; H \times \{0\}] = (M \backslash (H \times \{0\}) \sqcup S_+N(H \times \{0\}).$$

The single surgery space is equipped with a blow-down map $\beta_s : M_s \to M$ and thus with a projection $\pi_{s,\varepsilon} : M_s \to [0, \varepsilon_0]$. The set on the right hand side of the formula above is given the minimal $\mathcal{C}^\infty$ structure containing both the lift of $\mathcal{C}^\infty(M)$ and also the polar coordinate functions (r, θ) (with $x = r\cos\theta, \varepsilon = r\sin\theta$).

Besides the uninteresting boundary at $\varepsilon = \varepsilon_0$ the single surgery space has two boundary hypersurfaces: the *b-boundary* $\overline{X}$, corresponding to the original boundary at $\varepsilon = 0$, $\overline{X}$ =(closure of $\beta^{-1}(\{\varepsilon = 0\}\backslash H)$)= $\overline{X_-} \sqcup \overline{X_+}$, and the new boundary hypersurface created by the blow-up, the *surgery boundary* $\overline{H} = S_+N(H \times \{0\}) \cong [-1, 1] \times H$. By construction, the singular vector field $\sqrt{(x^2 + \varepsilon^2)}\partial_x$ lifts to be smooth on M_s; in fact the lift belongs to $\mathcal{V}_b(M_s)$. The latter space is the span, over $\mathcal{C}^\infty(M_s)$, of lifts of the vector fields on M that are tangent to $H \times \{0\}$. Clearly the Dirac operator $\eth_\varepsilon$ is in the algebra of operators generated by vector fields in $\mathcal{V}_b(M_s)$. However, $\eth_\varepsilon$ does not differentiate in the direction of the (lift of the) vector field $\varepsilon\partial_\varepsilon$, and so we restrict our attention to the somewhat smaller class of *surgery vector fields* on M_s,

$$\mathcal{V}_s(M_s) = \{V \in \mathcal{V}_b(M_s) : (\pi_{s,\varepsilon})_* V = 0\}.$$

The lift of the family of metrics g_ε to M_s is smooth and non-degenerate on $\mathcal{V}_s(M_s)$; moreover its restriction to $\overline{X}$ is precisely the exact b-metric g_0. The lifted metric may also be restricted to $\overline{H}$, and gives another exact b-metric there.

The surgery differential operators on X, $\mathrm{Diff}_s^*(X)$, are now defined as the differential operators generated over $\mathcal{C}^\infty(M_s)$ by the vector fields $\mathcal{V}_s(M_s)$. The notation $\mathrm{Diff}_s^*(X)$, referring to X instead of M_s, is meant to indicate that these operators should be regarded as acting on X (or rather, the fibres of π_ε) and depending parametrically in a precise manner on ε. The Dirac operator $\eth_\varepsilon$ is a surgery differential operators of order one. In fact it is *surgery-elliptic*, in the sense that may be locally expressed by an elliptic combination of basis of sections of $\mathcal{V}_s(M_s)$. Similarly $\eth_\varepsilon^2 \in \mathrm{Diff}_s^2$ and it is elliptic as well. (Henceforth we shall merely write elliptic rather than surgery-elliptic).

The main goal now is to define the surgery calculus, a pseudodifferential calculus naturally containing the inverses (when they exist) of the elliptic surgery differential operators. Surgery pseudodifferential operators are defined in terms of their Schwartz kernel on $X^2 \times [0, \varepsilon_0]$. These are distributions on $X^2 \times [0, \varepsilon_0]$ with specific singularities along the submanifolds $\Delta \times [0, \varepsilon_0]$, $H \times H \times \{0\}$, $H \times X \times \{0\}$, and $X \times H \times \{0\}$. It is convenient to introduce the notation

$$H_R = X \times H \qquad H_L = H \times X.$$

The Schwartz kernels of surgery operators are best described as being pushed forward from the *surgery double space* M_s^2, which is obtained from $X \times X \times [0, \varepsilon_0]$ by blowing up these various submanifolds. The order in which we perform these blow-ups is important. First we blow up $H \times H \times \{0\}$, obtaining the space $[X^2 \times [0, \varepsilon_0]; H^2 \times \{0\}]$ with its blow-down map $\hat{\beta}^2$. Then we blow up in $[X^2 \times [0, \varepsilon_0]; H^2 \times \{0\}]$ the lifts by $\hat{\beta}^2$ of $H_R \times \{0\}$ and $H_L \times \{0\}$. This defines M_s^2, and we denote this two-step blow-up process more succinctly by

$$M_s^2 = [X^2 \times [0, \varepsilon_0]; H^2 \times \{0\}; H_R \times \{0\} \sqcup H_L \times \{0\}].$$

The total blow-down map is $\beta_s^2 : M_s^2 \to X^2 \times [0, \varepsilon_0]$.

As a general note about iterated blow-ups, the order in which various submanifolds of a manifold with corners are blown up is important and will in general affect the final space. There are various conditions on the submanifolds, however, which ensure that the iterated blow-up may be performed in any order.

The blow-ups in the definition of M_s^2 define three new boundary hypersurfaces. The blow-up of $H^2 \times \{0\}$ produces the face B_{ds}, and the blow-up of $H_R \times \{0\}$ and $H_L \times \{0\}$ produces the hypersurfaces B_{rs} and B_{ls}. We also have the boundary hypersurface coming from the original boundary at $\varepsilon = 0$ which is denoted by B_{db}. Finally the diagonal $\Delta \times [0, \varepsilon_0]$ lifts through β_s^2 to a submanifold $\Delta_s \subset M_s^2$. Notice that both B_{ds} and B_{db} have non-empty intersection with the lifted diagonal (the "d" in the subscript is meant to suggest this).

The calculus of surgery pseudodifferential is the sum of two pieces:

$$\Psi_s^{*,\mathcal{E}}(X) = \Psi_s^*(Z) + \Psi_s^{-\infty,\mathcal{E}}(Z).$$

The *small surgery calculus* $\Psi_s^*(X)$ consists of operators with Schwartz kernels on $X^2 \times [0, \varepsilon_0]$ which are pushforwards from M_s^2 of distributions which exhibit the usual conormal singularities along the lifted diagonal Δ_s and which vanish to infinite order at the boundary hypersurfaces B_{rs} and B_{ls} (the ones not intersecting the lifted diagonal). By construction, $\mathrm{Diff}_s^*(X) \subset \Psi_s^*(Z)$.

The second piece of the calculus contains operators with nontrivial boundary terms; their Schwartz kernels are smooth in the interior of M_s^2 but have polyhomogeneous conormal expansions of the type (16) at the various boundary faces. As in the discussion of the b-calculus, the exponents allowed in these expansions are given by an index family

$$\mathcal{E} = \{E_{\mathrm{ds}}, E_{\mathrm{ls}}, E_{\mathrm{rs}}, E_{\mathrm{db}}\}$$

The boundary faces B_{ls} and B_{rs} are given index sets with strictly positive real part, which ensures that the corresponding kernels vanish at these faces; the boundary faces B_{ds} and B_{db}, the ones meeting Δ_s, are given index sets with non-negative real part and with the first term in the expansion equal to $(0, 0)$, which ensures that the kernels can be restricted to these faces. It is possible to consider index sets depending on a complex parameter, which will be the case for the resolvent $(\eth_\varepsilon^2 - \lambda)^{-1}$, and it also possible to discuss holomorphy in this context.

Again as with the b-calculus, it is most important to establish how surgery pseudodifferential operators behave under composition. There are composition formulæ of the type

$$\Psi_s^{m,\mathcal{E}}(X) \circ \Psi_s^{m',\mathcal{E}'}(X) \subset \Psi_s^{m+m',\mathcal{E}''}(X),$$

with $\mathcal{E}''$ given explicitly by $\mathcal{E}, \mathcal{E}'$. These are proved using the general results on pushforwards of polyhomogeneous distributions in [38].

4.4 – The surgery resolvent

Having now defined the surgery calculus, one would like to show that the resolvent $(\eth_\varepsilon^2 - \lambda)^{-1}$ lies in it for a suitable choice of the index family $\mathcal{E} = \mathcal{E}(\lambda)$. This is proved by constructing a good parametrix for the resolvent in this calculus, which we now sketch.

First consider the case where $\lambda \in \Omega$, $\Omega \cap [0, +\infty) = \emptyset$. We wish to construct an element $E(\lambda)$ in the surgery calculus which is an inverse of $(\eth_\varepsilon^2 - \lambda)$ modulo a "small" remainder:

$$(\eth_\varepsilon^2 - \lambda) \circ E(\lambda) = \mathrm{Id} - R(\lambda). \tag{19}$$

Provided that the remainder term is sufficiently residual, the right hand side of (19) can be inverted using Neumann series, and after some work we can conclude that the resolvent itself is a surgery pseudodifferential operator.

Using the symbol calculus, a version of which exists for the small surgery calculus, we obtain an initial parametrix $E_\sigma(\lambda) \in \Psi_s^{-2}(X)$, with

$$(\eth_\varepsilon^2 - \lambda) \circ E_\sigma(\lambda) = \mathrm{Id} - R_\sigma(\lambda) \quad R_\sigma(\lambda) \in \Psi_s^{-\infty}(X).$$

The remainder term $R_\sigma(\lambda)$ is compact when $\varepsilon > 0$, but not when $\varepsilon = 0$. The problem comes from its nonvanishing restriction to the two boundary hypersurfaces meeting the lifted diagonal, B_{db} and B_{ds}.

Exactly as we did in the b-calculus, we must then find a correction term which cancels the first term in the Taylor series of $R(\lambda)$ at B_{db} and B_{ds}. In order to implement this argument, we use two *normal homomorphisms*, given by restrictions of Schwartz kernels to these two boundary faces. To be more specific, there are two natural identifications

$$B_{\mathrm{ds}} = [\overline{H}^2; \partial \overline{H}^2] \quad B_{\mathrm{db}} = [\overline{X}^2; \partial \overline{X}^2]. \tag{20}$$

Restriction to each of these hypersurfaces defines, in turn, two surjective homomorphisms

$$N_s : \Psi_s^*(X) \to \Psi_b^*(\overline{H}) \qquad N_b : \Psi_s^*(X) \to \Psi_b^*(\overline{M}), \tag{21}$$

the surgery normal and b-normal homomorphism, respectively.

Notice that in (20) we are blowing up the entire corner, not just that component of it which intersects the lifted diagonal. The resulting b-calculi in (21) are therefore slightly larger than the ones considered in §4.2; the differences in their properties, however, are negligible.

These normal homomorphisms are also natural with respect to the geometry. For example,

$$N_s(\eth_\varepsilon) = \eth_{\overline{H}} \qquad N_b(\eth_\varepsilon) = \eth_{\overline{X}} \equiv \eth_0,$$

where $\eth_{\overline{H}}$ is defined in terms of the restriction of the lift of g_ε to $\overline{H}$.

Returning to the construction of a good parametrix, we must modify $E_\sigma(\lambda)$ by an operator $E(\lambda)' \in \Psi_s^{-\infty,\mathcal{E}}$ for some index family $\mathcal{E}$ and such that

$$(\eth_\varepsilon^2 - \lambda) \circ (E_\sigma(\lambda) - E(\lambda)') = \mathrm{Id} - (R_\sigma(\lambda) - R'(\lambda)) \quad \text{with}$$

$$K_{R_\sigma(\lambda)}|_{\mathrm{ds}} = K_{R'(\lambda)}|_{\mathrm{ds}} \qquad K_{R_\sigma(\lambda)}|_{\mathrm{db}} = K_{R'(\lambda)}|_{\mathrm{db}}.$$

This is equivalent to solving

$$\begin{aligned} N_s(\eth_\varepsilon^2 - \lambda) \circ N_s(E(\lambda)') &= N_s(R_\sigma(\lambda)) \\ N_b(\eth_\varepsilon^2 - \lambda) \circ N_b(E(\lambda)') &= N_b(R_\sigma(\lambda)). \end{aligned} \tag{22}$$

In other words, once again we need to *invert* the two normal operators: $(\eth_{\overline{H}}^2 - \lambda)$ and $(\eth_0^2 - \lambda)$. For λ away from the spectrum of $\eth_{\overline{H}}^2$ and $\eth_0^2$, as we are at present assuming, this is possible. The solutions of these problems in (22) always have nontrivial asymptotic expansions at ∂M_s^2.

This argument fixes the Schwartz kernel of $E(\lambda)'$ on B_{ds} and B_{db} respectively, and we then find some extension $E(\lambda)'$ to the entire space M_s^2. The remainder term after this correction term has been added is now sufficiently residual that we can iterate it away without difficulty.

In conclusion, we reemphasize that the fundamental step in proving that the resolvent $(\eth_\varepsilon^2 - \lambda)$, for $\lambda \in \Omega$, is an element of the surgery calculus $\Psi_s^{-2,\mathcal{E}_\lambda}(X)$ is the *inversion* of the two normal homomorphisms N_s and N_b.

4.5 – Small eigenvalues in the nondegenerate case

In analyzing the large time behaviour of the heat kernel $\exp(-t\eth_\varepsilon^2)$, uniformly in ε, it is necessary to understand the structure of the resolvent $(\eth_\varepsilon^2 - \lambda)$ for λ near 0. The simplest case to understand is when we impose the assumption that

$$\text{the Dirac operator } \eth_H \text{ is invertible.} \tag{23}$$

This hypothesis is called *nondegeneracy.* As indicated in §4.2, under this assumption the operator $\eth_0^2$ induced on $\overline{X}$ by the limiting metric g_0 is Fredholm on the ordinary (unweighted) L^2 space. In particular, spec$(\eth_0^2)$ is discrete near 0. This can be sharpened: if σ_0^2 is the smallest eigenvalue of the boundary operator $\eth_H$ associated to $\eth_0$, then the spectrum of $\eth_0$ is discrete in the interval $[0, \sigma_0^2)$. It is continuous, possibly with embedded discrete spectrum, in $[\sigma_0^2, +\infty)$. The full spectral and scattering theory of such operators is described, using the b-calculus, in [36]. A similar, but easier, analysis shows that assuming (23), the surgery normal operator $N_s(\eth_\varepsilon)$ has only continuous spectrum contained in $[\sigma_0^2, +\infty)$.

Choose δ so that spec$(\eth_0^2) \cap (-\delta, \delta) = \emptyset$. For λ in a δ-neighbourhood of 0, the resolvent of $\eth_0$ can be written as

$$(\eth_0^2 - \lambda)^{-1} = \mathrm{Res}_0(\lambda) + \frac{1}{\lambda}\Pi_0,$$

where $\mathrm{Res}_0(\lambda)$ a parametrix depending holomorphically on λ and with a finite rank error term, and Π_0 is the orthogonal projection onto the null space of $\eth_H^2$. We let $N = \dim \mathrm{null}(\eth_0)$.

We can modify the construction of the resolvent in the surgery calculus to take into account this refined structure of the inverse of the b-normal operator. In fact, it is not hard to produce a surgery pseudodifferential operator $G(\lambda)$, depending holomorphically on λ near zero, such that

$$(\eth_\varepsilon^2 - \lambda) \circ G(\lambda) = \mathrm{Id} - \Pi(\lambda) \tag{24}$$

with $\Pi(\lambda)$ a surgery pseudodifferential operator depending holomorphically on λ and of uniform finite rank $= N$, and with $N_b(\Pi(\lambda)) = \Pi_0$.

Projecting the operator $\Pi(\lambda)$ onto its range, it is clear that the invertibility of the right hand side of (24) is equivalent to the invertibility of an $N \times N$-matrix of the form $(\delta_{ij} - a_{ij}(\lambda))$. If $q(\varepsilon, \lambda)$ is the determinant of this matrix, then it is holomorphic in λ for each fixed ε and polyhomogeneous conormal in ε. Moreover $q(0, \lambda) = \lambda^N$. For each fixed $\varepsilon \in [0, \varepsilon_0]$, the function $q(\varepsilon, \cdot)$ has precisely N zeros, counting multiplicity; these are the *small eigenvalues* of $\eth_\varepsilon^2$. The orthogonal projection Π_ε onto the small eigenvalues of $\eth_\varepsilon^2$ is therefore of uniform finite rank and it follows from this construction that it too is a surgery pseudodifferential operator. The small eigenvalues themeselves have polyhomogeneous expansion in ε.

In summary, assuming the nondegeneracy condition (23), for λ in a small neighbourhood of zero, the resolvent $(\eth_\varepsilon^2 - \lambda)$ is a meromorphic family of surgery pseudodifferential operators, with poles at the small eigenvalues of $\eth_\varepsilon^2$. The orthogonal projection onto the small eigenvalues is a surgery pseudodifferential operator of uniformly finite rank $N = \dim(\mathrm{null}(\eth_0))$.

4.6 – The logarithmic surgery calculus

The nondegeneracy condition (23) is strong, and often not satisfied in applications. To proceed further without this assumption requires substantially more work, unfortunately. Although we will not be able to describe this in anywhere near the amount of detail we have been going into up until now, we wish to indicate a few of the new features in this general case.

The main problem is already seen in the very simplest example of surgery degeneration, namely the one-dimensional example of the interval $X = I = [-1, 1]_x$ with the family of metrics $g_\varepsilon = dx^2/(x^2 + \varepsilon^2)$. (The boundaries at $x = \pm 1$ are unimportant here, and we could well have considered the surgery degeneration of a circle at the risk of slightly more complicated notation.) The total length of X with respect to g_ε is $2L_\varepsilon$, where $L_\varepsilon = \operatorname{arcsinh}(1/\varepsilon)$, and the (Dirichlet) eigenfunctions of the Laplacian Δ_ε are of the form $u_k(r, \varepsilon) \equiv \sin(\pi k r / L_\varepsilon)$, $k \in \mathbb{Z}$, where $r = \operatorname{arcsinh}(x/\varepsilon)$. Already we see the new length scale: each of these quantities is most naturally expressed not in terms of the parameter ε, but rather in terms of the inverse logarithm of ε, $\operatorname{ilg}\varepsilon = 1/\log(1/\varepsilon)$.

Proceeding further with this example, we next examine the lifts of the eigenfunctions above on the single surgery space M_s. After a brief calculation, we see that u_k lifts to a function obviously smooth in the interior of M_s, equal to 1 on the surgery front face (the lift of $\{0\} \times \{0\}$), and equal to $(-1)^k$ on the adjacent boundaries at $\varepsilon = 0$. The lift is not polyhomogeneous on M_s, and does not behave uniformly in ε. In fact, the oscillations of these eigenfunctions somehow disappear into the corners, at the intersection of the surgery front face and the other b-faces at $\varepsilon = 0$.

These various issues must be dealt with simultaneously, and again the idea is to resolve these new singular phenomena geometrically by performing some new blow-ups. We describe these only for the singular surgery space, and shall now define the single *logarithmic* surgery space M_{Ls}. The double logarithmic surgery space M^2_{Ls} is unfortunately much more complicated than the (already none-too-simple) space M^2_s, and we must refer the interested reader to [25] for its definition.

To deal with the new length scale we first define the logarithmic blow-up of M_s. This is obtained by simply replacing the boundary defining function ρ of each boundary hypersurface (at $\varepsilon = 0$) by $\operatorname{ilg} \rho$. In effect, this defines a new (but equivalent) $\mathcal{C}^\infty$ structure on M_s. Smooth functions in this new structure are those which are smooth in the various 'new' functions $\operatorname{ilg} \rho$ on the original space. Although this may not appear to be a blow-up in the sense we have been describing this concept, it may be recast in this language, cf. [25]. Next we blow up the corners, i.e. the intersections of the b-faces and the surgery face at $\varepsilon = 0$. The resulting space is now called the single logarithmic surgery space M_{Ls}. It has four boundary faces at $\varepsilon = 0$, instead of the two possessed by M_s.

Rather than entering into any more details of this construction, suffice it to say that the overall strategy is much the same as before. A double logarithmic surgery space M^2_{Ls} is defined, and is equipped with blow-down maps to M_{Ls}. The space $\Psi^{*,\mathcal{E}}_{\mathrm{Ls}}$ of logarithmic surgery pseudodifferential operators is again defined as containing operators, the Schwartz kernels of which are pushed forward from M^2_{Ls}, and these kernels on the double logarithmic surgery space are conormal at all boundary faces (note that now this implies the existence of expansions in powers of $\operatorname{ilg} \rho$ at any face with boundary defining function ρ). The main theorem, proved by an explicit parametrix construction, is that in the degenerate case the resolvent $(\eth^2_\varepsilon - \lambda)^{-1}$ is an element of this surgery calculus, in a precise sense uniformly even as λ approaches zero.

The one aspect of this that we shall discuss slightly more is the new normal operator that must be considered. This is the *reduced normal operators*, $\mathrm{RN}(\eth_\varepsilon)$. To define it, first recall that since 0 is now an eigenvalue of $\ker(\eth_H)$, the limit operator $\eth_0^2$ is not Fredholm on L^2 (the weight 0 is one of the omitted ones). However, it is Fredholm on $x^{\pm\delta}L^2$ for δ sufficiently small, and has a parametrix $G(\pm\delta)$ in the b-calculus. The structure of this parametrix can be used to prove that near H

$$\eth_0^2 v=0, v\in x^{-\delta}L_b^2(\overline{X}_+\sqcup\overline{X}_-) \Longrightarrow v\sim v_1\log x+v_0+v',\ \eth_H^2(v_i)=0\ ,v'\in L^2.$$

These asymptotic boundary values define two pairs of subspaces in $\ker(\eth_H^2)$, analogous to the scattering Lagrangians considered earlier for the Dirac operator. These are

$$\Lambda_\pm^N=\{v_1;\exists\, v\sim v_1(y)\log x+v_0(y)+v',\eth_0^2 v=0,v'\in L^2\}$$
$$\Lambda_\pm^D=\{v_0;\exists\, v\sim v_0(y)+v',v'\in L^2\}.$$

(The subscripts here refer to $X_\pm$.) The reduced normal operator $RN(\eth_\varepsilon)$ is the operator D_s^2 on $[-1,1]$ acting on $\ker(\eth_H^2)$-values funtions with the boundary conditions

$$u|_{s=-1}\in\Lambda_-^D\quad D_s u|_{s=-1}\in\Lambda_-^N$$
$$u|_{s=+1}\in\Lambda_+^D\quad D_s u|_{s=+1}\in\Lambda_+^N.$$

Just as in the simpler parametrix construction in the nondegenerate case, we need to invert the various normal operators, which now includes this new one. The inversion of the reduced normal operator can be done quite explicitly, and we can also see at least in very vague outline how the scattering Lagrangians enter into the analysis.

5 – Applications of the surgery calculus

In this final section of this survey, we present four applications of the surgery calculus. The first is purely analytic, and is the detailed description of how eigenvalues of $\eth_{X,\varepsilon}$ accumulate as $\varepsilon\to 0$. The second is a final discussion of the proof of the gluing formula for the eta invariant from this point of view. After that we discuss an interesting application of this gluing formula, which is the signature formula for manifolds with corners of codimension two. We finally discuss the analytic torsion and some aspects of the proof of the gluing formula for determinant bundles.

5.1 – Accumulation of eigenvalues

One of the immediate consequences of the construction of the resolvent for the family of operators $\eth_X^2$ is a formula for the rate of accumulation of its eigenvalues as $\varepsilon \to 0$. We shall only state the result here and say very little about its proof, which involves the full intricacies of the logarithmic surgery calculus.

Consider the eigenvalues $\lambda_j(\varepsilon)$ of $\eth_X^2$. We are particularly interested in the eigenvalues tending to zero as $\varepsilon \to 0$. In order to study them, we rescale by setting $\lambda_j(\varepsilon) = (\operatorname{ilg}\varepsilon) z_j(\varepsilon)$, where 'ilg' stands for the inverse logarithm, i.e. $\operatorname{ilg}\varepsilon = 1/\log(1/\varepsilon)$ as in §4.6. (Of course other eigenvalues $\lambda_j(\varepsilon)$ tend to finite nonzero limits or to infinity, and it may be possible to study them by analogous methods, but this has not been carried out.) Recall also the reduced normal operator $\mathrm{RN}\,(\eth_\varepsilon^2)$ introduced at the end of the last section.

THEOREM 2. *Assuming that the eigenvalues $\lambda_j(\varepsilon)$, and hence $z_j(\varepsilon)$, are listed in increasing order, with multiplicity, and similarly for the eigenvalues μ_j of the reduced normal operator $\mathrm{RN}\,(\eth_\varepsilon^2)$, then as $\varepsilon \to 0$, either $z_j(\varepsilon) \to \mu_j$, or else $z_j(\varepsilon) \to \infty$. The number of eigenvalues converging to zero is the same as the dimension of the nullspace of the limiting operator $\eth_{X,0}^2$ on $X_+ \sqcup X_-$, and for each μ_j, there is exactly one family of eigenvalues $z_j(\varepsilon)$ converging to it.*

The eigenvalues $z_j(\varepsilon)$ converging to 0 are somewhat special: it can be proved that they are rapidly decreasing in $\operatorname{ilg}\varepsilon$ – in fact, they vanish as a power of ε – and are therefore called the *very small eigenvalues*.

This result shows that the bottom of the spectrum of $\eth_{X,0}^2$ is somehow 'granular', at least inasmuch as it is obtained as a limit of eigenvalues accumulating at a very slow rate.

The only point of the proof we wish to mention is that it involves considering the resolvent with rescaled spectral parameter

$$R(z,\varepsilon) = (\eth_{X,\varepsilon}^2 - (\operatorname{ilg}\varepsilon)^2 z^2)^{-1}.$$

The uniformity of this object is considered as $\varepsilon \to 0$. The rescaling of the spectral parameter corresponds essentially to blowing up the λ-spectral plane at $\lambda = 0$.

5.2 – The surgery formula for the eta invariant

We next consider the eta invariant for the Dirac operator associated to the metric g_ε:

$$\eta(\eth_\varepsilon) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \mathrm{Tr}(\eth_\varepsilon e^{-t\eth_\varepsilon^2}) dt.$$

According to the third approach to surgery, our main concern is to describe as precisely as possible the behaviour of $\eta(\eth_\varepsilon)$ as $\varepsilon \to 0$.

First we assume that

$$\ker(\eth_H) = \{0\}. \tag{25}$$

The large time behaviour of $e^{-t\eth_\varepsilon^2}$ can be analyzed using the results of the §4.5 : since the spectrum of $\eth_\varepsilon^2$ remains discrete near $\lambda = 0$, down to $\varepsilon = 0$, it follows from the contour integral representation

$$e^{-t\eth_\varepsilon^2} = \frac{i}{2\pi} \int_\gamma e^{-t\lambda} (\eth_\varepsilon^2 - \lambda)^{-1} d\lambda \tag{26}$$

(with γ enclosing the spectrum) that $e^{-t\eth_\varepsilon^2}$ is exponentially decreasing as $t \to \infty$ uniformly in ε, up to a uniformly finite rank operator.

Next one needs to understand the heat kernel uniformly for finite times. Consider first the heat kernel associated to a Dirac Laplacian on a closed Riemannian manifold X. For each $t > 0$, $e^{-t\eth_X^2}$ is a smoothing operator. Bearing in mind the initial condition it must satisfy we see that as $t \to 0$ the heat kernel must develop some sort of singularity along the diagonal. For the Laplacian in $\mathbb{R}^n$, for example, the heat kernel is explicitly given by

$$\frac{1}{(2\pi t)^{\frac{n}{2}}} e^{-|x-x'|^2/4t}.$$

In other words, viewed as a distribution on $X \times X \times [0, \infty)_t$, the heat-kernel is singular along the submanifold $\Delta \times \{t = 0\}$. It is possible to encode the full short-time asymptotics of the heat kernel, i.e. the expansion of this singularity, using the language of blow-ups; this is explained in detail in [36]. Because of the different homogeneities of the space and time variables, we must use *parabolic blow-up* instead of the normal blow-ups to which we have mostly been referring. (In fact, these parabolic blow-ups are akin to the logarithmic blow-ups of §4.6.) The parabolic blow-up of a submanifold Y, which is defined relative to a subbundle $S \subset N^*Y$, appeared first in [20]. The *heat space* is the parabolic blow-up of $X^2 \times [0,\infty)$ along $\Delta \times \{0\}$ with respect to the subbundle $S = \mathrm{Span}\{dt\}$, and is denoted $[X^2 \times [0,\infty); \Delta \times \{0\}, S]$. It is defined as the disjoint union of $(X^2 \times [0,\infty))\backslash(\Delta \times \{0\})$ and a 'parabolic normal bundle' to $\Delta \times \{0\}$. We shall not give more details of its definition, but only state the basic fact that the fundamental solution of the heat equation lifts to a polyhomogeneous conormal distribution on this space.

Because this construction is essentially local in the space variables, it is also possible to define a b-heat space as well as a *surgery heat space* M^2_{hs}. This latter space is the parabolic blow up of $M^2_s \times [0,\infty)_t$ along $\Delta_s \times \{0\}$ relative to the analogous subbundle S spanned by dt along the diagonal at $\{t = 0\}$. The heat kernel of $\eth^2_\varepsilon$ is polyhomogeneous conormal on this space, which fully encodes its uniformity as $\varepsilon \to 0$. In order to remain within the category of *compact* manifolds with corners, we can even compactify the temporal variable at $t = \infty$, thus obtaining the *compactified surgery heat space* $M^2_{\mathrm{c-hs}}$. The notation $[0,1]_t$ denotes the compactified t-axis, although of course t is not the natural linear variable on this finite interval. The lifted diagonal embeds into M^2_{hs}, and hence in $M^2_{\mathrm{c-hs}}$, and this induces

$$i_{\Delta_s} : \Delta_s \times [0,1]_t \equiv M_s \times [0,1]_t \hookrightarrow M^2_{\mathrm{c-hs}}.$$

Using this notation, we can now reexpress the integral which define the eta invariant in terms of pull-backs and push-forwards. Thus

$$\eta(\eth_\varepsilon) = (\pi_t)_*(\pi_s)_* \left[i^*_{\Delta_s} \mathrm{tr} \left(\frac{t^{-1/2}}{\sqrt{\pi}} \mathrm{Tr}(\eth_\varepsilon e^{-t\eth^2_\varepsilon}) \right) \right].$$

Here tr denotes the trace on each fibre of the endomorphism bundle $\mathrm{Hom}(\$, \$)$. The map $\pi_s \equiv \pi_s \times \mathrm{Id} : M_s \times [0,1]_t \to [0,\varepsilon_0] \times [0,1]_t$ is the composition of the blow-down map $\beta_s : M_s \to X \times [0, \varepsilon_0]$ and the natural projection $X \times [0, \varepsilon] \to [0, \varepsilon_0]$; finally $\pi_t : [0, \varepsilon_0] \times [0, 1]_t \to [0, 1]_t$ is the obvious projection.

This formula expresses the eta invariant of $\eth_\varepsilon$ as the push-forward of the polyhomogeneous conormal distribution

$$i^*_{\Delta_s} \mathrm{tr} \left(\frac{t^{-1/2}}{\sqrt{\pi}} \mathrm{Tr}(\eth_\varepsilon e^{-t\eth_\varepsilon^2}) \right).$$

from the manifold with corners $M_s \times [0, 1]_t$ to the interval $[0, \varepsilon_0]$. Polyhomogeneity of the 'integrand' at the two temporal faces follows from the short-time and large-time behaviour of the surgery heat kenel. The short-time behaviour follows from Getzler rescaling, as in [36]; it actually implies smoothness up to the face $t = 0$. The large time behaviour has been already analyzed, and it implies the rapid vanishing at the corresponding temporal face, up to a uniformly finite rank operator.

The resulting distribution on $[0, \varepsilon_0]$ may be analyzed using the general results on push-forwards from [38]. In this manner, the behaviour of the eta invariant of $\eth_\varepsilon$ as $\varepsilon \to 0$ can be simply read off *geometrically.* To state the theorem recall that the projection Π_ε onto the small eigenvalues of $\eth_\varepsilon$ is a finite rank operator, uniformly in ε. Let $\tilde{\eta}(\varepsilon)$ be the signature of Π_ε. Then we have

Theorem 3 ([34]). *The eta invariant associated to the Dirac operator $\eth_\varepsilon$ subject to the condition* (25) *satisfies*

$$\eta(\eth_\varepsilon) = {}^b\eta(\eth_{\overline{X}_+}) + {}^b\eta(\eth_{\overline{X}_-}) + \tilde{\eta}(\varepsilon) + r_1(\varepsilon) + r_2(\varepsilon) \log \varepsilon$$

as $\varepsilon \to 0$, where $r_1, r_2 \in \mathcal{C}^\infty([0, \varepsilon_0])$ with $r_1(0) = r_2(0) = 0$.

When the operator $\eth_\varepsilon$ is no longer nondegenerate, a similar proof works. One must define a logarithmic heat surgery space, and then the eta invariant, as a function of ε, may be obtained as the push-forward from this space of a polyhomogeneous distribution. Let Π_ε denote the orthogonal projection onto the eigenspaces corresponding to the very small eigenvalues and let $\tilde{\eta}(\varepsilon)$ be the signature of Π_ε. The generalization of the previous result is

Theorem 4 ([25]). *The eta invariant associated to the Dirac operator* $\eth_\varepsilon$, *no longer necessarily satisfying the nondegeneracy hypothesis* (25), *satisfies*

$$\eta(\eth_\varepsilon) = {}^b\eta(\eth_{\overline{X}_+}) + {}^b\eta(\eth_{\overline{X}_-}) + \tilde{\eta}(\varepsilon) + \eta(\mathrm{RN}\,(\eth_\varepsilon)) + r_1(\mathrm{ilg}\,\varepsilon) + \log(\mathrm{ilg}\,\varepsilon) r_2(\mathrm{ilg}\,\varepsilon)$$

as $\varepsilon \to 0$. *Here, as before,* r_1 *and* r_2 *are smooth functions vanishing at* 0.

It is possible to calculate the eta invariant for the reduced normal operator in terms of finite dimensional data involving the scattering Lagrangian subspaces associated to the Dirac operators on $X_\pm$, as discussed in §2.2.

5.3 – The signature theorem on manifolds with corners

Suppose that X is a compact manifold with corners. As with manifolds with boundary, there are many possible choices for natural metrics to consider on X. Following our usual choice in this matter, we shall assume that the interior of X is endowed with an exact b-metric g. This may be described as follows. Assume that the codimension one boundary faces of X are listed as M_α, $\alpha = 1, \ldots N$, and that each such boundary face has a defining function x_α. Then near M_α,

$$g = \frac{dx_\alpha^2}{x_\alpha^2} + h_\alpha,$$

where h_α is some smooth nonnegative symmetric 2-tensor in a collar neighbourhood of M_α which restricts to a metric on M_α, and near each corner of codimension k, given as the intersection of boundary faces $M_{\alpha_1}, \ldots, M_{\alpha_k}$,

$$g = \frac{dx_{\alpha_1}^2}{x_{\alpha_1}^2} + \ldots + \frac{dx_{\alpha_k}^2}{x_{\alpha_k}^2} + h_{\alpha_1 \ldots \alpha_k},$$

where the final summand restricts to a metric on the corner.

Suppose that $\dim X = 4\ell$, and let $\eth_X$ denote the signature operator on X. This is simply the deRham-Hodge operator $d + d^*$, restricted to act between the $+1$ and -1 eigenspaces of the natural algebraic involution τ which equals $i^{p(p-1)+2\ell} *$ on p-forms. The question we discuss here is whether it is possible to obtain a signature formula for X, relative to a metric of this (or any other) type. In the case where X has only a boundary, i.e. has corners only up to codimension one, then this is precisely the celebrated signature formula for manifolds with boundary of Atiyah, Patodi and Singer [1]. We have already mentioned some extensions and generalizations of this result, in particular its recasting by Melrose [36] and the families index theorems of [44], [45]. In each of these papers, the signature formula is regarded as an index formula, and so it would seem that the most natural problem to study is whether it is possible to obtain an index formula for general Dirac-type operators associated to exact b-metrics on manifolds with corners.

At this stage we recall for the reader the fact that on a compact closed manifold there are in some sense two index theorems: one for Dirac-type operators and the other for general elliptic (pseudodifferential) operators. These are regarded as equivalent, because the latter may be deduced from the former using K-theory. One may consider these two types of index theorems for manifolds with boundary or corners as well, but the relationship between them is no longer so simple. In this context the index theorem for general elliptic operators was only very recently obtained, by Melrose and Nistor [41], [42], but this formula is stated and proved using Hochschild homology, and the terms in it do not translate readily to more familiar ones for specific geometric operators. Thus it is still an open problem to find an index formula for Dirac-type operators on manifolds with corners.

Some partial progress on this sort of index theorem was made by Müller [50] when X has corners only up to codimension two, assuming also some rather strong nondegeneracy conditions. If $\eth_X$ is a Dirac-type operator in the interior of X, then because of the nature of the metric, there are induced Dirac-type operators $\eth_\alpha$ on every codimension one boundary face M_α, each of which is now a manifold with boundary endowed with an exact b-metric, and also operators $\eth_{\alpha\beta}$ on the corners $H_{\alpha\beta} = M_\alpha \cap M_\beta$, whenever these intersections are nontrivial. Since these corners are compact, these latter operators have discrete spectrum, but the $\eth_\alpha$ and $\eth_X$ have continuous spectrum. The continuous spectrum for the $\eth_\alpha$ is fairly simple, since it is of locally finite multiplicity, with thresholds at points determined by the eigenvalues of the $\eth_{\alpha\beta}$. In particular, 0 is in the essential spectrum of $\eth_\alpha$ if and only if some $\eth_{\alpha\beta}$ is not invertible. The continuous spectrum for $\eth_X$ itself is much more complicated, since its multiplicity is no longer necessarily locally constant. In particular, if some $\eth_{\alpha\beta}$ is not invertible, then the spectrum of $\eth_X$ near zero is of this rather complicated type. In particular, it is unclear whether the basic method to understand this spectrum near zero, by analytically continuing the resolvent of $\eth_X^2$ to some branched cover of the plane, still works. Unfortunately, some sort of information about the spectrum of $\eth_X$ near zero is necessary to obtain a formula for the index, and this does not seem any too accessible. In any event, Müller's analysis assumes that each of the corner Dirac operators is invertible so that, while the continuous spectrum of $\eth_X$ may reach zero, it is of the simpler type there, of locally finite multiplicity. Müller does prove without this assumption, though, that when $\eth_X$ is the signature operator, then its L^2-index is well-defined and still yields the topological signature of the manifold X. Unfortunately, the nondegeneracy conditions are never satisfied for the signature operator, and so Müller cannot deduce the signature formula this way.

It turns out that it is possible to obtain the signature formula for manifolds with corners of codimension two following a somewhat different sort of argument. This was accomplished by the first author, Melrose and Hassell [26]. The idea is to obtain this formula not via an index calculation on all of X, but instead as a limit of index formulæ on a family of compact manifolds with smooth boundary X_ε which fill out X as ε tends to zero.

There are a few steps to this proof. In the first, an appropriate family of smoothings X_ε is defined. Then the APS signature theorem is applied to each of these manifolds with boundary. Denoting by $\eth_\varepsilon$ the restriction of $\eth_X$ to X_ε, we get

$$\mathrm{sign}\,(X) = \mathrm{ind}\,(\eth_\varepsilon) = \int_{X_\varepsilon} \omega - \frac{1}{2}\eta(\eth_{\partial X_\varepsilon}) + B.$$

The first term on the right here is the integral of the usual signature density, which is the $\mathcal{L}$-polynomial in the Pontrjagin forms, the second term is the eta invariant of the induced signature operator on the boundary, and the final term is an integral over ∂X_ε of a local expression involving the second fundamental form. This final term is necessitated by the fact that the metric g_ε is no longer of product type near the boundary.

The remainder of the proof involves calculating the limits of these various terms as $\varepsilon \to 0$. The left hand side, the signature, is topological, so obviously does not change with ε. Using the asymptotics of the metric g, the integral of ω over all of X is well-defined, and the first term on the right tends to this. The final term on the right tends to zero because of the specific construction of the smooth surfaces ∂X_ε.

Thus it remains to calculate the limit of the eta invariant. It turns out, again by the choice of smoothing, that the induced metric on ∂X_ε is simply undergoing surgery degeneration. Thus we already have the tools to analyze the limit of the eta invariant of the induced Dirac operator $\eth_{\partial X_\varepsilon}$. The only difficulty now is essentially combinatorial. The gluing formula for the eta invariant assumes only a single disconnecting hypersurface H, decomposing the manifold into two pieces. Here the relevant manifold ∂X_ε has some system of nonintersecting hypersurfaces $H_{\alpha\beta}$, and the metric is degenerating across each one of them. There are again several ways of expressing the defect term in the formula for the limit of the eta invariant. The most elegant of these is as follows. Associate to X a one-dimensional directed graph $\mathcal{G}$ by discarding the interior of X, replacing each codimension one boundary component M_α by a vertex v_α and each codimension two corner $M_\alpha \cap M_\beta$ by an edge $e_{\alpha\beta}$. These edges are directed by choosing arbitrarily some ordering of the M_α, then identifying $e_{\alpha\beta}$ in an orientation preserving manner with $[-1,1]$ if $\alpha < \beta$ in this ordering. We consider the trivial vector bundle V over $\mathcal{G}$ with fibre the direct sum of all the cohomologies of the corners, i.e. the direct sum of all $\ker(\eth_{\alpha\beta})$. There is a Dirac operator acting on sections of this trivial bundle. The 'boundary conditions' at the vertex v_α are given by the scattering Lagrangian $\Lambda^\alpha_{\mathrm{sc}}$ associated to M_α. The domain of the Dirac operator $\eth_{\mathcal{G}}$ is the space of sections ϕ which restrict along each edge $e_{\alpha\beta}$ to an element $\phi_{\alpha\beta}$ of the corresponding nullspace $\ker(\eth_{\alpha\beta})$, and such that at the vertex v_α, the sum of all $\phi_{\alpha\beta}$ for edges $e_{\alpha\beta}$ contiguous to that edge sum to an element of $\Lambda^\alpha_{\mathrm{sc}}$. Notice that in the simple case where there are only two vertices and one edge, this reduces to the operator introduced at the end of §2.2. In any case, the defect term may be expressed as the eta invariant of this operator $(\eth_{\mathcal{G}}, \Lambda_{\mathrm{sc}})$. The final signature formula then is

THEOREM 5. *Let X be a manifold of dimension 4ℓ with corners of codimension two, and suppose that g is an exact b-metric on the interior* *of X. Then with the preceding notation and conventions,*

$$\operatorname{sign}(X) = \int_X \omega - \frac{1}{2}\sum_\alpha \eta(\eth_{M_\alpha}) - \frac{1}{2}\eta(\eth_{\mathcal{G}}, \Lambda_{\mathrm{sc}}).$$

The correction term in this formula may once again be expressed in purely finite dimensional linear algebraic terms using the various scattering Lagrangians, cf. [26].

The one place where we have really used special features of the signature operator here is when we were able to rule out any extra integer terms when taking the limit of the eta invariant. This is because the rank of $\eth_{\partial X_\varepsilon}$ is determined topologically, hence is constant. In general, there might well be some spectral flow. The only thing we would be able to deduce by this method in general, then, is the mod $\mathbb{Z}$ reduction of this formula.

5.4 – The surgery formula for the analytic torsion

Behaviour of the analytic torsion under surgery was already studied in the fundamental work of Cheeger. Many of the subsequent proofs of the Cheeger-Müller theorem also exploit some form of this method in a basic way. In this section we look at the surgery problem for the analytic torsion from the point of view of the surgery calculus, as studied by Hassell [23].

Let $X = X_+ \cup_H X_-$ and $g_\varepsilon = h + dx^2/(x^2+\varepsilon^2)$ be a family of metrics on X undergoing surgery degeneration along H, as usual. Consider the analytic torsion $T(X, g_\varepsilon)$ associated to the metric g_ε. We can also consider the metric independent definition given in §2.3

$$T(X, \{\mu\}) = T(X, g_\varepsilon) \cdot \Lambda(g_\varepsilon, \{\mu\})$$

with $\{\mu\} = \{\mu^{(i)}\}$ a basis of $H^*(X) = \oplus H^i(X)$. More generally, if E is a flat unitary bundle we can define $T(X, E, g_\varepsilon)$ and $T(X, E, \{\mu\})$ using the de Rham complex twisted by E. A suitable understanding of the behaviour of the two terms appearing in this definition of $T(X, \{\mu\})$ will lead to a surgery formula for $T(X, \{\mu\})$. The analysis of the first factor $T(X, g_\varepsilon)$ is based directly on the uniform analysis of the heat kernel associated to Δ_ε, as in the case of the eta invariant. The construction of this heat kernel as a polyhomogeneous distribution on the logarithmic surgery heat space is again the main ingredient in this analysis. The second factor $\Lambda(g_\varepsilon, \{\mu\})$ can be understood using a Hodge-theoretic reinterpretation of the Mayer-Vietoris sequence for $X = X_+ \cup_H X_-$.

Putting these two results together HASSELL proves [23] that for suitable choices of $\{\mu\}$,

$$T(X, \{\mu\}) = {}^bT(\overline{X}_+, g_0) + {}^bT(\overline{X}_-, g_0) + \frac{1}{2}\sum_{q=0}^{n} q \log\det RN(\Delta_q).$$

The correct choice of set of bases $\{\mu\}$ is determined by properties of the very small eigenvalues. These are rather simple to understand here because, using the (Hodge-) Mayer-Vietoris sequence again, it can be seen that the multiplicity of $0 \in \operatorname{spec}(\Delta_\varepsilon)$ is *constant* in $\varepsilon \geq 0$. If Π_ε is the orthogonal projection onto $\ker(\Delta_\varepsilon)$, then the $\{\mu\}$ in this formula must be chosen in the image of Π_ε, which is by definition simply the Hodge cohomology of (X, g_ε).

It is also important to use the fact that

$$\frac{1}{2}\sum_{q=0}^{n} q \log\det RN(\Delta_q)$$

may be explicitly decribed in terms of the finite dimensional subspaces $\Lambda^N_\pm$, $\Lambda^D_\pm$ appearing in the definition of the boundary condition for the reduced normal operator, and thus ultimately from the cohomology of H. In fact, another cohomological computation shows that this finite dimensional geometric expression also appears in a 'surgery formula' for the combinatorial Reidemeister torsion $\tau(M, \{\mu\})$. Using this, it is possible to state the surgery formula for the analytic torsion in a particularly elegant way:

THEOREMA 6 ([23]). *If $X = X_+ \cup_H X_-$ is odd dimensional, the difference $\log T - \log\tau$ obeys the surgery formula*

$$\log\frac{T(X, g_\varepsilon)}{\tau(X, g_\varepsilon)} = \log\frac{{}^bT(\overline{X}, g_0)}{{}^b\tau(\overline{X}, g_0)} + \frac{1}{2}\chi(H)\log 2$$

with $\overline{X} = \overline{X}_+ \sqcup \overline{X}_-$ and $\chi(H)$ equal to the Euler characteristic of H.

This result can be applied to reprove the Cheeger-Müller theorem on the equality of the analytic and Reidemeister torsion on any closed compact manifold. Also, using a doubling argument it is also possible to prove an extension of the Cheeger-Müller theorem for manifolds with boundary.

THEOREM 7 ([23]). *For an odd-dimensional manifold with boundary with exact b-metric g,*

$$^{b}T(Z,g) = 2^{-\chi(\partial Z)/4}\,\tau(Z,g).$$

Similar results hold when we twist by a flat unitary bundle E.

5.5 – Determinant bundles and surgery

In this section we finally address the two questions raised at the end of §2.5, following the treatment given by the second author in [53], [54]. Recall the geometrical data: we are given a fibration $\phi : M \to B$ of compact manifolds with fibres even dimensional and endowed with smoothly varying metrics and smoothly varying spin structures. We denote by $g_{M/B}$ this family of fibre metrics and by $\eth_M$ the associated family of Dirac operators. These data define a determinant bundle $\mathcal{L}(\eth)$ with a Quillen metric $\|\cdot\|_Q$ and Bismut-Freed connection $\nabla^{\mathcal{L}}$. We assume that the fibration M is the union along a fibering hypersurface H of two fibration with boundary: $M = M_+ \cup_H M_-$. These data fix the families $\eth_{M_\pm}$ as well as the family $\eth_H$. First let us make the very strong assumption that $\ker(\eth_H)_z = \{0\}$ for each $z \in B$. In this particular case, assuming that the metrics are product-like near H, the two families of APS boundary value problems on the fibrations $M_\pm$ are well defined and vary smoothly with $z \in B$; since they are Fredholm they define two smooth determinant bundles, $\mathcal{L}(\eth_{M_+}, \Pi_0^+)$ and $\mathcal{L}(\eth_{M_-}, \Pi^-)$, with $\Pi_0^+(z)$ equal to the spectral projection for $(\eth_H)_z$.

These two determinant bundles can also be defined using the L^2 condition on the associated fibration with cylindrical ends: $\overline{M} = \overline{M}_+ \sqcup \overline{M}_-$. In other words they can be defined in terms of the associated b-Dirac families $\eth_{\overline{M}_+}$, $\eth_{\overline{M}_-}$. We shall also use the suggestive notation $\eth_{\overline{M}} = \eth_{\overline{M}_-} \sqcup \eth_{\overline{M}_+}$. We denote the associated determinant bundles by $^{b}\mathcal{L}(\eth_{\overline{M}_+})$ $^{b}\mathcal{L}(\eth_{\overline{M}_-})$. Notice that each Laplacian $(\Delta_{\overline{M}})_z$ has discrete spectrum near zero; this means that the description of the b-determinant bundle in terms of small eigenvalues, as given in §2.5, is still valid.

The determinant bundles in the two pictures (APS vs L^2_b) are canonically isomorphic (see §1.2); however there are substantial advantages to working with b-determinant bundles. Namely, the definition of Quillen metric and Bismut-Freed connection can be given directly on ${}^b\mathcal{L}$, provided that the trace functional appearing in the definition of the zeta function $\zeta(s, \Delta^+, \lambda) = \mathrm{Tr}(\Pi_{(\lambda,\infty)}(\Delta^+)^{-s})$ is replaced by the b-Trace and similarly for the second term, $\beta^+(\lambda)$, appearing in the Bismut-Freed connection (see §2.5). Here λ must be always chosen away from the discrete spectrum of the family of b-Laplacians $\Delta_{\overline{M}}$. The first term ∇^λ in the definition of the Bismut-Freed connection is defined directly in terms of the metric and thus extends to b-metrics with no effort. In summary, by using the b-Trace functional, we obtain in a natural way the b-Quillen metric $\|\cdot\|_{Q,b}$ and, more importantly, the b-Bismut-Freed connection

$$ {}^b\nabla^{\mathcal{L}}|_{U_\lambda} = {}^b\nabla^\lambda + {}^b\beta^+(\lambda). $$

This latter step is not at all obvious in the APS framework. We note that in proving the compatibility of ${}^b\nabla^{\mathcal{L}}$ with $\|\cdot\|_{Q,b}$, the commutator formula for the b-Trace is used in a crucial way.

Returning to the surgery problem, this discussion clarifies the *limit picture* at least under the assumption that $\mathrm{Ker}\,(\eth_H)_z = 0$. Thus let $x \in \mathcal{C}^\infty(M)$ be a defining function for H and consider the family of vertical metrics

$$ g_{M/B}(\varepsilon) = \frac{dx^2}{x^2 + \varepsilon^2} + g_{M/B}. $$

Let $\eth_M(\varepsilon)$ be the associated Dirac family on the closed fibration $(M \to B, g_{M/B}(\varepsilon))$. We denote by $\nabla^{\mathcal{L},\varepsilon}$ the associated Bismut-Freed connection. We denote by ${}^b\nabla^{\mathcal{L}}_+$ and ${}^b\nabla^{\mathcal{L}}_-$ the b-Bismut-Freed connections induced by the limit metric $g_{M/B}(0)$ on the fibrations $\overline{M}_+$, $\overline{M}_-$.

The following theorem is proved in [53] under the assumption $\mathrm{Ker}\,(\eth_H)_z = 0$ for all $z \in B$.

THEOREM 8. *There exists a natural explicit isomorphism of determinant bundles*

$$ S(\varepsilon) : \mathcal{L}(\eth_M(\varepsilon)) \longrightarrow {}^b\mathcal{L}(\eth_{\overline{M}_+}) \otimes {}^b\mathcal{L}(\eth_{\overline{M}_-}). $$

For the curvature and the holonomy of the corresponding Bismut-Freed connection the following formulæ hold:

$$\lim_{\varepsilon\to 0}\,(\nabla^{\mathcal{L},\varepsilon})^2 = ({}^b\nabla^{\mathcal{L}}_{+})^2 + ({}^b\nabla^{\mathcal{L}}_{-})^2$$
$$\lim_{\varepsilon\to 0}\,\mathrm{hol}_\gamma(\nabla^{\mathcal{L},\varepsilon}) = \mathrm{hol}_\gamma({}^b\nabla^{\mathcal{L}}_{+})\cdot \mathrm{hol}_\gamma({}^b\nabla^{\mathcal{L}}_{-}) \quad \forall\gamma\in \mathrm{Map}(S^1,B).$$

The proof is another application of the surgery calculus; the explicit isomorphism is induced by the projection Π_ε onto the *small* eigenvalues of $\Delta^{\pm}$. The behaviour of the curvature and the holonomy of the Bismut-Freed connection is obtained by working directly with a push-forward of the latter object. The first "metric" part of $\nabla^{\mathcal{L},\varepsilon}$ (see (10)) converges almost by definition; the second part, i.e. the term $\beta^+_\varepsilon(\lambda)$ can be analyzed using the heat-surgery calculus. One can prove that for λ small

$$\lim_{\varepsilon\to 0}\,\beta^+_\varepsilon(\lambda) + \log\varepsilon\cdot d\zeta'(0,\eth^2_H,0) = {}^b\beta^{+}_{M_+} + {}^b\beta^{+}_{M_-}$$

where the convergence is to be taken as C^k convergence of 1-forms on the set U_λ. The two formulæ in the theorem then follow readily.

This theorem successfully solves the surgery problem on determinant bundles under the nondegeneracy condition on $(\eth_H)_z$, $z\in B$. We now drop this assumption and consider the general case (see [54]). Since $\eth_H$ arises as a boundary family we can certainly fix a spectral section P for $\eth_H$ obtaining as in §2.1 and §2.5 the two Fredholm families $(\eth^+_{M_+},P)$, $(\eth^+_{M_-},\mathrm{Id}-P)$ and thus the two determinant bundles $\mathcal{L}(\eth_{M_+},P)$, $\mathcal{L}(\eth_{M_-},\mathrm{Id}-P)$. These should be thought of as APS-determinant bundles. However, in order to apply the surgery calculus and the b-calculus we need to consider the corresponding b-determinant bundles, as we did in the invertible case. This is indeed possible provided we allow the use of pseudodifferential operators.

We shall now briefly pause to explain this fundamantal point. Let $D\!\!\!\!/ = (D\!\!\!\!/_z)_{z\in B}$ be any family of Dirac operators on manifolds with boundary. Let us denote by $D\!\!\!\!/_\partial$ the boundary family and let P be a spectral section for $D\!\!\!\!/_\partial$. It is proved in [43] that there exists a smooth family of operators A^P, with $A^P(z)\in\Psi_b^{-\infty}$ with the following properties:

- The family $(D\!\!\!\!/ + A^P)_\partial$ and the family of indicial operators $I(D\!\!\!\!/ + A^P)$ are both invertible; according to the b-calculus the family $(D\!\!\!\!/ + A^P)$ is then Fredholm on L^2.

- The two families of Fredholm operators, one defined by the generalized APS-boundary value problem $(\not{D}, P)$, and the other fixed by the b-family $(\not{D} + A^P)$, are *homotopic*.
- The family $(A^P)_\partial$ is finite rank and self-adjoint, and P_z is equal the projection onto the non-negative part of the spectrum of $((\not{D}+A^P)_\partial)_z$ for each $z \in B$.

We refer to A^P as a P-regularizing perturbation. These properties establish the following important priciple: the general APS family index theory defined by a spectral section P can always be reduced to the *invertible* case but only by passing to a *larger* class of operators. In any case, using these properties, we now have a b-determinant bundle defined in terms of the Fredholm family $(\not{D}+A^P)$, which we denote by ${}^b\mathcal{L}(\not{D}+A^P)$. Since we are again in the invertible case, we can use the b-calculus and introduce a b-Quillen metric and a b-Bismut-Freed connection, essentially as in the previous case.

Returning again to the surgery problem, this discussion shows that there are b-determinant bundles, ${}^b\mathcal{L}(\eth_{\overline{M}_+} + A^P_+)$ and ${}^b\mathcal{L}(\eth_{\overline{M}_-} + A^{(\mathrm{Id-P})}_-)$ endowed with b-Quillen metrics and Bismut-Freed connections, ${}^b\nabla^P_+$, ${}^b\nabla^{(\mathrm{Id-P})}_-$. The surgery calculus can be used to show the existence of an element in the (fibre) surgery calculus $A(\varepsilon) \in \Psi^{-\infty}_s$ with the property that as $\varepsilon \to 0$

$$\eth(\varepsilon) + A(\varepsilon) \longrightarrow (\eth_{\overline{M}_+} + A^P_+) \sqcup (\eth_{\overline{M}_-} + A^{(\mathrm{Id-P})}_-)$$

(in the precise sense of §4.3). Since the family $\eth(\varepsilon) + A(\varepsilon)$, $\varepsilon > 0$ is a perturbation by a family of smoothing operators of the family $\eth(\varepsilon)$, it is certainly Fredholm. The associated determinant bundle $\mathcal{L}(\eth(\varepsilon) + A(\varepsilon))$ can be endowed with a Quillen metric and Bismut-Freed connection. The arguments leading to the theorem above can now be extended (using the full force of the surgery and b-pseudodifferential calculi), resulting in the explicit isomorphism

$$S_P(\varepsilon) : \mathcal{L}(\eth(\varepsilon) + A(\varepsilon)) \longrightarrow {}^b\mathcal{L}(\eth_{\overline{M}_+} + A^P_+) \otimes {}^b\mathcal{L}(\eth_{\overline{M}_-} + A^{(\mathrm{Id-P})}_-)$$

and the (asymptotic) additivity of the curvatures and multiplicativity of the holonomies.

The final step is to show that these surgery formulæ for the curvature and the holonomy are *independent* of the particular choice of perturbations $A_+^P \sqcup A_-^{(\mathrm{Id}-\mathrm{P})}$ and $A(\varepsilon)$.

Consider first the closed case. Let $\eth$ be a Dirac family and A^0, A^1 two *smoothing* perturbations. We obtain two determinant bundles, $\mathcal{L}(\eth + A^0)$ and $\mathcal{L}(\eth + A^1)$, endowed with their hermitian structures. The space of smoothing perturbations is clearly simply connected; let $A(r), r \in [0,1]$, be a path of perturbations. Consider the family $\mathcal{D}$ on $B \times [0,1]$ given by $(\mathcal{D})_{(z,r)} = \eth_z + (A(r))_z$. This is a family of Fredholm operators and we can consider the associated determinant bundle. The latter is endowed with a Bismut-Freed connection $\nabla^{\mathcal{D}}$; the local anomaly formula of Bismut-Freed can now be applied, and it shows explicitly that the curvature of $\nabla^{\mathcal{D}}$ is zero in the dr-direction. Since the space of smoothing perturbations is simply connected, this shows that parallel transport defined by $\nabla^{\mathcal{D}}$ gives a *canonical* isomorhism $\tau : \mathcal{L}(\eth + A^0) \to \mathcal{L}(\eth + A^1)$ which preserves curvature and holonomy. This property can be applied to the pair of families $\eth(\varepsilon)$ and $\eth(\varepsilon) + A(\varepsilon)$ as well as the pair $\eth(\varepsilon) + A(\varepsilon)$ and $\eth(\varepsilon) + B(\varepsilon)$ for a different choice of perturbation $B(\varepsilon)$.

Consider now the boundary case and, as above, let $\not{D} = (\not{D}_z)_{z \in B}$ be a family of Dirac operator on manifolds with boundary. Denote by $\not{D}_\partial$ the boundary family and let P be a spectral section for $\not{D}_\partial$. It is not difficult to see, using the b-calculus, that the space of P-regularizing perturbations is simply connected. Let A_0^P and A_1^P two P-regularizing perturbations, $A^P(r)$ a path joining them and $\mathcal{D}_P$ the induced family on $B \times [0,1]$. The Bismut-Freed curvature formula is extended to manifolds with boundary in [54]. Applying the formula we discover that the dr-component of the curvature of the b-Bismut-Freed connection of the determinant bundle associated to $\mathcal{D}_P$ is *not* zero; however, and this is the key point, *it only depends on the boundary family* $(\mathcal{D}_P)_\partial$. When this argument is applied to the fibration $\overline{M} = \overline{M}_+ \sqcup \overline{M}_-$ the two contributions cancel out because of the different orientation of the normals; thus the parallel transport defined by the Bismut-Freed connection produces, as in the closed case, a *canonical* isomorphism preserving curvature and holonomy. This means that the surgery results established for the P-regularizing perturbation $A_+^P \sqcup A_-^{(\mathrm{Id}-\mathrm{P})}$ on $\overline{M} = \overline{M}_+ \sqcup \overline{M}_-$ and the surgery perturbation $A(\varepsilon)$, only depend on the family $\eth_M(\varepsilon)$, the limit families $\eth_{\overline{M}_\pm}$ and on the choice of spectral section P for the family of operators $\eth_H$ induced on the fibering hypersurface defining our decomposition $M = M_+ \cup_H M_-$. These results answer the questions raised at the end of §2.5 in the framework of the b-calculus. It is still an open problem as to whether the APS-framework, and the other two approaches to surgery, can be used to give similar answers.

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Indirizzo degli Autori:

Rafe Mazzeo – Stanford University – Departement of Mathematics – Stanford – California 94305. Supported by the NSF with a Young Investigator Fellowship and by grant #DMS-9226382
E-mail: mazzeo@math.stanford.edu

Paolo Piazza – Università "La Sapienza" – Dipartimento di Matematica "G. Castelnuovo" – P.le A. Moro 2 – 00185 Roma – Italia. This article was written while the author was visiting *Université Paris-Sud.* He would like to thank that institution for hospitality and the European Community for financial support.
E-mail: piazza@mat.uniroma1.it