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Strongly nonlinear elliptic unilateral problems having natural growth terms and L^1 data

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Riassunto: *Si dimostra un teorema di esistenza per una disequazione ellittica fortemente non lineare in L^1 con una condizione naturale di monotonia per la parte non lineare.*

Abstract: *An existence theorem for a strongly nonlinear elliptic inequality with an exact natural growth condition on the nonlinearity and an L^1 data is proved.*

1 – Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ and let $A(u) = -\operatorname{div} a(x, u, \nabla u)$ be a Leray-Lions operator defined on $W^{1,p}(\Omega)$, $1 < p < \infty$. Let $f \in L^1(\Omega)$.

L. Boccardo and T. Gallouët [8] proved the existence of at least one solution for the following nonlinear Dirichlet problem:

$$(1.1)\qquad \begin{aligned} &A(u) + g(x, u, \nabla u) = f \ \text{ in } \mathcal{D}'(\Omega),\\ &u \in W_0^{1,p}(\Omega) \quad \text{and } g(x, u, \nabla u) \in L^1(\Omega) \end{aligned}$$

where g is a nonlinearity having an "exact natural growth" with respect to $|\nabla u|$ (of order p) and which satisfies the classical "sign condition" with

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respect to u (see also [1]-[5] and [11] for related topics in the setting of Orlicz-Sobolev spaces).

It's our purpose in this paper to prove an existence theorem for the corresponding obstacle problem. Indeed, we prove the existence of at least one solution of the following unilateral problem:

$$(1.2)\quad \begin{cases} u \in K_\psi,\ g(x,u,\nabla u) \in L^1(\Omega)\,, \\ \langle A(u), T_k(v-u)\rangle + \int_\Omega g(x,u,\nabla u) T_k(v-u)dx \geq \int_\Omega f T_k(v-u)dx, \\ \text{for all } v \in K_\psi \text{ and all } k > 0, \end{cases}$$

where $K_\psi = \left\{ v \in W_0^{1,p}(\Omega) : v \geq \psi \text{ a.e. in } \Omega \right\}$ with ψ a measurable function on Ω such that $\psi^+ \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, and where T_k is the truncation operator at height $k > 0$, defined on $\mathbb{R}$ by:

$$(1.3)\qquad T_k(s) = s \quad \text{if} \quad |s| \leq k\,, \qquad T_k(s) = k\frac{s}{|s|} \quad \text{if} \quad |s| > k.$$

Note that the use of the truncation operator in (1.2) is justified by the following facts: *i)* If $f \in L^1(\Omega)$, the solution does not in general belong to $L^\infty(\Omega)$ (see [8] and Remark 2.3 below); *ii)* If $f \in W^{-1,p'}(\Omega)$ then, a solution of (1.2) is also a solution for the classical variational inequality, and conversely (see Remark 2.2 below). If $g \equiv 0$, existence results can be found in [12], [13].

2 – Main result

Let Ω be a bounded open subset of $\mathbb{R}^N$, and let

$$K_\psi = \left\{ v \in W_0^{1,p}(\Omega) : v \geq \psi \text{ a.e. in } \Omega \right\}$$

where $\psi : \Omega \to \overline{\mathbb{R}}$ is a measurable function on Ω such that

$$\psi^+ \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)\,.$$

Let $A :\ W_0^{1,p}(\Omega) \longrightarrow W^{-1,p'}(\Omega)$ be a mapping given by

$$A(u) = -\operatorname{div} a(x,u,\nabla u)$$

where $\frac{1}{p}+\frac{1}{p'}=1$ and where $a:\Omega\times\mathbb{R}\times\mathbb{R}^N\longrightarrow\mathbb{R}^N$ is a Carathéodory function satisfying for a.e. $x\in\Omega$, for all $s\in\mathbb{R}$ and ξ,ξ^* in $\mathbb{R}^N$ with $\xi\neq\xi^*$:

$$|a(x,s,\xi)|\leq c(x)+k_1\,|s|^{p-1}+k_2\,|\xi|^{p-1} \tag{2.1}$$

$$[a(x,s,\xi)-a(x,s,\xi^*)]\,[\xi-\xi^*]>0 \tag{2.2}$$

$$\alpha\,|\xi|^p\leq a(x,s,\xi)\xi \tag{2.3}$$

where $c(x)$ belongs to $L^{p'}(\Omega)$, $c\geq 0$, $k_1\geq 0, k_2\geq 0$ and $\alpha>0$.

Note that taking $\xi=te$, with $|e|=1$ and letting t tend to ± 0, (2.3) implies that $a(x,s,0)=0$.

Furthermore let $g(x,s,\xi):\Omega\times\mathbb{R}\times\mathbb{R}^N\longrightarrow\mathbb{R}$ be a Carathéodory function such that for a.e. $x\in\Omega$ and for all $s\in\mathbb{R}$, $\xi\in\mathbb{R}^N$:

$$g(x,s,\xi)s\geq 0 \tag{2.4}$$

$$|g(x,s,\xi)|\leq b(|s|)(c'(x)+(|\xi|^p) \tag{2.5}$$

$$|g(x,s,\xi)|\geq\beta\,|\xi|^p \quad \text{for } |s|\geq\gamma \tag{2.6}$$

where $b:\mathbb{R}^+\to\mathbb{R}$ is a continuous and non decreasing function and $c'(x)$ belongs to $L^1(\Omega)\,,c'\geq 0$ and $\beta>0,\gamma\geq 0$. Remark that in view of (2.5) and (2.6), $|g(x,s,\xi)|$ has a growth *exactly* of order $|\xi|^p$ when $|\xi|\to\infty$.

Finally let

$$f\in L^1(\Omega) \tag{2.7}$$

THEOREM 2.1. *Under the assumptions* (2.1)-(2.7)*, there exists at least one solution of* (1.2)*.*

REMARK 2.2. Let us remark that if the data f lies in $W^{-1,p'}(\Omega)$ and if u is a solution of the following variational inequality (see [6] for the existence of u):

$$(2.8)\qquad \begin{cases} u \in K_\psi,\ g(x,u,\nabla u) \in L^1(\Omega),\ g(x,u,\nabla u)u \in L^1(\Omega) \\ \langle A(u), v-u\rangle + \int_\Omega g(x,u,\nabla u)(v-u)dx \geq \langle f\ , v-u\rangle \\ \qquad\qquad \forall v \in K_\psi \cap L^\infty(\Omega), \end{cases}$$

then u is also a solution of (1.2). Conversely, if u is a solution of (1.2) and if $f \in W^{-1,p'}(\Omega)$ then u is a solution of (2.8).

Indeed, assume first that u is a solution of (2.8). Take $k > 0$, $v \in K_\psi$ and let h large enough such that $h \geq k + \|\psi^+\|_\infty$. Define $w = T_h(u) + T_k(v-u)$. It is easy to see that $w \in K_\psi \cap L^\infty(\Omega)$.

Using in (2.8) w as test function yields:

$$(2.9)\ \begin{cases} \langle A(u), T_h(u)-u+T_k(v-u)\rangle + \int_\Omega g(x,u,\nabla u)(T_h(u)-u+T_k(v-u))dx \\ \qquad\qquad \geq \langle f\ , T_h(u)-u+T_k(v-u)\rangle. \end{cases}$$

Note that $T_h(u) \to u$ strongly in $W_0^{1,p}(\Omega)$ as $h \to \infty$ and that for a.e. $x \in \Omega$

$$|g(x,u,\nabla u)(T_h(u)-u+T_k(v-u))| \leq 2g(x,u,\nabla u)u + k\,|g(x,u,\nabla u)| \in L^1(\Omega).$$

Passing to the limit as $h \to \infty$ in both sides of (2.9) gives:

$$\langle A(u), T_k(v-u)\rangle + \int_\Omega g(x,u,\nabla u)T_k(v-u)dx \geq \langle f, T_k(v-u)\rangle,$$

and u is thus a solution of (1.2).

Conversely, assume now that u is a solution of (1.2). Let us first prove that $g(x,u,\nabla u)u \in L^1(\Omega)$.

Using $v = \psi^+$ in (1.2) we obtain

$$\int_\Omega g(x,u,\nabla u)T_k(u-\psi^+)dx \leq \langle A(u), T_k(\psi^+-u)\rangle - \langle f, T_k(\psi^+-u)\rangle.$$

Since $(u-\psi^+)$ and u have the same sign, we have $g(x,u,\nabla u)(u-\psi^+)\geq 0$ for a.e. $x\in\Omega$. Letting $k\to\infty$ we get by using Fatou's lemma

$$\int_\Omega g(x,u,\nabla u)(u-\psi^+)dx \leq \langle A(u),\psi^+-u\rangle - \langle f,\psi^+-u\rangle < +\infty$$

which implies, since $g(x,u,\nabla u)\in L^1(\Omega)$ and $\psi^+\in L^\infty(\Omega)$ that

$$g(x,u,\nabla u)u\in L^1(\Omega).$$

Note that for a.e. $x\in\Omega$ we have

$$|g(x,u,\nabla u)T_k(v-u)| \leq |g(x,u,\nabla u)|\,\|v\|_\infty + g(x,u,\nabla u)u \in L^1(\Omega)$$

when $v\in L^\infty(\Omega)$. Going back to (1.2) and letting $k\to\infty$ we obtain by using Lebesgue's theorem in the second term and $T_k(v-u)\to v-u$ in $W_0^{1,p}(\Omega)$ in the first and third one:

$$\langle A(u),v-u\rangle+\int_\Omega g(x,u,\nabla u)(v-u)dx\geq\langle f,v-u\rangle\,,\ \text{for all } v\in K_\psi\cap L^\infty(\Omega)\,,$$

and u is thus a solution of (2.8).

Remark 2.3. The condition $\beta>0$ is necessary in order to obtain $u\in W_0^{1,p}(\Omega)$, since in the case of equations, the solution of $A(u)=f$ does not belong to $W_0^{1,p}(\Omega)$ when f belongs only to $L^1(\Omega)$: indeed it is well known that the solution belongs only to $W_0^{1,q}(\Omega)$ with $q<\frac{N(p-1)}{N-1}$ (see [9]).

Proof of Theorem 2.1. Some tools of this proof are inspired by [6], [7], [8] and [10].

Step 1: Consider the sequence of approximate problems:

$$(2.10)\quad \begin{cases} u_n\in K_\psi,\ g(x,u_n,\nabla u_n)\in L^1(\Omega),\ g(x,u_n,\nabla u_n)u_n\in L^1(\Omega)\\ \langle A(u_n),v-u_n\rangle+\int_\Omega g(x,u_n,\nabla u_n)(v-u_n)dx\geq\int_\Omega f_n(v-u_n)dx,\\ \qquad \forall v\in K_\psi\cap L^\infty(\Omega), \end{cases}$$

where f_n is a sequence of smooth functions which converges strongly to f in $L^1(\Omega)$ with $\|f_n\|_1\leq C_0$ for some constant C_0.

By Theorem 3.1 of [6], there exists at least one solution u_n of (2.10). Applying Remark 2.2, we also have:

$$(2.11)\quad \begin{cases} \langle A(u_n), T_k(v-u_n)\rangle + \int_\Omega g(x,u_n,\nabla u_n)T_k(v-u_n)dx \geq \int_\Omega f_n T_k(v-u_n)dx, \\ \forall v \in K_\psi \text{ and } \forall k > 0. \end{cases}$$

We shall prove that $(u_n)_n$ is bounded in $W_0^{1,p}(\Omega)$. For that we fix k for the remainder of this step, with $k \geq \gamma$ (where γ is given by (2.6)) (take for example $k = \gamma$).

Applying (2.11) with $v = \psi^+$ as test function one has

$$(2.12)\qquad \begin{cases} \langle A(u_n), T_k(u_n - \psi^+)\rangle + \int_\Omega g(x,u_n,\nabla u_n)T_k(u_n - \psi^+)dx \\ \leq \int_\Omega f_n T_k(u_n - \psi^+)dx. \end{cases}$$

Since $(u_n - \psi^+)$ and u_n have the same sign one has $g(x,u_n,\nabla u_n)T_k(u_n - \psi^+) \geq 0$ and since $a(x,s,0) = 0$, we have then:

$$\int_{\{|u_n-\psi^+|\leq k\}} a(x,u_n,\nabla u_n)\nabla(u_n - \psi^+)dx = \langle A(u_n), T_k(u_n - \psi^+)\rangle$$
$$\leq \int_\Omega f_n T_k(u_n - \psi^+)dx \leq C_0 k.$$

We deduce that:

$$\int_{\{|u_n-\psi^+|\leq k\}} a(x,u_n,\nabla u_n)\nabla u_n dx \leq$$
$$\leq C_0 k + \int_{\{|u_n-\psi^+|\leq k\}} |a(x,u_n,\nabla u_n)|\, |\nabla\psi^+|\, dx$$

which gives by using Young's inequality

$$\int_{\{|u_n-\psi^+|\leq k\}} a(x,u_n,\nabla u_n)\nabla u_n dx \leq$$
$$\leq C_0 k + \frac{1}{p'}\varepsilon^{p'} \int_{\{|u_n-\psi^+|\leq k\}} \left|\frac{1}{\mu} a(x,u_n,\nabla u_n)\right|^{p'} dx +$$
$$+ \frac{1}{p}\left(\frac{\mu}{\varepsilon}\right)^p \int_{\{|u_n-\psi^+|\leq k\}} |\nabla\psi^+|^p\, dx$$

where we choose $\mu = 3\max(k_1, k_2)$ and $\frac{\varepsilon^{p'}}{p'} = \frac{\alpha}{2}$. This implies

$$\begin{aligned}\int_{\{|u_n-\psi^+|\leq k\}} a(x,u_n,\nabla u_n)\nabla u_n dx \leq C_1 + \\ + \frac{\varepsilon^{p'}}{p'}\int_{\{|u_n-\psi^+|\leq k\}}\left[\frac{1}{\mu}\left|a(x,u_n,\nabla u_n)\right|\right]^{p'} dx\end{aligned} \tag{2.13}$$

where C_i $(i = 1, 2, ...)$ are various constants which do not depend on n (but which can depend on $k, \varepsilon, \psi^+, c(x), k_1, k_2, \beta$ and α).

Using (2.1) in (2.13) yields since $(a+b+c)^{p'} \leq 3^{p'}\left(a^{p'} + b^{p'} + c^{p'}\right)$

$$\begin{aligned}&\int_{\{|u_n-\psi^+|\leq k\}} a(x,u_n,\nabla u_n)\nabla u_n dx \leq \\ &\leq C_2 + \frac{\varepsilon^{p'}}{p'}\int_{\{|u_n-\psi^+|\leq k\}}|u_n|^p dx + \frac{\varepsilon^{p'}}{p'}\int_{\{|u_n-\psi^+|\leq k\}}|\nabla u_n|^p dx \leq \\ &\leq C_2 + \frac{\varepsilon^{p'}}{p'}\int_\Omega |k+\psi^+|^p dx + \frac{\varepsilon^{p'}}{p'}\int_{\{|u_n-\psi^+|\leq k\}}|\nabla u_n|^p dx \leq \\ &\leq C_3 + \frac{\varepsilon^{p'}}{p'}\int_{\{|u_n-\psi^+|\leq k\}}(k_4|\nabla u_n|)^p dx.\end{aligned} \tag{2.14}$$

Consequently using the coercivity (2.3) and $\frac{\varepsilon^{p'}}{p'} = \frac{\alpha}{2}$

$$\frac{\alpha}{2}\int_{\{|u_n-\psi^+|\leq k\}}|\nabla u_n|^p dx \leq C_3. \tag{2.15}$$

On the other hand one has because of (2.12) and $a(x,s,\xi)\xi \geq 0$

$$\begin{aligned}&\int_\Omega g(x,u_n,\nabla u_n)T_k(u_n-\psi^+)dx \leq \\ &\leq C_0 k - \int_{\{|u_n-\psi^+|\leq k\}} a(x,u_n,\nabla u_n)\nabla(u_n-\psi^+)dx \leq \\ &\leq C_0 k + \int_{\{|u_n-\psi^+|\leq k\}} a(x,u_n,\nabla u_n)\nabla\psi^+ dx.\end{aligned}$$

Since $\left(u_n\chi_{\{|u_n-\psi^+|\le k\}}\right)_n$ is bounded in $L^\infty(\Omega)$ and $\left(\nabla u_n\chi_{\{|u_n-\psi^+|\le k\}}\right)_n$ is bounded in $(L^p(\Omega))^N$ then in view of (2.1), $\left(a(x,u_n,\nabla u_n)\chi_{\{|u_n-\psi^+|\le k\}}\right)_n$ is bounded in $(L^{p'}(\Omega))^N$. This implies that

$$\left(\int_{\{|u_n-\psi^+|\le k\}} a(x,u_n,\nabla u_n)\nabla\psi^+ dx\right)_n$$

is bounded. So

$$\int_\Omega g(x,u_n,\nabla u_n)T_k(u_n-\psi^+)dx \le C_4,$$

which gives

$$k\int_{\{|u_n-\psi^+|>k\}} |g(x,u_n,\nabla u_n)|\, dx \le C_4.$$

Since $|u_n|\ge k$ when $|u_n-\psi^+|\ge k$, and since we have fixed $k\ge\gamma$, one has $|u_n|>\gamma$ whenever $|u_n-\psi^+|>k$. Consequently by (2.6)

$$\beta k\int_{\{|u_n-\psi^+|>k\}} |\nabla u_n|^p\, dx \le C_4. \tag{2.16}$$

Combining (2.15) and (2.16) we deduce that $(u_n)_n$ is bounded in $W_0^{1,p}(\Omega)$.

Passing to a subsequence, if necessary, we can assume that:

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega),\ \text{strongly in } L^p(\Omega) \text{ and a.e. in } \Omega. \tag{2.17}$$

Note that $u\in K_\psi$, which is the first statement in (1.2).

Step 2: Let now k such that $k\ge\|\psi^+\|_\infty$ and let $\delta=\left(\frac{b(k)}{2\alpha}\right)^2$. Let $\varphi(s)=s\,e^{\delta s^2}$, $z_n=T_k(u_n)-T_k(u)$, $\eta=e^{-4\delta k^2}$and $v=u_n-\eta\varphi(z_n)$. It is easy to see that we have $v\in K_\psi$ and that when $\delta\ge\left(\frac{b(k)}{2\alpha}\right)^2$ one has for all $s\in\mathbb{R}$:

$$\varphi'(s)-\frac{b(k)}{\alpha}|\varphi(s)|\ge\frac{1}{2}. \tag{2.18}$$

Using v as test function in (2.11) we get, for all $h>0$:

$$\langle A(u_n),T_h(\eta\varphi(z_n))\rangle+\int_\Omega g(x,u_n,\nabla u_n)T_h(\eta\varphi(z_n))dx\le\int_\Omega f_nT_h(\eta\varphi(z_n))dx.$$

Choosing $h > 2k$ one has since $|\eta\varphi(z_n)| \le |z_n| \le 2k$

$$\langle A(u_n), \varphi(z_n)\rangle + \int_\Omega g(x, u_n, \nabla u_n)\varphi(z_n)dx \le \int_\Omega f_n\varphi(z_n)dx. \tag{2.19}$$

Denote by $\varepsilon_1(n), \varepsilon_2(n), ...$ various sequences of real numbers which converge to zero when n tends to infinity.

Note that in (2.19), $\int_\Omega f_n\varphi(z_n)dx \to 0$ when $n \to \infty$, since $\varphi(z_n) \rightharpoonup 0$ weak* in $L^\infty(\Omega)$ and $f_n \to f$ strongly in $L^1(\Omega)$.

Since $g(x, u_n, \nabla u_n)\varphi(z_n) \ge 0$ on the subset $\{|u_n(x)| > k\}$ we deduce from (2.19) that

$$\langle A(u_n), \varphi(z_n)\rangle + \int_{\{|u_n| \le k\}} g(x, u_n, \nabla u_n)\varphi(z_n)dx \le \varepsilon_1(n). \tag{2.20}$$

On the one hand writing $\Omega = \{|u_n| \le k\} \cup \{|u_n| > k\}$ and using $a(x, s, 0) = 0$, we have:

$$\begin{aligned}\langle A(u_n), \varphi(z_n)\rangle &= \int_\Omega a(x, u_n, \nabla u_n)(\nabla T_k(u_n) - \nabla T_k(u))\varphi'(z_n)dx \\ &= \int_\Omega a(x, u_n, \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(u))\varphi'(z_n)dx + \\ &\quad - \int_{\{|u_n(x)|>k\}} a(x, u_n, \nabla u_n)\nabla T_k(u)\varphi'(z_n)dx.\end{aligned}$$

Since $\nabla T_k(u)\chi_{\{|u_n(x)|<k\}} \to 0$ in $(L^p(\Omega))^N$ strongly while $(a(x, u_n, \nabla u_n)\varphi'(z_n))_n$ is bounded in $\left(L^{p'}(\Omega)\right)^N$, and since $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ in $(L^p(\Omega))^N$ weak, we have

$$\begin{aligned}\langle A(u_n), \varphi(z_n)\rangle &= \int_\Omega [a(x, u_n, \nabla T_k(u_n)) - a(x, u_n, \nabla T_k(u))] \times \\ &\qquad \times [\nabla T_k(u_n) - \nabla T_k(u)]\,\varphi'(z_n)dx + \\ &\quad + \int_\Omega a(x, u_n, \nabla T_k(u))\,[\nabla T_k(u_n) - \nabla T_k(u)] \times \\ &\qquad \times \varphi'(z_n)dx + \varepsilon_2(n) = \\ &= \int_\Omega [a(x, u_n, \nabla T_k(u_n)) - a(x, u_n, \nabla T_k(u))] \times \\ &\qquad \times [\nabla T_k(u_n) - \nabla T_k(u)]\,\varphi'(z_n)dx + \varepsilon_3(n).\end{aligned} \tag{2.21}$$

On the other hand

$$
\begin{aligned}
&\left|\int_{\{|u_n|\le k\}} g(x,u_n,\nabla u_n)\varphi(z_n)dx\right| \le \\
&\le \int_{\{x\in\Omega:|u_n|\le k\}} b(k)(c'(x)+|\nabla u_n|^p)\,|\varphi(z_n)|\,dx \le \\
&\le \varepsilon_4(n)+b(k)\int_\Omega |\nabla T_k(u_n)|^p\,|\varphi(z_n)|\,dx \le \qquad (2.22)\\
&\le \varepsilon_4(n)+\frac{b(k)}{\alpha}\int_\Omega a(x,u_n,\nabla T_k(u_n))\nabla T_k(u_n)\,|\varphi(z_n)|\,dx \le \\
&\le \frac{b(k)}{\alpha}\int_\Omega [a(x,u_n,\nabla T_k(u_n))-a(x,u_n,\nabla T_k(u))]\times \\
&\quad \times[\nabla T_k(u_n)-\nabla T_k(u)]\,|\varphi(z_n)|\,dx+\varepsilon_5(n)
\end{aligned}
$$

Combining (2.20), (2.21) and (2.22) yields

$$
\begin{aligned}
&\int_\Omega [a(x,u_n,\nabla T_k(u_n))-a(x,u_n,\nabla T_k(u))]\times \\
&\times[\nabla T_k(u_n)-\nabla T_k(u)]\,(\varphi'(z_n)-\frac{b(k)}{\alpha}\,|\varphi(z_n)|\,dx\le\varepsilon_6(n)
\end{aligned}
\tag{2.23}
$$

which gives by using (2.18)

$$
\begin{aligned}
0\le \int_\Omega [a(x,u_n,\nabla T_k(u_n))-a(x,u_n,\nabla T_k(u))]\,[\nabla T_k(u_n)-\nabla T_k(u)] \le \\
\le 2\varepsilon_6(n)\to 0
\end{aligned}
$$

Therefore Lemma 5 of [7] implies

$$
\nabla T_k(u_n)\to\nabla T_k(u) \text{ strongly in } W_0^{1,p}(\Omega) \text{ for any fixed } k\ge\|\psi^+\|_\infty . \tag{2.24}
$$

Consequently there exists a subsequence, still denoted by $(u_n)_n$, such that:

$$
\nabla u_n\to\nabla u \quad \text{a.e. in } \Omega. \tag{2.25}
$$

STEP 3: We shall prove in this step that

$$
g(x,u_n,\nabla u_n)\to g(x,u,\nabla u) \text{ strongly in } L^1(\Omega)
$$

by using Vitali's theorem. Since $g(x,u_n,\nabla u_n)\to g(x,u,\nabla u)$ a.e. $x\in\Omega$ thanks to (2.17) and (2.25), we only have to prove that $(g(x,u_n,\nabla u_n))_n$ is uniformly equi-integrable in Ω.

Let $E\subset\Omega$ be a measurable subset of Ω. We have for any $m>0$:

$$\tag{2.26} \begin{cases} \int_E |g(x,u_n,\nabla u_n)|\,dx & \le \int_{E\cap\{|u_n|\le m\}} |g(x,u_n,\nabla u_n)|\,dx+ \\ & \quad + \int_{E\cap\{|u_n|>m\}} |g(x,u_n,\nabla u_n)|\,dx, \end{cases}$$

$$\int_{E\cap\{|u_n|\le m\}} |g(x,u_n,\nabla u_n)|\,dx \le b(m)\int_E (c'(x)+|\nabla T_m(u_n)|^p)dx.$$

Let $\varepsilon>0$ be given. In virtue of the strong convergence (2.24), there exists some $\rho(\varepsilon,m)>0$ which depends only on ε and m such that

$$\tag{2.27} \begin{aligned} &E \text{ measurable, } |E|<\rho(\varepsilon,m)\Rightarrow \\ &\quad\Rightarrow \int_{E\cap\{|u_n|\le m\}} |g(x,u_n,\nabla u_n)|\,dx \le \frac{\varepsilon}{2}, \quad \forall n. \end{aligned}$$

We now turn to the second term of the right-hand side of (2.26). Define $v_n=u_n-S_m(u_n)$, where for $m>1$,

$$\begin{cases} S_m(s)=0 & \text{if } |s|\le m-1 \\ S_m(s)=\dfrac{s}{|s|} & \text{if } |s|\ge m \\ S'_m(s)=1 & \text{if } m-1\le |s|\le m. \end{cases}$$

If $u_n\le m-1$ then $S_m(u_n)\le 0$ and $v_n\ge u_n\ge\psi$; if $u_n\ge m-1$ then since $0\le S_m(u_n)\le 1$ one has $v_n\ge u_n-1\ge m-2\ge\psi$ for $m\ge 2+\|\psi^+\|_\infty$.

Consequently v_n belongs to K_ψ. Using v_n as test function in (2.11) yields:

$$\langle A(u_n),T_k(S_m(u_n))\rangle+\int_\Omega g(x,u_n,\nabla u_n)T_k(S_m(u_n))dx\le\int_\Omega f_nT_k(S_m(u_n))dx$$

which implies by choosing $k\ge 1$

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n S'(u_n)dx+\int_\Omega g(x,u_n,\nabla u_n)S_m(u_n)dx\le\int_\Omega f_nS_m(u_n)dx.$$

So
$$\int_{\{|u_n|>m\}} |g(x,u_n,\nabla u_n)|\,dx \le \int_{\{|u_n|>m-1\}} |f_n|\,dx.$$

Since $f_n \to f$ strongly in $L^1(\Omega)$ and since $|\{|u_n| > m-1\}| \to 0$ uniformly in n when $m \to \infty$, there exists some $m(\varepsilon) > 1$ which only depends on ε such that:

$$\int_{\{|u_n|>m(\varepsilon)-1\}} |f_n|\,dx \le \frac{\varepsilon}{2}, \ \ \forall n,$$

and thus

$$\int_{\{|u_n|>m(\varepsilon)\}} |g(x,u_n,\nabla u_n)|\,dx \le \frac{\varepsilon}{2}, \ \ \forall n. \tag{2.28}$$

Fixing first $m = m(\varepsilon)$, and combining (2.26), (2.27) and (2.28), we obtain that there exists $\rho'(\varepsilon) = \rho(\varepsilon, m(\varepsilon))$ such that

$$\int_E |g(x,u_n,\nabla u_n)|\,dx \le \varepsilon \ \ ,\forall n \ \text{ when } |E| < \rho'(\varepsilon), E \text{ measurable}$$

which shows that $g(x,u_n,\nabla u_n)$ are uniformly equi-integrable in Ω as required.

Step 4: Go back to approximate problems (2.11). We have in particular:

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(v-u_n)dx + \int_\Omega g(x,u_n,\nabla u_n)T_k(v-u_n)dx \ge \ge \int_\Omega f_n T_k(v-u_n)dx \quad \forall v \in K_\psi \cap L^\infty(\Omega) \text{ and } \forall k > 0. \tag{2.29}$$

Since f_n tends to f in $L^1(\Omega)$ strongly and $g(x,u_n,\nabla u_n)$ tends to $g(x,u,\nabla u)$ in $L^1(\Omega)$ strongly, there is no problem to pass to the limit in the last and second terms of (2.29). For what concerns the first one, note that $a(x,u_n,\nabla u_n) \rightharpoonup a(x,u,\nabla u)$ weakly in $\left(L^{p'}(\Omega)\right)^N$, since $a(x,u_n,\nabla u_n)$ is bounded in $\left(L^{p'}(\Omega)\right)^N$ and $u_n \to u$, $\nabla u_n \to \nabla u$ a.e. in Ω; on the other hand, since $v \in L^\infty(\Omega)$, set $h = k + \|v\|_\infty$; then

$$|\nabla T_k(v-u_n)| = \chi_{\{|v-u_n|\le k\}} |\nabla v - \nabla u_n| \le \le \chi_{\{|u_n|\le h\}} |\nabla v - \nabla u_n| \le |\nabla v| + |\nabla T_h(u_n)|,$$

which implies, using Vitali's theorem with (2.24) and (2.25) that

$$\nabla T_k(v-u_n) \to \nabla T_k(v-u) \text{ in } \left(L^p\left(\Omega\right)\right)^N \text{strongly}$$

for any $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Letting $n \to \infty$ in both sides of (2.29), we get

$$\langle A(u), T_k(v-u)\rangle + \int_\Omega g(x,u,\nabla u) T_k(v-u)dx \geq \int_\Omega f T_k(v-u))dx \quad \forall v \in K_\psi \cap L^\infty(\Omega) \text{ and } \forall k > 0. \tag{2.30}$$

Taking for any $v \in K_\psi$ the test function $T_m(v)$ which belongs to $K_\psi \cap L^\infty(\Omega)$ for $m \geq \|\psi^+\|_\infty$ and passing to the limit in (2.30) as m tends to infinity completes the proof of Theorem 2.1.

Remark 2.4 Since we only know that $f \in L^1(\Omega)$ we can not hope for the existence of a solution u of

$$\begin{cases} u \in K_\psi, \ g(x,u,\nabla u) \in L^1(\Omega), \\ \langle A(u), v-u\rangle + \displaystyle\int_\Omega g(x,u,\nabla u)(v-u)dx \geq \int_\Omega f(v-u)dx, \\ \forall v \in K_\psi \cap L^\infty(\Omega) \end{cases} \tag{2.31}$$

because in general neither the right-hand side nor the term $\int_\Omega g(x,u,\nabla u)(v-u)dx$ are defined. Note indeed that in general $g(x,u,\nabla u)u \notin L^1(\Omega)$ (see Remark 3 of [8]).

Remark 2.5 If $\psi \geq 0$ then we can show that in addition to (1.2) we have

$$\langle A(u), v-T_k(u)\rangle + \int_\Omega g(x,u,\nabla u)(v-T_k(u))dx \geq \int_\Omega f(v-T_k(u))dx \quad \forall v \in K_\psi \cap L^\infty(\Omega).$$

Indeed, the use in (2.10) of the test function $v + T_h(u_n) - T_k(u_n)$ with $h \geq k$, which belongs to $K_\psi \cap L^\infty(\Omega)$ when $v \in K_\psi \cap L^\infty(\Omega)$, yields

$$\begin{aligned} &\langle A(u_n), v + T_h(u_n) - u_n - T_k(u_n)\rangle + \\ &\quad + \int_\Omega g(x,u_n,\nabla u_n)(v + T_h(u_n) - u_n - T_k(u_n))dx \geq \\ &\geq \int_\Omega f(v + T_h(u_n) - u_n - T_k(u_n))dx, \qquad \forall v \in K_\psi \cap L^\infty(\Omega). \end{aligned}$$

Since for n fixed $T_h(u_n)$ tends to u_n in $W_0^{1,p}(\Omega)$ strongly when h tends to infinity, passing to the limit in h for n fixed gives

$$\langle A(u_n), v-T_k(u_n)\rangle + \int_\Omega g(x,u_n,\nabla u_n)(v-T_k(u_n))dx \geq \int_\Omega f_n(v-T_k(u_n))dx, \quad \forall v \in K_\psi \cap L^\infty(\Omega),$$

in which it is easy to pass to the limit as n tends to infinity. This proves (2.32).

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