Rendiconti di Matematica, Serie VII
Volume 18, Roma (1998), 305-326

Local boundedness for minima of functionals with nonstandard growth conditions

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RIASSUNTO: *In questo articolo proviamo la locale limitatezza dei minimi locali di funzionali della forma*

$$\mathcal{F}(u) = \int_\Omega f(x, u, \nabla u)\, dx\,,$$

dove f soddisfa opportune ipotesi di convessità e la sua crescita rispetto al gradiente è controllata da una funzione di Young di classe Δ_2 e dalla sua coniugata di Sobolev. I risultati ottenuti estendono quelli già noti per funzionali con crescita p, q.

ABSTRACT: *We prove the local boundedness of local minimizers of functionals of the form*

$$\mathcal{F}(u) = \int_\Omega f(x, u, \nabla u)\, dx\,,$$

where f satisfies some convexity assumptions and its growth with respect to the gradient is controlled in terms of a Young function of Δ_2 class and its Sobolev conjugate. The results extend some boundedness theorems for minimizers of functionals satisfying the so called p, q-growth conditions.

1 – Introduction

Let Ω be an open subset of $\mathbb{R}^n$. Let us consider the variational

KEY WORDS AND PHRASES: *Integral functionals – Nonstandard growth conditions – Minimizers – Local boundedness.*
A.M.S. CLASSIFICATION: 49N60

integral

$$\mathcal{F}(u) = \int_\Omega f(x, u, \nabla u)\, dx \tag{1}$$

and the local minimizers for $\mathcal{F}$, i.e., functions $u \in W^{1,1}_{\text{loc}}(\Omega)$ such that $f(x, u, \nabla u) \in L^1_{\text{loc}}(\Omega)$, and

$$\int_{\text{supp}\,\varphi} f(x, u, \nabla u)\, dx \le \int_{\text{supp}\,\varphi} f(x, u+\varphi, \nabla u + \nabla\varphi)\, dx$$

for every $\varphi \in W^{1,1}(\Omega)$ with $\text{supp}\,\varphi \subset\subset \Omega$.

The problem of regularity of minimizers is one of the main questions concerning functionals of the form (1), and it has been widely studied in the last decades.

In the literature the integrand f is very often required to satisfy some growth conditions like, for instance,

$$c_0|\xi|^p - c_1 \le f(x, s, \xi) \le c_2(|\xi|^q + 1) \tag{2}$$

for almost every $x \in \Omega$, for every $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, where c_0, c_1, c_2 are positive constants and $1 < p \le q$.

If $p = q$, that is, if f satisfies the so-called natural growth conditions, there are so many significant contributions to the theory of regularity, starting from the pioneering papers by De Giorgi (see [3]), that it would be impossible to give an exhaustive list. If $p < q$, we say that f satisfies p, q-growth conditions. This case has been studied extensively in the last years, starting from some papers by Marcellini (see [11]–[14] and the references therein contained). In most of these papers, p and q must not be too far from each other in order to obtain regularity of local minimizers. For instance, if f does not depend on x and u, the condition $q < p^* = \frac{np}{n-p}$ assures that minimizers are locally bounded (see [17]). A result of higher integrability of the gradients of minimizers has been obtained by Fusco and Sbordone in [5] for functionals which depend only on the modulus of the gradient.

We will say that f satisfies general growth conditions if there exist two positive functions g_1 and g_2 and two positive constants c_1, c_2 such that

$$g_1(|\xi|) - c_1 \le f(x, s, \xi) \le c_2[g_2(|\xi|) + 1] \tag{3}$$

for almost every $x \in \Omega$, for every $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$. Some regularity results in the case where $g_1 = g_2$ have been given by Lieberman in [10], and Mascolo, Papi in [15] and [16]. Moreover Marcellini in [13] and [14] has proved the local Lipschitz-continuity of minimizers under general growth conditions on the second derivatives of f.

In this paper we consider general growth conditions on f, more precisely we assume that there exist a Young function g belonging to the class $\Delta_2^{(m)} \cap \nabla_2^{(r)}$ (see Definitions 2.2 and 2.4), and three positive constants c_1, c_2 and β, with $\beta < 1$, such that

$$g(|\xi|) - c_1 \leq f(x, s, \xi) \leq c_2 \left[1 + g^*(|s|) + g^*(|\xi|)\right]^\beta , \tag{4}$$

where g^* is the Sobolev conjugate function of g (see Definition 2.6). We allow f to depend also on x and u, with some convexity assumptions on f (see Section 3). By assuming (4), we will prove that the local minimizers of (1) are locally bounded. We point out that, if $g(t) = t^p$, $p > 1$, our result contains the local boundedness results previously known in the p, q-growth case.

On the other hand, condition (4) actually extends the class of the admissible integrands. Indeed there are significant cases in which (4) is satisfied but it is not possible to obtain any p, q-growth condition with $1 < p \leq q < p^*$. Some examples of this type, which cannot be treated using the previously known results, but which satisfy the hypotheses of our boundedness theorem, are considered in Section 3.

The plan of the paper is the following. In Section 2 we introduce the Young functions and the related Orlicz spaces, and give some properties which will be used in the sequel. In Section 3 we give the precise statement of the boundedness theorem, and some applications. Finally in Section 4 we give the proof of the result, which is based on a new version of Caccioppoli's inequality and on a suitable iteration method.

2 – Young functions and Orlicz spaces

In this section we will recall some definitions and well known results on Young functions and Orlicz-Sobolev spaces. A convex function $g : [0, +\infty) \to [0, +\infty)$ is called a Young function if $g(0) = 0$ and

$\lim_{t\to+\infty} g(t) = +\infty$. Let φ be the left derivative of g. Then φ is nondecreasing and left-continuous, and

$$g(t) = \int_0^t \varphi(s)\, ds\,, \quad t \in [0, +\infty)\,. \tag{5}$$

Moreover, by the convexity of g, one has

$$g(t) \le t\varphi(t)\,, \quad \text{for every } t > 0. \tag{6}$$

In this paper we only consider positive Young functions, that is, Young functions which are zero only for $t = 0$. A useful class of positive Young functions is that of the so called *N-functions*, that is, the positive Young functions such that

$$\lim_{t\to 0^+} \frac{g(t)}{t} = 0\,, \qquad \lim_{t\to+\infty} \frac{g(t)}{t} = +\infty\,.$$

If g is an N-function, (5) holds with φ nondecreasing, left-continuous, such that $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$ and $\lim_{t\to+\infty} \varphi(t) = +\infty$.

We will say that two positive Young functions g and h are *equivalent near infinity* if there exist positive constants t_0, k_1 and k_2 such that

$$h(k_1 t) \le g(t) \le h(k_2 t)$$

for every $t \ge t_0$.

We will now introduce some classes of positive Young functions which refine the notion of Δ_2 class (see [9] and [18]). A simple well known preliminary result will be needed. For sake of completeness we will give its proof.

PROPOSITION 2.1. *Let g be a positive Young function, and let φ be its left derivative. For $m \ge 1$, the following properties are equivalent:*

(i) *$t\varphi(t) \le m g(t)$, for every $t \ge 0$;*

(ii) *$g(\lambda t) \le \lambda^m g(t)$, for every $t \ge 0$, for every $\lambda > 1$;*

(iii) *the function $t^{-m} g(t)$ is nonincreasing.*

Proof. (i) ⇒ (ii): From (i) we obtain, for every $s > 0$:

$$\frac{\varphi(s)}{g(s)} \le \frac{m}{s}\,.$$

Integrating this between t and λt, we have

$$\ln \frac{g(\lambda t)}{g(t)} \le m \ln \lambda\,,$$

and therefore (ii) follows.

(ii) ⇒ (iii): If $\tau > t$, from (ii) we obtain

$$g(\tau) = g\left(\frac{\tau}{t}\,t\right) \le \frac{\tau^m}{t^m}\,g(t)\,,$$

that is,

$$\frac{g(\tau)}{\tau^m} \le \frac{g(t)}{t^m}\,.$$

(iii) ⇒ (i): For every t such that $\varphi(t)$ is continuous (that is, for all t except at most a countable number), by taking the derivative of $t^{-m}g(t)$, we obtain

$$0 \ge \frac{d}{dt}\frac{g(t)}{t^m} = \frac{t\varphi(t) - mg(t)}{t^{m+1}}\,.$$

Therefore (i) holds for these values of t. The result then follows by the left continuity of φ. □

Definition 2.2 Let $m \ge 1$. We will say that a positive Young function g belongs to the class $\Delta_2^{(m)}$ if any of the three conditions (i), (ii) or (iii) is satisfied.

It is easy to check that the usual Δ_2 class of positive Young functions (see [9] or [18] for the definition) is the union of the classes $\Delta_2^{(m)}$, for $m > 1$. The N-function t^m belongs to $\Delta_2^{(m)}$, while the N-function $g(t) = t^m \log^a(1+t)$, where $m > 1$ and $a \ge 1$, is equivalent near infinity, for every $\varepsilon > 0$, to a N-function in $\Delta_2^{(m+\varepsilon)}$. The same is true for $g(t) = t^m - \log(1+t)$. Moreover we observe that it follows from (iii) that every $g \in \Delta_2^{(m)}$ satisfies $g(t) \le c\,t^m$ for all $t \ge 1$. It is easy to see that the class $\Delta_2^{(1)}$ contains only linear Young functions.

The following proposition can be proved similarly to Proposition 2.1.

PROPOSITION 2.3. *Let g be a positive Young function, and let φ be its left derivative. Let r be a constant such that $r \geq 1$. Then the following conditions are equivalent:*

(i)′ $t\varphi(t) \geq rg(t)$, *for every* $t \geq 0$*;*

(ii)′ $g(\lambda t) \geq \lambda^r g(t)$, *for every* $t \geq 0$, *for every* $\lambda > 1$*;*

(iii)′ *the function* $t^{-r} g(t)$ *is nondecreasing.*

DEFINITION 2.4. Let $r \geq 1$. We will say that a positive Young function g belongs to the class $\nabla_2^{(r)}$ if any of the three conditions (i)′, (ii)′ or (iii)′ is satisfied.

By (6), every positive Young function belongs to $\nabla_2^{(1)}$. As before, one can show that the usual class ∇_2 (see [9] and [18]) is the union of the classes $\nabla_2^{(r)}$ for $r > 1$. Moreover, by (iii)′, if $g \in \nabla_2^{(r)}$, then $g(t) \geq ct^r$ for $t \geq 1$, with $c > 0$. For $r \geq 1$, the N-functions $g(t) = t^r$ and $g(t) = t^r \log^a (1 + t)$, with $a \geq 1$, belong to $\nabla_2^{(r)}$.

Let us introduce the Orlicz space associated with a Young function $g \in \Delta_2^{(m)}$, for $m \geq 1$ (see [9], [1], [18]). Let Ω be a measurable subset of $\mathbb{R}^n$ with finite Lebesgue measure. We denote by $L_g(\Omega)$ the set of all measurable functions u defined on Ω satisfying

$$\int_\Omega g(|u|)\, dx < +\infty\,.$$

As usual, we identify functions which differ on a set of zero measure. Since $g \in \Delta_2^{(m)}$, the set $L_g(\Omega)$ is a vector space. $L^g(\Omega)$ is a Banach space when endowed with the norm

$$||u||_{g,\Omega} = \inf \left\{ k > 0 : \int_\Omega g\left(\frac{|u|}{k}\right) dx \leq 1 \right\}. \tag{7}$$

For $||u||_{g,\Omega} > 0$, the infimum in (7) is attained and

$$\int_\Omega g\left(\frac{|u|}{||u||_{g,\Omega}}\right) dx = 1\,. \tag{8}$$

Moreover, it can be shown that Young functions which are equivalent near infinity generate the same Orlicz space.

Proposition 2.5. *Assume that* $g \in \Delta_2^{(m)}$, *for* $m \geq 1$, *and let* u *be a function in* $L_g(\Omega)$. *Then*

(*a*) $||u||_{g,\Omega} \leq 1$ *if and only if* $\int_\Omega g(|u|)\,dx \leq 1$;

(*b*) *If* $||u||_{g,\Omega} \leq 1$, *then*

$$||u||_{g,\Omega}^m \leq \int_\Omega g(|u|)\,dx\,; \tag{9}$$

(*c*) *If* $g \in \nabla_2^{(r)}$ *and* $||u||_{g,\Omega} \leq 1$, *then*

$$\int_\Omega g(|u|)\,dx \leq ||u||_{g,\Omega}^r\,. \tag{10}$$

Proof. (*a*) follows from (7) and (8).

By (ii) of Proposition 2.1 and (8), we have:

$$1 = \int_\Omega g\left(\frac{|u|}{||u||_{g,\Omega}}\right) dx \leq \frac{1}{||u||_{g,\Omega}^m} \int_\Omega g(|u|)\,dx\,,$$

which implies (9).

If $g \in \nabla_2^{(r)}$, (ii)′ of Proposition 2.3 implies

$$1 = \int_\Omega g\left(\frac{|u|}{||u||_{g,\Omega}}\right) dx \geq \frac{1}{||u||_{g,\Omega}^r} \int_\Omega g(|u|)\,dx\,,$$

which gives (10). □

Using the Young functions and the correspondent Orlicz spaces, it is possible to extend the classical notion of Sobolev spaces.

Let Ω be a bounded open set of $\mathbb{R}^n$. The Orlicz-Sobolev space $W^1L_g(\Omega)$ consists of those functions u in $L_g(\Omega)$ whose distributional derivatives belong to $L_g(\Omega)$. This is a Banach space under the norm[(1)]

$$||u||_{W^1L_g(\Omega)} = ||u||_{g,\Omega} + ||\nabla u||_{g,\Omega}\,. \tag{11}$$

[(1)] For simplicity of notation we write $||\nabla u||_{g,\Omega}$ instead of $||\,|\nabla u|\,||_{g,\Omega}$.

As for the usual Sobolev spaces, $W_0^1 L_g(\Omega)$ will denote the closure of $C_0^\infty(\Omega)$ with respect to the norm (11). An extensive study of these spaces can be found in [1], [4], [18].

We now recall the notion of Sobolev conjugate of a positive Young function (see [1], [4], [2]). For sake of simplicity, we will only consider the case of a function in $\Delta_2^{(m)}$.

DEFINITION 2.6. Assume that $g \in \Delta_2^{(m)}$, with $m < n$. We define the *Sobolev conjugate function* of g as the Young function g^* whose inverse is defined by

$$(g^*)^{-1}(t) = \int_0^t \frac{g^{-1}(s)}{s^{1+\frac{1}{n}}}\, ds\,. \tag{12}$$

It is easy to check, using condition (ii) of Proposition 2.1, that $g^{-1}(s) \leq cs^{1/m}$ for $s \leq 1$. Therefore the integral in (12) is finite, and it is easy to verify that g^* is a positive Young function. For every number $m \in [1, n)$, we denote by m^* the usual Sobolev conjugate exponent of m, i.e., $m^* = mn/(n - m)$. In the case where $g(t) = t^m$, with $m < n$, then $g^*(t) = (t/m^*)^{m^*}$. Moreover, if $g(t)$ is equivalent near infinity to $t^m (\log(1 + t))^a$, with $1 \leq m < n$ and $a \geq 1$, then $g^*(t)$ is equivalent near infinity to $t^{m^*} (\log(1 + t))^{na/(n-m)}$ (see [2]).

PROPOSITION 2.7. *Let g be a positive Young function in $\Delta_2^{(m)} \cap \nabla_2^{(r)}$, with $r, m \in [1, n)$. Then $g^* \in \Delta_2^{(m^*)} \cap \nabla_2^{(r^*)}$.*

PROOF. Since $g \in \nabla_2^{(r)}$, for every $\lambda > 1$ one has $g^{-1}(\lambda t) \leq \lambda^{1/r} g^{-1}(t)$. Therefore

$$(g^*)^{-1}(\lambda t) = \int_0^{\lambda t} \frac{g^{-1}(s)}{s^{1+\frac{1}{n}}}\, ds = \lambda \int_0^t \frac{g^{-1}(\lambda s)}{(\lambda s)^{1+\frac{1}{n}}}\, ds \leq \lambda^{\frac{1}{r}-\frac{1}{n}} (g^*)^{-1}(t)\,,$$

which implies $g^*(\lambda t) \geq \lambda^{r^*} g^*(t)$. Thus inequality (ii)$'$ of Proposition 2.3 holds with exponent r^*. If $g \in \Delta_2^{(m)}$, since in this case $g^{-1}(\lambda t) \geq \lambda^{1/m} g^{-1}(t)$ for every $\lambda > 1$, we obtain

$$(g^*)^{-1}(\lambda t) \geq \lambda^{\frac{1}{m}-\frac{1}{n}} \int_0^t \frac{g^{-1}(s)}{s^{1+\frac{1}{n}}}\, ds\,,$$

which implies that g^* satisfies (ii) of Proposition 2.1, with exponent m^*. □

The following embedding theorem holds (see [1], [4], [2]):

THEOREM 2.8. *Let Ω be an open set of $\mathbb{R}^n$ with smooth boundary, and let g be a positive Young function in $\Delta_2^{(m)}$, with $1 \leq m < n$. Then*

$$W^1 L_g(\Omega) \subset L_{g^*}(\Omega)\,,$$

and the embedding is continuous. Moreover, if $u \in W_0^1 L_g(\Omega)$, one has

$$||u||_{g^*,\Omega} \leq c\,||\nabla u||_{g,\Omega}\,,$$

where c depends on n, g and Ω.

3 – Main result and applications

Let Ω be an open set, $\Omega \subset \mathbb{R}^n$. Let $f = f(x,s,\xi)$ be a function defined for $x \in \Omega$, $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, with values in $\mathbb{R}$. We will assume that f is a Caratheodory function, that is, it is continuous with respect to (s,ξ) for almost every $x \in \Omega$, and it is measurable with respect to x for every $s \in \mathbb{R}$ and every $\xi \in \mathbb{R}^n$.

We will assume that f satisfies

(f_1) there exist a positive Young function $g \in \Delta_2^{(m)} \cap \nabla_2^{(r)}$, with $r \geq 1$ and $1 \leq m < \min\{r^*, n\}$, and three positive constants c_1, c_2 and β, with $\beta < 1$, such that

$$g(|\xi|) - c_1 \leq f(x,s,\xi) \leq c_2[1 + g^*(|s|) + g^*(|\xi|)]^\beta\,, \tag{13}$$

for almost every $x \in \Omega$, for every $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$.

Moreover we will assume that f satisfies one of the following properties

(f_2) for almost every $x \in \Omega$, $f(x,\cdot,\cdot)$ is a convex function;

$(f_2)'$ for almost every $x \in \Omega$ and for every $s \in \mathbb{R}$, $f(x,s,\cdot)$ is a convex function; moreover, there exists a constant $c_3 > 0$ such that

$$f(x,s_1,\xi) \leq c_3 f(x,s_2,\xi)$$

for almost every $x \in \Omega$, for every pair $s_1, s_2 \in \mathbb{R}$, with $|s_1| \leq |s_2|$, and for every $\xi \in \mathbb{R}^n$.

Let us define the integral functional

$$\mathcal{F}(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx \,. \tag{14}$$

We shall consider local minimizers of $\mathcal{F}$, i.e., functions $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ such that

$$\int_{\Omega'} f(x, u(x), \nabla u(x))\, dx < +\infty \tag{15}$$

for every $\Omega' \subset\subset \Omega$, and

$$\int_{\mathrm{supp}\,\varphi} f(x, u, \nabla u)\, dx \leq \int_{\mathrm{supp}\,\varphi} f(x, u + \varphi, \nabla u + \nabla\varphi)\, dx \,, \tag{16}$$

for every $\varphi \in W^{1,1}(\Omega)$, with $\mathrm{supp}\,\varphi \subset\subset \Omega$.

REMARK 3.1. We observe that, if (15) holds, then u belongs to $W^1_{\mathrm{loc}}L_g(\Omega)$. Indeed, by the growth condition (13), $|\nabla u| \in L_{g,\mathrm{loc}}(\Omega)$ and this in turn implies that $u \in L_{g,\mathrm{loc}}(\Omega)$. □

We now state the main result of this paper. For $R > 0$, let Q_R be a cube of $\mathbb{R}^n$ with center $x_0 \in \Omega$ and side $2R$, such that $Q_R \subset\subset \Omega$.

THEOREM 3.2. *Let u be a local minimizer of the functional $\mathcal{F}$, with f satisfying either conditions $(f_1), (f_2)$ or $(f_1), (f_2)'$. Then u is locally bounded in Ω, and there exists $R_0 > 0$ such that, for every R, $0 < R < R_0$, and for every cube Q_R with closure contained in Ω, the following estimate holds:*

$$\sup_{Q_{R/2}} g^*(|u|) \leq 1 + C \left[\int_{Q_R} [g^*(|u|)]^\beta\, dx \right]^{\frac{r^*-m}{m(1-\beta)}} \,, \tag{17}$$

where C depends on R, g, n, Ω.

In the rest of this section we will make some comments and give some examples to which this theorem applies.

REMARK 3.3. If we consider a function f of the form

$$f(x, s, \xi) = h(|s|)g(\xi)\,,$$

with h and g nonnegative, then condition $(f_2)'$ is satisfied if g is convex and h is nondecreasing, or more generally if h satisfies

$$a(s) \le h(s) \le ca(s)\,,$$

for every $s > 0$, where a is a positive, nondecreasing function, and c is a positive constant. □

REMARK 3.4. Assume that f satisfies p, q-growth conditions of the form (2). Then Theorem 3.2, applied with $g(t) = t^p \in \Delta_2^{(p)} \cap \nabla_2^{(p)}$, assures local boundedness of local minimizers of $\mathcal{F}$ as soon as $q < p^*$. This result generalizes the boundedness theorem proved in [17] in the case where f does not depend on x and u and satisfies conditions of the Δ_2 type. □

REMARK 3.5. Let $g(t) = t \ln^a(1+t)$, with $a \ge 1$. We recall that, for every $\varepsilon > 0$, g is equivalent near infinity to a function in $\Delta_2^{(1+\varepsilon)}$, but g is not equivalent to any function in $\nabla_2^{(r)}$ with $r > 1$. The Sobolev conjugate of g is equivalent near infinity to the function $h(t) = t^{1^*}(\ln(1+t))^{\frac{na}{n-1}}$. If f satisfies (13) with this choice of g, then Theorem 3.2 can be applied since one can take ε such that $1 + \varepsilon < 1^*$. A similar remark applies to the cases $g(t) = t$ or $g(t) = t - \log(1+t)$. □

REMARK 3.6. Let us consider a positive Young function $g \in \Delta_2^{(m)} \cap \nabla_2^{(r)}$ satisfying the following growth condition:

$$c_1 t^p - c_2 \le g(t) \le c_3(t^q + 1)\,,$$

with $1 \le p \le q < n$. It follows from the definition that its Sobolev conjugate g^* satisfies:

$$c_1' t^{p^*} - c_2' \le g^*(t) \le c_3'(t^{q^*} + 1)\,. \tag{18}$$

for suitable constants $c_1', c_2', c_3' > 0$. In Example 3.7 below, we exhibit a function g for which the powers appearing in (18) are sharp. Therefore, if we consider an integrand $f(x,s,\xi)$ satisfying condition (13) with this choice of g, then f satisfies the following power growth condition:

$$c_1'' t^p - c_2'' \leq f(x,s,\xi) \leq c_3''(t^{q^*\beta}+1)\,.$$

Therefore, if one wants to apply the existing results proved for p,q-growth conditions, recalled in Remark 3.4, one has to impose that $q^*\beta < p^*$, that is, $\beta < p^*/q^*$, while our results allows the weaker condition $\beta < 1$.

For instance one can take

$$f(x,s,\xi) = a(x,s)\left[g(|\xi|) + g^*(|\xi|)^\beta\right]\,,$$

where $\beta < 1$ and $a(x,s)$ is a Carathéodory function satisfying $\lambda \leq a(x,s) \leq \Lambda$, for some positive constants λ, Λ.

As a further example, one could consider an integrand of the form

$$f(x,s,\xi) = a(x,s)g(|\xi|)\,,$$

with $0 < \lambda \leq a(x,s) \leq \Lambda[1 + g^*(|s|)^\gamma]$, $\gamma < \frac{1}{n}$. Since one can check that $g(t) \leq c(1 + g^*(t)^{\frac{n-1}{n}})$, f satisfies the growth condition (f_1) for a suitable $\beta \in (0,1)$. □

EXAMPLE 3.7. For every p, q such that $1 < p < q < n$, and for every $\varepsilon > 0$, we construct an N-function $g \in \Delta_2^{(q+\varepsilon)} \cap \nabla_2^{(p-\varepsilon)}$ such that

$$t^p \leq g(t) \leq 1 + t^q \tag{19}$$

for every $t \in \mathbb{R}$ and such that its Sobolev conjugate satisfies

$$c_1 t^{p^*} \leq g^*(t) \leq c_2(1 + t^{q^*})\,, \tag{20}$$

with sharp exponents. Define $a = (p+q)/2$, $b = (q-p)/2$, and consider the function

$$h(t) = \begin{cases} t^p & \text{if } t \leq \tau_0\,, \\ t^{a+b\sin\ln\ln\ln t} & \text{if } t > \tau_0\,, \end{cases} \tag{21}$$

where τ_0 is such that $\sin\ln\ln\ln\tau_0 = -1$. First of all, we observe that the function $h(t)$ oscillates between the function t^p, to which it is tangent for

t such that $\sin\ln\ln\ln t = -1$, and the function t^q, to which it is tangent for $\sin\ln\ln\ln t = 1$. It is easy to see that, for every $\varepsilon > 0$, it is possible to choose τ_0 large enough such that h is convex and $h \in \Delta_2^{(q+\varepsilon)} \cap \nabla_2^{(p-\varepsilon)}$. We now modify the function h to obtain a function g with the same growth (19) such that g^* satisfies (20) with sharp exponents. To do this, we construct four increasing sequences of positive numbers $\{s_k\}$, $\{t_k\}$, $\{\sigma_k\}$, $\{\tau_k\}$ such that

$$\ldots < s_k < t_k < \sigma_k < \tau_k < s_{k+1} < \ldots$$

and $\sin\ln\ln\ln s_k = \sin\ln\ln\ln t_k = 1$, $\sin\ln\ln\ln \sigma_k = \sin\ln\ln\ln \tau_k = -1$.

The new function $g(t)$ will be equal to t^q in $[s_k, t_k]$, and to t^p in $[\sigma_k, \tau_k]$. In the remaining intervals we take $g(t) = h(t)$. Then $g \in C^1(\mathbb{R}^+)$ and is convex. We will show that it is possible to choose the four sequences so that the Sobolev-conjugate g^* satisfies

$$\limsup_{t\to+\infty} \frac{g^*(t)(q^*)^{q^*}}{t^{q^*}} = 1\,, \qquad \liminf_{t\to+\infty} \frac{g^*(t)(p^*)^{p^*}}{t^{p^*}} = 1\,. \tag{22}$$

We take as s_1 the first value greater than τ_0 such that $\sin\ln\ln\ln s_1 = 1$. Similarly, once τ_k is fixed, we will take as s_{k+1} the first value greater than τ_k such that $\sin\ln\ln\ln s_{k+1} = 1$. For fixed s_k, we show how to choose t_k. Since $g(t) = t^q$ in $[s_k, t_k]$, we have

$$\begin{aligned}(g^*)^{-1}(t_k^q) &= \int_0^{t_k^q} \frac{g^{-1}(t)}{t^{1+\frac{1}{n}}}\,dt = \\ &= \int_0^{s_k^q} \frac{g^{-1}(t)}{t^{1+\frac{1}{n}}}\,dt + \int_{s_k^q}^{t_k^q} \frac{g^{-1}(t)}{t^{1+\frac{1}{n}}}\,dt = C_k + q^* t_k^{q/q^*}\,,\end{aligned} \tag{23}$$

where C_k depends on s_k and the values of $g(t)$ for $t \le s_k$. If we set $\xi_k = C_k + q^* t_k^{q/q^*}$, (23) implies

$$g^*(\xi_k) = \left(\frac{\xi_k - C_k}{q^*}\right)^{q^*}.$$

We now choose t_k satisfying $\sin\ln\ln\ln t_k = 1$ so large that $\xi_k \ge k$ (so that $\xi_k \to +\infty$) and

$$\left(\frac{q^*}{\xi_k}\right)^{q^*} g^*(\xi_k) \ge 1 - \frac{1}{k}\,.$$

We now take as σ_k the first value greater than t_k such that $\sin\ln\ln\ln\sigma_{k+1} = -1$. With the same argument used for the choice of t_k, once σ_k has been fixed we can take τ_k large enough in order to find a number η_k such that $\eta_k \geq k$ and

$$\left(\frac{p^*}{\eta_k}\right)^{p^*} g^*(\eta_k) \leq 1 + \frac{1}{k}\,.$$

Since

$$\left(\frac{t}{p^*}\right)^{p^*} \leq g^*(t) \leq \left(\frac{t+c_3}{q^*}\right)^{q^*}\,,$$

(22) is satisfied. Finally, it is easy to check, using statements (iii) and (iii)′ of Propositions 2.1 and 2.3, that $g \in \Delta_2^{(q+\varepsilon)} \cap \nabla_2^{(p-\varepsilon)}$. ∎

4 – Caccioppoli's inequality and proof of the main theorem

The first result of this section is a modified version of Caccioppoli's inequality. For $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and $k \geq 0$, we define the set

$$A(k,R) = \{x \in Q_R : u(x) > k\}\,.$$

THEOREM 4.1. *Assume that f satisfies one of the hypotheses (f_2), $(f_2)'$, and that, for almost every $x \in \Omega$, for every $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$,*

$$-c_1 \leq f(x,s,\xi) \leq c_2[1 + H(|\xi|) + H(|s|)]\,, \tag{24}$$

where c_1 and c_2 are positive constants, and $H(t)$ is a positive, increasing function such that, for every $t > 0$ and $\lambda > 1$,

$$H(\lambda t) \leq \lambda^m H(t)\,,$$

where $m \geq 1$. Let u be a local minimizer of the functional $\mathcal{F}$. Then there exists a constant C depending only on c_1, c_2, m and diam Ω *such that, for every cube $Q_R \subset\subset \Omega$, for every $0 < \sigma < 1$ and $k > 0$, u satisfies*

$$\begin{aligned}\int_{A(k,R(1-\sigma))} & f(x,u,\nabla u)\,dx \leq \\ & \leq C\left[\frac{1}{(R\sigma)^m}\int_{A(k,R)} H(u-k)\,dx + (1+H(k))|A(k,R)|\right]\,,\end{aligned} \tag{25}$$

where we denote by $|E|$ the Lebesgue measure of a measurable set E.

To prove this result we will need the following lemma (see [6], Section V, Lemma 3.1)

LEMMA 4.2. *Let $\phi(t)$ be a nonnegative bounded function, defined in $[\tau_0, \tau_1]$. Suppose that, for $\tau_0 \le s < t \le \tau_1$, ϕ satisfies*

$$\phi(s) \le \frac{A}{(t-s)^m} + B + \theta\phi(t)\,,$$

where A, B, m, θ are nonnegative constants with $0 \le \theta < 1$. Then for all ρ and R such that $\tau_0 \le \rho < R \le \tau_1$ one has

$$\phi(\rho) \le c\,\left[\frac{A}{(R-\rho)^m} + B\right]\,,$$

where c is a constant depending on m and θ.

PROOF OF THEOREM 4.1. By replacing f with $f + c_1$, we can assume $f \ge 0$. For $\sigma \in (0,1)$, let s and t be positive real numbers such that $R(1-\sigma) \le s < t \le R$, and let η be a function in $C_0^\infty(Q_t)$ such that $0 \le \eta \le 1$, $\eta = 1$ on Q_s and $|\nabla\eta| \le \frac{2}{t-s}$. Consider $\varphi = -\eta^m w$, where $w = (u-k)_+ = \max\{u-k, 0\}$. It follows that $\varphi = 0$ outside of $A(k,t)$, and that, on this set,

$$\nabla\varphi = -\eta^m \nabla u - m\eta^{m-1}(u-k)\nabla\eta\,.$$

Using the definition (16) of local minimizer with this choice of φ, we obtain

$$\begin{aligned}\int_{A(k,t)} f(x,u,\nabla u)\,dx &\le \int_{A(k,t)} f(x, u+\varphi, \nabla u + \nabla\varphi)\,dx = \\ &= \int_{A(k,t)} f(x, (1-\eta^m)u + \eta^m k, (1-\eta^m)\nabla u + \eta^m[\frac{m}{\eta}(k-u)\nabla\eta])\,dx\,.\end{aligned}$$

If (f_2) holds, the convexity of f and the properties of H imply:

$$\begin{aligned}&f(x, (1-\eta^m)u + \eta^m k, (1-\eta^m)\nabla u + \eta^m[\frac{m}{\eta}(k-u)\nabla\eta]) \le \\ &\le (1-\eta^m) f(x,u,\nabla u) + \eta^m f(x, k, \frac{m}{\eta}(k-u)\nabla\eta) \le \\ &\le (1-\eta^m) f(x,u,\nabla u) + c_2\eta^m\left[H(\frac{m}{\eta}(u-k)|\nabla\eta|) + 1 + H(k)\right] \le \\ &\le (1-\eta^m) f(x,u,\nabla u) + c_2\left[\frac{(2m\,\max\{1,R\})^m}{(t-s)^m} H(u-k) + 1 + H(k)\right]\,,\end{aligned}$$

where in the last passage we have used the bound $t-s \leq R$.

On the other hand, if $(f_2)'$ is satisfied, then, using the convexity with respect to ξ, and recalling that $u > k$ on $A(k,t)$, we obtain that on this set

$$\begin{aligned} & f(x,(1-\eta^m)u+\eta^m k,(1-\eta^m)\nabla u+\eta^m(\frac{m}{\eta}(k-u)\nabla\eta)) \leq \\ & \quad \leq (1-\eta^m)f(x,(1-\eta^m)u+\eta^m k,\nabla u)+ \\ & \quad\quad +\eta^m f(x,(1-\eta^m)u+\eta^m k,\frac{m}{\eta}(k-u)\nabla\eta) \leq \\ & \quad \leq c_3(1-\eta^m)f(x,u,\nabla u)+c_2\eta^m\left[H(\frac{m}{\eta}(u-k)|\nabla\eta|)+1+H(u)\right]. \end{aligned}$$

Since H is increasing, for every $a, b \geq 0$ one has $H(a+b) \leq H(2a)+H(2b)$, and therefore, for almost every x in $A(k,t)$,

$$H(u) \leq H(2(u-k))+H(2k) \leq 2^m[H(u-k)+H(k)]\,.$$

We thus obtain

$$\begin{aligned} & f(x,(1-\eta^m)u+\eta^m k,(1-\eta^m)\nabla u+\eta^m(\frac{m}{\eta}(k-u)\nabla\eta)) \leq \\ & \leq c_3(1-\eta^m)f(x,u,\nabla u)+c_4\left[\frac{(2m\ \max\{1,R\})^m}{(t-s)^m}H(u-k)+1+H(k)\right], \end{aligned}$$

where c_4 depends only on c_2 and m.

In both cases, since $R < \frac{1}{2}\operatorname{diam}\Omega$, supp $(1-\eta^m) \subset A(k,t) \setminus A(k,s)$ and $A(k,s) \subset A(k,t) \subset A(k,R)$, we obtain

$$\begin{aligned} \int_{A(k,s)} f(x,u,\nabla u)\,dx & \leq c_5 \int_{A(k,t)\setminus A(k,s)} f(x,u,\nabla u)\,dx+ \\ & + \frac{c_5}{(t-s)^m}\int_{A(k,R)} H(u-k)\,dx + c_5(1+H(k))|A(k,R)|\,. \end{aligned}$$

Adding $c_5 \int_{A(k,s)} f(x,u,\nabla u)\,dx$ to both sides of the inequality, we obtain

$$\begin{aligned} \int_{A(k,s)} f(x,u,\nabla u)\,dx & \leq c_6 \int_{A(k,t)} f(x,u,\nabla u)\,dx+ \\ & + \frac{c_6}{(t-s)^m}\int_{A(k,R)} H(u-k)\,dx + c_6(1+H(k))|A(k,R)|\,, \end{aligned}$$

where $c_6 = c_5/(c_5+1) < 1$. Applying Lemma 4.2 with

$$\begin{aligned} \phi(t) = \int_{A(k,t)} f(x,u,\nabla u)\,dx, &\qquad A = c_6 \int_{A(k,R)} H(u-k)\,dx, \\ B = c_6(1+H(k))|A(k,R)|, &\qquad \theta = c_6, \\ \tau_0 = \rho = R(1-\sigma), &\qquad \tau_1 = R, \end{aligned}$$

we obtain the desired result. □

In order to give the proof of Theorem 3.2 we recall a well known iteration lemma (see [8], Lemma 7.1):

Lemma 4.3. *Let $\{J_h\}$ be a sequence of positive real numbers, such that*

$$J_{h+1} \le C B^h J_h^{1+\alpha},$$

with $C > 0$, $\alpha > 0$, $B > 1$. Then, if J_0 satisfies

$$J_0 \le C^{-\frac{1}{\alpha}} B^{-\frac{1}{\alpha^2}},$$

one has

$$J_h \le B^{-\frac{h}{\alpha}} J_0,$$

and therefore J_h tends to zero as $h \to +\infty$.

Proof of Theorem 3.2. Let u be a local minimizer of $\mathcal{F}$, and let $Q_R \subset\subset \Omega$. We consider the following sequences of radii:

$$\rho_h = \frac{R}{2}\left(1 + \frac{1}{2^h}\right), \qquad \overline{\rho}_h = \frac{\rho_h + \rho_{h+1}}{2}, \qquad h = 0,1,2,\dots$$

Let d be a positive constant to be chosen later, and define the following sequence of levels of u:

$$k_h = d\left(1 - \frac{1}{2^{h+1}}\right), \qquad h = 0,1,2,\dots$$

For $h \in \mathbb{N}$, let us consider

$$J_h = \int_{A(k_h,\rho_h)} [g^*(u-k_h)]^\beta\,dx \qquad h = 0,1,2,\dots \tag{26}$$

Since $\{\rho_h\}$ is decreasing and $\{k_h\}$ is increasing, the sequence $\{J_h\}$ decreases with h. By the absolute continuity of the integral, choosing R_0 small enough we can assume that $J_h \leq J_0 \leq 1$ for every h and that $R_0 < 1$. Let now η_h be a smooth function such that

$$\begin{aligned} &\operatorname{supp}\eta_h \subset Q_{\overline{\rho}_h}\,, \quad 0 \leq \eta_h \leq 1\,, \\ &\eta_h = 1 \text{ on } Q_{\rho_{h+1}}, \quad |\nabla\eta_h| \leq \frac{2^{h+4}}{R}\,. \end{aligned} \tag{27}$$

By Hölder's inequality, we get

$$\begin{aligned} J_{h+1} \leq& \int_{A(k_{h+1},\overline{\rho}_h)} [g^*(\eta_h(u-k_{h+1}))]^\beta\,dx \leq \\ \leq& |A(k_{h+1},\overline{\rho}_h)|^{1-\beta}\left(\int_{A(k_{h+1},\overline{\rho}_h)} g^*(\eta_h(u-k_{h+1}))\,dx\right)^\beta\,. \end{aligned} \tag{28}$$

Once again, for R_0 small enough, we can assume that

$$||\eta_h(u-k_{h+1})||_{g^*,A(k_{h+1},\overline{\rho}_h)} \leq 1\,.$$

Then, using Proposition 2.7 and part (c) of Proposition 2.5, and applying the Orlicz-Sobolev embedding Theorem 2.8, we obtain:

$$\begin{aligned} J_{h+1} \leq& |A(k_{h+1},\overline{\rho}_h)|^{1-\beta}\,||\eta_h(u-k_{h+1})_+||^{r^*\beta}_{g^*,Q_{\overline{\rho}_h}} \leq \\ \leq& c\,|A(k_{h+1},\overline{\rho}_h)|^{1-\beta}||\nabla(\eta_h(u-k_{h+1}))||^{r^*\beta}_{g,A(k_{h+1},\overline{\rho}_h)}\,, \end{aligned}$$

where $c = c(\Omega, n, g, \beta)$ (in the following, we allow c to assume different values from line to line). By the properties of η_h, we have

$$\begin{aligned} &||\nabla(\eta_h(u-k_{h+1}))||_{g,A(k_{h+1},\overline{\rho}_h)} \leq \\ &\quad \leq ||\nabla u||_{g,A(k_{h+1},\overline{\rho}_h)} + ||\nabla\eta_h(u-k_{h+1})||_{g,A(k_{h+1},\overline{\rho}_h)} \leq \\ &\quad \leq ||\nabla u||_{g,A(k_{h+1},\overline{\rho}_h)} + \frac{2^{h+4}}{R}\,||u-k_{h+1}||_{g,A(k_{h+1},\overline{\rho}_h)}\,. \end{aligned}$$

As before, by taking R_0 small, we can assume that $||u-k_{h+1}||_{g,A(k_{h+1}\overline{\rho}_h)} \leq 1$, and that $||\nabla u||_{g,A(k_{h+1},\overline{\rho}_h)} \leq 1$. Using part (b) of Proposition 2.5, we obtain

$$J_{h+1} \le c\,|A(k_{h+1},\overline{\rho}_h)|^{1-\beta} \left\{ \left[\int_{A(k_{h+1},\overline{\rho}_h)} g(|\nabla u|)\,dx \right]^{\frac{1}{m}} + \right. \tag{29}$$
$$\left. + \frac{2^{h+4}}{R} \left[\int_{A(k_{h+1},\overline{\rho}_h)} g(u-k_{h+1})\,dx \right]^{\frac{1}{m}} \right\}^{r^*\beta} .$$

Using Caccioppoli's inequality (25), with $H=(g^*)^\beta$, $k=k_{h+1}$, $R=\rho_h$, $\sigma=[4(2^h+1)]^{-1}$ and m replaced by $m^*\beta$, and recalling the first inequality of (13), we have

$$\int_{A(k_{h+1},\overline{\rho}_h)} g(|\nabla u|)\,dx \le c\left[\left(\frac{2^{h+3}}{R}\right)^{m^*\beta} \int_{A(k_{h+1},\rho_h)} [g^*(u-k_{h+1})]^\beta dx + \right. \tag{30}$$
$$\left. + (1+[g^*(k_{h+1})]^\beta)|A(k_{h+1},\rho_h)| \right].$$

Therefore, since (f_1) implies $g(t)\le c(1+[g^*(t)]^\beta)$ for every $t\ge 0$, and since $R<1$, (29) and (30) imply

$$J_{h+1} \le c\,|A(k_{h+1},\overline{\rho}_h)|^{1-\beta} \times$$
$$\times \left\{ \frac{\lambda^h}{R^{1+\frac{m^*\beta}{m}}} \left(J_h^{\frac{1}{m}} + |A(k_{h+1},\rho_h)|^{\frac{1}{m}} (1+[g^*(k_{h+1})]^\beta)^{\frac{1}{m}} \right) \right\}^{r^*\beta},$$

where we denote by λ a constant depending on m and β which may assume different values from line to line. On the other hand

$$J_h \ge \int_{A(k_{h+1},\rho_h)} [g^*(u-k_h)]^\beta\,dx \ge [g^*(k_{h+1}-k_h)]^\beta\,|A(k_{h+1},\rho_h)| = \tag{31}$$
$$= \left[g^*\left(\frac{d}{2^{h+2}}\right)\right]^\beta |A(k_{h+1},\rho_h)|\,.$$

Since $g^*\in\Delta_2^{(m^*)}$, from (ii) of Proposition 2.1 we obtain

$$\left[g^*\left(\frac{d}{2^{h+2}}\right)\right]^\beta \ge \frac{[g^*(d)]^\beta}{2^{(h+2)m^*\beta}}. \tag{32}$$

Combining (31) and (32) we get

$$|A(k_{h+1}, \rho_h)| \leq \frac{\lambda^h J_h}{[g^*(d)]^\beta}\,.$$

Therefore we have

$$(33)\quad J_{h+1} \leq \frac{c\, c_1(R)\lambda^h}{[g^*(d)]^{\beta(1-\beta)}} J_h^{1-\beta} \left\{ J_h^{1/m} + J_h^{1/m} \left(\frac{1+[g^*(d)]^\beta}{[g^*(d)]^\beta} \right)^{1/m} \right\}^{r^*\beta},$$

where

$$c_1(R) = \frac{1}{R^{(1+\frac{m^*\beta}{m})r^*\beta}}\,.$$

If we assume that d satisfies

$$(34)\qquad g^*(d) \geq 1\,,$$

which implies

$$\frac{1+[g^*(d)]^\beta}{[g^*(d)]^\beta} \leq 2\,,$$

inequality (33) becomes

$$(35)\qquad J_{h+1} \leq \frac{c\, c_1(R)\lambda^h}{[g^*(d)]^{\beta(1-\beta)}}\, J_h^{1+\beta(\frac{r^*}{m}-1)}\,.$$

Applying Lemma 4.3, with

$$B = \lambda \qquad \alpha = \beta\,\frac{r^*-m}{m} \qquad C = \frac{c\, c_1(R)}{[g^*(d)]^{\beta(1-\beta)}}\,,$$

we obtain $\lim_{h\to\infty} J_h = 0$ if

$$(36)\qquad J_0 \leq c\, c_2(R)[g^*(d)]^{\frac{m(1-\beta)}{r^*-m}}\,.$$

Since

$$J_0 \leq \int_{Q_R} [g^*(|u|)]^\beta\, dx\,,$$

it is easy to see that (36) is satisfied if we choose d such that

$$g^*(d) \geq \left(\frac{1}{c\,c_2(R)} \int_{Q_R} [g^*(|u|)]^\beta \, dx\right)^{\frac{r^*-m}{(1-\beta)m}} . \tag{37}$$

Hence, since $\lim_{h\to\infty} J_h = 0$ implies $|A(d, \frac{R}{2})| = 0$, we conclude that

$$\sup_{Q_{\frac{R}{2}}} u \leq d \, . \tag{38}$$

On the other hand, since $(-u)$ is a local minimizer of the functional

$$\tilde{\mathcal{F}}(v) = \int_\Omega \tilde{f}(x, v, \nabla v) dx \, ,$$

where $\tilde{f}(x, v, \xi) = f(x, -v, -\xi)$ satisfies the same assumptions as f, we obtain

$$\sup_{Q_{\frac{R}{2}}} (-u) \leq d \, . \tag{39}$$

Taking (34) and (37) into account, inequalities (38) and (39) give (17). Theorem 3.2 is then proved. □

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Lavoro pervenuto alla redazione il 20 marzo 1997
ed accettato per la pubblicazione il 31 dicembre 1997.
Bozze licenziate il 16 aprile 1998

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