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On the Neumann problem for a hyperbolic partial differential equation of second order

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Riassunto: *Si studia un problema di Neumann per l'equazione* $u_{xy} = c$*. Estendendo il metodo introdotto da G. Fichera* [5] *per un problema di Dirichlet, si stabiliscono la condizioni necessarie e sufficienti per l'esistenza della soluzione.*

Abstract: *The paper concerns the Neumann problem for the equation* $u_{xy} = c$*. By using the method of G. Fichera, introduced in paper* [5] *devoted to the Dirichlet problem, necessary and sufficient conditions for the existence of the solutions are found.*

1 – Introduction

Neumann-type problems have been intensively examined for second-order hyperbolic partial differential equations, or systems of such equations, whose leading parts correspond to the *second canonical form* $\dfrac{\partial^2 u}{\partial \xi^2} - \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial \eta^2}$ (cf. [1], [4], [8], [10]-[17] and references). Let us mention briefly the main results of these investigations: In paper [8] of M. Ikawa a mixed (i.e. initial-boundary) value problem for a hyperbolic equation with the boundary condition of Neumann type is considered. The author proves the existence, uniqueness and regularity of a solution assuming first that the coefficients of the equation are independent of t and then

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using the method of Cauchy's polygonal line. Paper [13] of S. MIYATAKE is devoted to a nonlinear problem in a quarter — plane with the boundary conditions given on the line $x = 0$ which is, by assumption, noncharacteristic for the considered partial differential equation.

A lot of important results have been obtained by Y. SHIBATA and his collaborators (cf. [12] and [14]-[17]). In papers [14] and [15] the existence and uniqueness of solutions to the mixed problems for linear hyperbolic systems of second order with nonhomogeneous Neumann boundary conditions is proved. Also, it is examined how the constants appearing in the energy inequalities depend on the coefficients of the operators given in the problems. In paper [16] of Y.S. and M. KIKUCHI the local existence in time is proved for classical solutions of second-order systems of hyperbolic equations with nonlinear boundary conditions. The proof is conducted via the reduction of the problem to a hyperbolic-elliptic coupled system with the unknowns u and $D_t u$. Paper [17] of Y.S. and G. NAKAMURA is devoted to the study of the Neumann problem for a quasilinear hyperbolic system. The local existence in time of classical solutions is proved. The result has applications to the equation of motion describing the small deformation of a homogenous isotropic, hyperelastic material under the action of gravity and small pressure. In paper [12] of A. MILANI and Y.S. the authors present a direct method to construct compatible regularizing data in the two model cases of a linear second order hyperbolic equation with the Neumann and Dirichlet boundary conditions. In papers [10] of I. LASIECKA and A. STAHEL, and [11] of I.L. the mixed problems for the wave equation with nonlinear Neumann boundary conditions are examined. In [10] the existence and uniqueness of a local solution is proved and under additional assumptions the global solution is also obtained. In [11] the stability of solutions is studied. It is shown that the boundary damping produces a uniform global stability of the corresponding solutions. BEIRÃO DA VEIGA in a number of papers (cf. [1] and references) presented and developed a general method to prove the strong continuous dependence with respect to the data of the solutions to nonlinear hyperbolic problems (including the Neumann problems considered in [16], [17]) as well as several problems of nonlinear fluid dynamics. Finally, W. DAN [4] proved the existence and uniqueness of solutions to some Neumann problems for linear hyperbolic-parabolic coupled systems with coefficients in Sobolev spaces.

Neumann-type problems for second-order hyperbolic partial differential equations with the leading parts corresponding to the *first canonical form* $\frac{\partial^2 u}{\partial\xi\partial\eta}$ have not been considered (despite the fact that analogous Dirichlet-type problems have been examined - cf. G. Fichera [5], [6], B. Firmani [7] and references) except in paper [3] of A. Borzymowski. In the said paper a nonlinear Neumann-type for a system of hyperbolic integral-differential equations of order $2p$; $p \geq 1$ with two independent variables is examined by reducing it (via some technique originating in paper [18] of Z. Szmydt) to a system of integral-functional equations and hence the local existence and uniqueness of a solution is proved on the basis of the Banach fixed point theorem.

In this paper we examine a linear Neumann problem for the equation $u_{xy} = c$ by adopting the method introduced by G. Fichera [5] in his investigation of the Dirichlet problem and by using functional equations. We find necessary and sufficient conditions for the global existence of a solution and get this solution in series form. We also discuss the uniqueness of the solution. To the best of our knowledge, the said results have not been obtained so far.

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2 – The problem and the assumptions

Let Y be a Banach space with norm $\|\cdot\|$, and $\mathbf{P}$ the rectangle $\mathbf{P} = [\mathbf{0},\mathbf{1}] \times [\mathbf{0},\sigma]$, where $0 < \sigma < \infty$.

We consider the system of two curves, Γ and $\widetilde{\Gamma}$, of equations $y = \alpha(x)$ and $y = \beta(x)$, respectively, where $\alpha, \beta : [0,1] \to [0,\sigma]$, and we introduce the class $\mathcal{K}$ of all functions $\mathbf{P} \to Y$ possessing continuous derivatives $D_x^r D_y^s u$, where $D_x = \frac{\partial}{\partial x}$; $D_y = \frac{\partial}{\partial y}$ and $r, s = 0, 1$.

Let $\mathbf{n}$ and $\tilde{\mathbf{n}}$ denote the unit normal vectors to Γ and $\widetilde{\Gamma}$, respectively, denote $L = D_x D_y$ and assume that $c : \mathbf{P} \to Y$ is a given function.

The aim of this paper is to examine the following problem $(\mathcal{P})$:

Find a solution of the equation

$$Lu = c \tag{2.1}$$

in $\mathbf{P}$, that is a function $u \in \mathcal{K}$ satisfying (2.1) at each point $(x,y) \in \mathbf{P}$, fulfilling the boundary conditions

$$\begin{aligned} &\frac{d}{d\mathbf{n}}u[x,\alpha(x)] = M(x) \\ &\frac{d}{d\tilde{\mathbf{n}}}u[x,\beta(x)] = N(x) \end{aligned} \tag{2.2}$$

where $x \in [0,1]$, and $M, N : [0,1] \to Y$ are given functions.

We make the following assumption that will be in force throughout the whole paper:

I. The functions α and β are of class C^1, strictly increase and satisfy the conditions

$$\begin{aligned} &\alpha(0) = \beta(0) = 0; \quad \alpha(1) = \beta(1) = \sigma \\ &\alpha_* < \beta_*; \quad \widehat{\alpha} > \widehat{\beta}; \quad \min(\alpha_*, \widehat{\beta}) > 0 \end{aligned} \tag{2.3}$$

where $\alpha_* = \alpha^{'}(0)$; $\beta_* = \beta^{'}(0)$; $\widehat{\alpha} = \alpha^{'}(1)$; $\widehat{\beta} = \beta^{'}(1)$.

Moreover, the curves Γ and $\widetilde{\Gamma}$ have no common points except $O(0,0)$ and $Q_0(1,\sigma)$.

II. The functions M and N are continuous and satisfy the condition

$$\max(\|M(x)\|, \|N(x)\|) \leq m_1[\min(x, 1-x)]^{1+\Theta_1} \tag{2.4}$$

for $x \in [0,1]$, where m_1 and Θ_1 are positive constants.

III. The function c is of the form

$$c(x,y) = d(x)h(y) \tag{2.5}$$

where $d : [0,1] \to \mathbb{R}$ and $h : [0,\sigma] \to Y$ are continuous functions satisfying the following relations

$$|d(x)| \leq m_2[\min(x, 1-x)]^{\Theta_2}; \quad \|h(y)\| \leq m_2[\min(y, \sigma - y)]^{\Theta_2} \tag{2.6}$$

for $(x,y) \in \mathbf{P}$, where m_2 and Θ_2 are positive constants;

$$\int_0^1 d(\xi)d\xi = \int_0^\sigma h(\xi)d\xi = 0 \tag{2.7}$$

REMARK 2.1. Denote

$$R(x,y)=\int_0^x\int_0^y c(\xi,\eta)d\eta\,d\xi;\quad R_*(x,y)=\int_x^1\int_y^\sigma c(\xi,\eta)d\eta\,d\xi\,. \tag{2.8}$$

It follows from the equality

$$\begin{aligned}R(x,y)=R_*(x,y)+\int_0^1\int_0^\sigma c(\xi,\eta)d\eta\,d\xi+\\ -\Big(\int_0^1\int_y^\sigma c(\xi,\eta)d\eta\,d\xi+\int_x^1\int_0^\sigma c(\xi,\eta)d\eta\,d\xi\Big)\end{aligned}$$

and from Assumption III that

$$R(x,y)=R_*(x,y)\,. \tag{2.9}$$

Evidently, the following Lemma holds good

LEMMA 2.1. *If u is of the form*

$$u(x,y)=R(x,y)+\varphi(x)+\psi(y) \tag{2.10}$$

where $\varphi:[0,1]\to Y$ and $\psi:[0,\sigma]\to Y$ are functions of class C^1 then u is a solution to equation (2.1) *in* $\mathbf{P}$*. Conversely, for any given solution u to equation* (2.1) *in* $\mathbf{P}$ *there are functions $\varphi:[0,1]\to Y$ and $\psi:[0,\sigma]\to Y$ of class C^1 such that equality* (2.10) *is satisfied in* $\mathbf{P}$*.*

3 – The main result

We explain the notation before stating the result.

Let $\sigma_0\in(0,\sigma)$ be arbitrarily fixed, set $x_0=\alpha^{-1}(\sigma_0)$; $\tilde{x}_0=\beta^{-1}(\sigma_0)$ and introduce the following function classes:

a) The class $\mathcal{K}_1$ of all continuous functions $\tilde{\omega}:[\tilde{x}_0,1]\to Y$ such that

$$\|\tilde{\omega}(x)\|\le\widetilde{C}_{\tilde{\omega}}(1-x)^{1+\Theta} \tag{3.1}$$

where $\widetilde{C}_{\tilde{\omega}}$ is a positive constant depending in general on $\tilde{\omega}$, and $\Theta=\min(\Theta_1,\Theta_2)$;

b) The class $\mathcal{K}_2$ of all continuous functions $\omega_* : [0, \sigma_0] \to Y$ such that

$$\|\omega_*(y)\| \leq C^*_{\omega_*} y^{1+\Theta} \tag{3.2}$$

where $C^*_{\omega_*}$ is a positive constant depending in general on ω_*, and Θ is as in (3.1).

We consider the functions (cf. [5])

$$\tau(y) = \alpha \circ \beta^{-1}(y); \qquad \mu(x) = \alpha^{-1} \circ \beta(x) \tag{3.3}$$

$$e(x) = (1 + \alpha'^2(x))^{\frac{1}{2}}; \quad \tilde{e}(x) = (1 + \beta'^2(x))^{\frac{1}{2}}; \tag{3.4}$$

$$\begin{aligned} V(x) &= -\frac{e(x)}{\alpha'(x)} \Big\{ M(x) - \Big[\frac{d}{d\mathbf{n}} R(x,y) \Big]_{y=\alpha(x)} \Big\} \\ W(x) &= \frac{\tilde{e}(x)}{\beta'(x)} \Big\{ N(x) - \Big[\frac{d}{d\tilde{\mathbf{n}}} R(x,y) \Big]_{y=\beta(x)} \Big\} \end{aligned} \tag{3.5}$$

We shall use the following series

$$\widetilde{S}(x) = \sum_{n=0}^{\infty} \tilde{a}_n(x) \qquad x \in [\tilde{x}_0, 1] \tag{3.6}$$

$$S_*(y) = \sum_{n=0}^{\infty} a_n(y) \qquad y \in [0, \sigma_0] \tag{3.7}$$

with:

$$\tilde{a}_n(x) = B_n(x) \widetilde{F} \circ \mu^n(x) \tag{3.8}$$

$$a_n(y) = A_n(y) F \circ \tau^n(y) \tag{3.9}$$

$$B_n(x) = \prod_{m=0}^{n-1} \tilde{b} \circ \mu^m(x); \quad \tilde{b}(x) = \frac{\alpha' \circ \mu(x)}{\beta'(x)} \tag{3.10}$$

$$A_n(y) = \prod_{m=0}^{n-1} b \circ \tau^m(y); \quad b(y) = \frac{\beta' \circ \beta^{-1}(y)}{\alpha' \circ \beta^{-1}(y)} \tag{3.11}$$

$$\widetilde{F}(x) = W(x) - \tilde{b}(x) V \circ \mu(x) \tag{3.12}$$

$$F(y) = \beta' \circ \beta^{-1}(y)[V \circ \beta^{-1}(y) - W \circ \beta^{-1}(y)] \tag{3.13}$$

Finally, we shall also employ the condition

$$(3.14)\quad V(x)=\sum_{n=0}^{\infty}\{-(\alpha'(x))^{-1}A_n\circ\alpha(x)F\circ\tau^n[\alpha(x)]+B_n(x)\widetilde{F}\circ\mu^n(x)\}$$

for $x\in[\tilde{x}_0,1]$, and set for $x\in[\tilde{x}_0,1]$, $y\in[\sigma_0,\sigma]$, $t\in[0,\tilde{x}_0]$, $z\in[0,\tilde{\sigma}_0]$:

$$(3.15)\qquad \varphi(x)=\tilde{\varphi}(x):=\int_1^x\widetilde{S}(\xi)d\xi+C^1\,,$$

$$(3.16)\quad \psi(y)=\tilde{\psi}(y):=\int_\sigma^y\alpha'\circ\alpha^{-1}(\xi)[\widetilde{S}\circ\alpha^{-1}(\xi)-V\circ\alpha^{-1}(\xi)]d\xi+C^2\,,$$

$$(3.17)\qquad \varphi(t)=\varphi_*(t):=\int_0^t[(\beta'(\xi))^{-1}S_*\circ\beta(\xi)+W(\xi)]d\xi+C^1+a\,,$$

$$(3.18)\qquad \psi(z)=\psi_*(z):=\int_0^zS_*(\xi)d\xi+C^2+b\,,$$

where C^1, C^2 are arbitrary constants and

$$(3.19)\quad \begin{aligned} a&=-\Big(\int_{\tilde{x}_0}^1\widetilde{S}(\xi)d\xi+\int_0^{\tilde{x}_0}[(\beta'(\xi))^{-1}S_*\circ\beta(\xi)+W(\xi)]d\xi\Big)\\ b&=-\Big(\int_{\sigma_0}^{\sigma}\alpha'\circ\alpha^{-1}(\xi)[\widetilde{S}\circ\alpha^{-1}(\xi)-V\circ\alpha^{-1}(\xi)]d\xi+\int_0^{\sigma_0}S_*(\xi)d\xi\Big)\,.\end{aligned}$$

Now we state our result

THEOREM 3.1. *If Assumptions* I–III *are satisfied then condition* (3.14) *is necessary and sufficient for the existence of a solution of problem* ($\mathcal{P}$) *such that the first derivatives of the functions* φ *and* ψ *appearing in formula* (2.10) *are continuous and belong to the classes* $\mathcal{K}_1$ *and* $\mathcal{K}_2$ *on* $[\tilde{x}_0,1]$ *and* $[0,\sigma_0]$, *respectively. The said solution is given by formula* (2.10) *together with relations* (3.15)-(3.19).

4 – Auxiliary theorems

LEMMA 4.1. *For every number $\varepsilon_0 \in (0,1)$ there is a sufficiently small number $\delta \in (0, \min(1,\sigma))$such that the inequalities:*

$$\begin{matrix} (1-\varepsilon_0)\alpha_* < \alpha'(x) < (1+\varepsilon_0)\alpha_* \\ (1-\varepsilon_0)\beta_* < \beta'(x) < (1+\varepsilon_0)\beta_* \end{matrix} \quad x \in [\delta, 1] \tag{4.1}$$

$$\begin{matrix} (1-\varepsilon_0)\widehat{\alpha} < \alpha'(x) < (1+\varepsilon_0)\widehat{\alpha} \\ (1-\varepsilon_0)\widehat{\beta} < \beta'(x) < (1+\varepsilon_0)\widehat{\beta} \end{matrix} \quad x \in [1-\delta, 1] \tag{4.2}$$

hold good.

The validity of Lemma 4.1 follows directly from Assumption I.

The following lemmas are valid.

LEMMA 4.2 (cf. [2], Lemma 2). *The relations*

$$\tau^n \rightarrow 0 \ \text{on} \ [0,\sigma); \quad \mu^n \rightarrow 1 \ \text{on} \ (0,1] \tag{4.3}$$

hold good when n $(\in \mathcal{N})$ tends to infinity, where $\rightarrow$ denotes the almost uniform convergence.

LEMMA 4.3. *The following inequalities*

$$\Big\| \Big[\frac{d}{d\widehat{\mathbf{n}}} R_*(x,y)\Big]_{y=\gamma(x)} \Big\| \le \text{const}[(1-x)(\sigma-\gamma(x))]^{\Theta_2} \times \\ \times [\sigma - \gamma(x) + (1-x)] \tag{4.4}$$

$$\Big\| \Big[\frac{d}{d\widehat{\mathbf{n}}} R(x,y)\Big]_{y=\gamma(x)} \Big\| \le \text{const}[x\gamma(x)]^{\Theta_2}[\gamma(x)+x] \tag{4.5}$$

are satisfied where $\widehat{\mathbf{n}} = \mathbf{n}$, $\gamma = \alpha$ or $\widehat{\mathbf{n}} = \tilde{\mathbf{n}}$, $\gamma = \beta$, respectively.

Proof. The proof, being similar for relation (4.5), will be given only for inequality (4.4).

Let $\widehat{\mathbf{n}} = \mathbf{n}$; $\gamma(x) = \alpha(x)$. Basing on (2.8) and using Assumption III, we get

$$\begin{aligned}\Big\|\Big[\frac{d}{d\mathbf{n}}R_*(x,y)\Big]_{y=\alpha(x)}\Big\| &\leq \int_{\alpha(x)}^{\sigma}\|c(x,\eta)\|d\eta + \int_x^1 \|c[\xi,\alpha(x)]\|d\xi \leq \\ &\leq \frac{m_2}{1+\Theta_2}[(1-x)^{\Theta_2}(\sigma-\alpha(x))^{1+\Theta_2} + \\ &\quad + (1-x)^{1+\Theta_2}(\sigma-\alpha(x))^{\Theta_2}] \leq \\ &\leq \mathrm{const}[(1-x)(\sigma-\alpha(x))]^{\Theta_2}[(\sigma-\alpha(x))+(1-x)],\end{aligned}$$

as required. The argument for $\widehat{\mathbf{n}} = \widetilde{\mathbf{n}}$; $\gamma(x) = \beta(x)$ is analogous.

5 – Proof of the main result

This section is devoted to the proof of Theorem 3.1.

To this end we adopt the argument of G. Fichera (cf. [5]), according to which the considered problem is first examined, by using functional equations, in the subrectangles $\Delta = [0,\tilde{x}_0]\times[0,\sigma_0]$ and $\Omega = [\tilde{x}_0,1]\times[\sigma_0,\sigma]$ of the rectangle $\mathbf{P}$ (see fig. 1) and hence the necessary and sufficient conditions for the existence of a solution in the whole domain $\mathbf{P}$ are found.

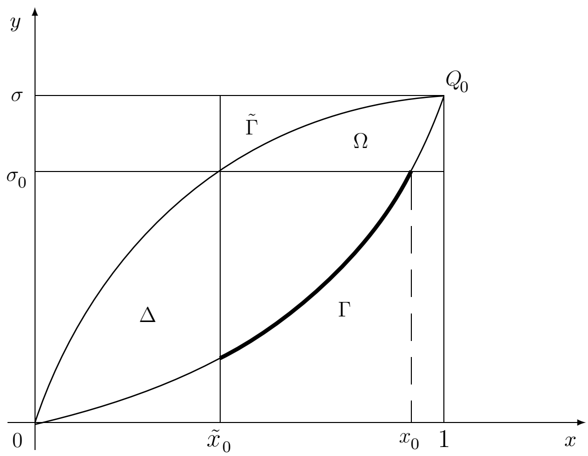

Fig. 1

We are going to consider problem $(\mathcal{P})$ in the domains Ω and Δ, successively, begining with Ω.

Let us assume that the direction cosines of the normals $\mathbf{n}$ and $\tilde{\mathbf{n}}$ are given by

$$\begin{aligned} \cos(x, \mathbf{n}) &= -\frac{\alpha'(x)}{e(x)}; \quad \cos(y, \mathbf{n}) = \frac{\mathbf{1}}{\mathbf{e(x)}} \\ \cos(x, \tilde{\mathbf{n}}) &= \frac{\beta'(x)}{\tilde{e}(x)}; \quad \cos(y, \tilde{\mathbf{n}}) = -\frac{1}{\tilde{e}(x)} \end{aligned} \tag{5.1}$$

Imposing on function u (cf. (2.10)) the boundary conditions (2.2), we get the following system of functional equations

$$\begin{aligned} (a) \qquad & \varphi'(x) - (\alpha'(x))^{-1}\psi' \circ \alpha(x) = V(x) \\ (b) \qquad & \varphi'(x) - (\beta'(x))^{-1}\psi' \circ \beta(x) = W(x) \end{aligned} \tag{5.2}$$

$(x \in [0,1])$, where φ' and ψ' are the unknowns sought in the class C^0, and V and W are given by formulae (3.5), respectively.

Thus, problem $(\mathcal{P})$ reduces in the rectangle Ω to solving system (5.2) for $x \in [\tilde{x}_0, 1]$; $y \in [\sigma_0, \sigma]$.

Replacing x by $\mu(x)$ in (5.2)(a) and combining the obtained equation with (5.2)(b), we get

$$\psi'(y) = \alpha' \circ \alpha^{-1}(y)[\varphi' \circ \alpha^{-1}(y) - V \circ \alpha^{-1}(y)] \quad y \in [\sigma_0, \sigma]\,; \tag{5.3}$$

$$\varphi'(x) - \tilde{b}(x)\varphi' \circ \mu(x) = \widetilde{F}(x) \quad x \in [\tilde{x}_0, 1] \tag{5.4}$$

Evidently, it is sufficient to solve the functional equation (5.4) and then substitute its solution φ' to (5.3) and find ψ'. We shall examine equation (5.4) by using the well known iteration method (cf. [9], chapt. II).

Proposition 5.1. *Equation* (5.4) *has a solution given by the formula*

$$\varphi'(x) = \tilde{\varphi}'(x) := \widetilde{S}(x) \quad x \in [\tilde{x}_0, 1] \tag{5.5}$$

It is the unique solution of (5.4) *in the class* $\mathcal{K}_1$. *The function* φ' *given by* (5.5) *belongs to* $\mathcal{K}_1$.

Proof. First of all we shall prove that the series representing $\widetilde{S}(x)$ (cf. (3.6)) is uniformly convergent in $[\tilde{x}_0, 1]$.

Let ε_0 be a number such that

$$0 < \varepsilon_0 < \frac{1 - q^\chi}{1 + q^\chi} \tag{5.6}$$

where (cf. (2.3))

$$q = \frac{\widehat{\beta}}{\widehat{\alpha}} \in (0,1)\,; \tag{5.7}$$

$$\chi = \frac{\Theta}{4 + \Theta} \tag{5.8}$$

It follows from Lemma 4.2 that there is a number $n_0 \in \mathcal{N}$ such that for all $n_0 \le n \in \mathcal{N}$ and $x \in [\tilde{x}_0, 1]$ the relation $\mu^n(x) \in [1 - \delta, 1]$ holds good. In what follows we shall assume that $n > n_0$. By (3.10) we have

$$B_n(x) = \prod_{m=0}^{n_0-1} \tilde{b} \circ \mu^m(x) \prod_{m=n_0}^{n-1} \tilde{b}_0 \circ \mu^m(x) \le \text{const} \prod_{m=n_0}^{n-1} \frac{\alpha' \circ \mu^{m+1}(x)}{\beta' \circ \mu^m(x)}$$

whence and from Lemma 4.1 (with ε_0 satisfying (5.6)) we obtain the estimate

$$B_n(x) \le \text{const} \Big[\frac{(1+\varepsilon_0)\widehat{\alpha}}{(1-\varepsilon_0)\widehat{\beta}}\Big]^n \tag{5.9}$$

Let us observe that (cf. (2.9), (3.5) and (3.12))

$$\begin{aligned} \|\widetilde{F} \circ \mu^n(x)\| \le \text{const} \Big(&\|M \circ \mu^{n+1}(x)\| + \|N \circ \mu^n(x)\| + \\ &+ \Big\|\Big[\frac{d}{d\mathbf{n}} R_*(x,y)\Big]_{y=\alpha\circ\mu^{n+1}(x)}\Big\| + \\ &+ \Big\|\Big[\frac{d}{d\tilde{\mathbf{n}}} R(x,y)\Big]_{y=\beta\circ\mu^n(x)}\Big\|\Big) \end{aligned} \tag{5.10}$$

In virtue of (2.4), (3.3) and Lemma 4.1, we have the following sequence of relations

$$\begin{aligned}\|M\circ\mu^{n+1}(x)\| &\le m_1\prod_{\nu=0}^{n}\Big[\frac{\beta'\circ\mu^{\nu}(\xi)}{\alpha'\circ\mu^{\nu+1}(\xi)}\Big]^{1+\Theta_1}(1-x)^{1+\Theta_1}\le\\ &\le \mathrm{const}\Big[\frac{(1+\varepsilon_0)\widehat{\beta}}{(1-\varepsilon_0)\widehat{\alpha}}\Big]^{n(1+\Theta_1)}(1-x)^{1+\Theta_1}\end{aligned}\tag{5.11}$$

$(x<\xi<1)$, and in a similar way we get

$$\|N\circ\mu^{n}(x)\|\le \mathrm{const}\Big[\frac{(1+\varepsilon_0)\widehat{\beta}}{(1-\varepsilon_0)\widehat{\alpha}}\Big]^{n(1+\Theta_1)}(1-x)^{1+\Theta_1}\tag{5.12}$$

Furthermore, Lemma 4.3 yields

$$\begin{aligned}\Big\|\Big[\frac{d}{d\mathbf{n}}R_*(x,y)\Big]_{y=\alpha\circ\mu^{n+1}(x)}\Big\| &\le \mathrm{const}(1-\mu^{n+1}(x))^{1+2\Theta_2}\le\\ &\le \mathrm{const}\Big[\frac{(1+\varepsilon_0)\widehat{\beta}}{(1-\varepsilon_0)\widehat{\alpha}}\Big]^{n(1+2\Theta_2)}(1-x)^{1+2\Theta_2};\end{aligned}\tag{5.13}$$

$$\Big\|\Big[\frac{d}{d\tilde{\mathbf{n}}}R_*(x,y)\Big]_{y=\beta\circ\mu^{n}(x)}\Big\|\le \mathrm{const}\Big[\frac{(1+\varepsilon_0)\widehat{\beta}}{(1-\varepsilon_0)\widehat{\alpha}}\Big]^{n(1+2\Theta_2)}(1-x)^{1+2\Theta_2}\tag{5.14}$$

On joining relations (5.10)-(5.14) we obtain the inequality

$$\|\widetilde{F}\circ\mu^{n}(x)\|\le \mathrm{const}\Big[\frac{(1+\varepsilon_0)\widehat{\beta}}{(1-\varepsilon_0)\widehat{\alpha}}\Big]^{n(1+\Theta)}(1-x)^{1+\Theta}\tag{5.15}$$

which, together with (5.9), implies (cf. (3.8))

$$\|\tilde{a}_n(x)\|\le \mathrm{const}\,q_1^n(1-x)^{1+\Theta}\le \mathrm{const}\,q_1^n\tag{5.16}$$

$(x\in[\tilde{x}_0,1])$ where

$$q_1=\Big(\frac{1+\varepsilon_0}{1-\varepsilon_0}\Big)^{2+\Theta}q^{\Theta}\tag{5.17}$$

with q given by (5.7).

It follows from (5.6) and (5.8) that $q_1 \in (0,1)$ and hence (cf. (5.16)) the series (3.6) is uniformly convergent, as required.

It is easily verified by direct calculation that the function $\varphi^{'}$ given by (5.5) is a solution to equation (5.4) for $x \in [\tilde{x}_0, 1]$.

In order to prove the uniqueness of the solution in the class $\mathcal{K}_1$, let us observe that if a function $\varphi^{'} : [\tilde{x}_0, 1] \to Y$ is a solution to equation (5.4) in the interval $[\tilde{x}_0, 1]$ then, for every $r \in \mathcal{N}$ and every $x \in [\tilde{x}_0, 1]$, the following equality

$$\varphi^{'}(x) = \sum_{n=0}^{r} \Big(\prod_{m=0}^{n-1} \tilde{b} \circ \mu^m(x) \Big) F \circ \mu^n(x) + \rho_r(x) \tag{5.18}$$

holds good where

$$\rho_r(x) = \Big(\prod_{m=0}^{r} \tilde{b} \circ \mu^m(x) \Big) \varphi^{'} \circ \mu^{r+1}(x) \tag{5.19}$$

Assuming that $r > n_0$, we have the following sequence of inequalities (cf. (3.1) and the derivation of (5.15))

$$\begin{aligned} \|\rho_r(x)\| &\leq \text{const} \prod_{m=n_0+1}^{r} \frac{\alpha^{'} \circ \mu^{m+1}(x)}{\beta' \circ \mu^m(x)} (1 - \mu^{r+1}(x))^{1+\Theta} \leq \\ &\leq \text{const} \Big[\frac{(1+\varepsilon_0)\widehat{\alpha}}{(1-\varepsilon_0)\widehat{\beta}} \Big]^r \Big[\frac{(1+\varepsilon_0)\widehat{\beta}}{(1-\varepsilon_0)\widehat{\alpha}} \Big]^{r(1+\frac{\Theta}{2})} (1 - \mu^{r+1}(x))^{\frac{\Theta}{2}} \leq \\ &\leq \text{const}\, q_2^r (1-x)^{1+\frac{\Theta}{2}} (1 - \mu^{r+1}(x))^{\frac{\Theta}{2}} \leq \\ &\leq \text{const}\, q_2^r (1-x)^{1+\frac{\Theta}{2}} (1 - \mu^{r+1}(x))^{\frac{\Theta}{2}} \end{aligned}$$

where (cf. (5.6))

$$0 < q_2 = \Big(\frac{1+\varepsilon_0}{1-\varepsilon_0} \Big)^{2+\frac{\Theta}{2}} q^{\frac{\Theta}{2}} \leq \Big[\Big(\frac{1+\varepsilon_0}{1-\varepsilon_0} \Big)^{4+\Theta} q^{\Theta} \Big]^{\frac{1}{2}} < 1$$

As a consequence we get

$$\|\rho_r(x)\| \leq \text{const}(1 - \mu^{r+1}(x))^{\frac{\Theta}{2}} \tag{5.20}$$

whence and from Lemma 4.2 it follows that

$$\rho_r(x) \underset{r\to\infty}{\longrightarrow} 0 \tag{5.21}$$

Relations (5.18) and (5.20) imply that $\varphi^{'}(x)$ is of the form (5.5) which ends the proof of uniqueness.

Finally, it follows from (3.6), (3.8)-(3.12) and (5.16) that the function $\varphi^{'}$ given by (5.5) is continuous and satisfies inequality (3.1), i.e. belongs to the class $\mathcal{K}_1$.

Thus, the proof of Proposition 5.1 is completed.

REMARK 5.1. It results from (2.4), (3.5) and (4.4) that if $\varphi^{'} \in \mathcal{K}_1$ then (5.3) implies the inequality

$$\|\psi^{'}(y)\| \le \widehat{C}_{\varphi'}(\sigma - y)^{1+\Theta} \tag{5.22}$$

where $\widehat{C}_{\varphi'}$ is a positive constant depending on $\widetilde{C}_{\varphi'}$ (cf. (3.1)).

COROLLARY 5.1. *It follows from Proposition* 5.1 *that the functions* $\tilde{\varphi}^{'}(x)$ *and*

$$\tilde{\psi}^{'}(y) = \alpha^{'} \circ \alpha^{-1}(y)[\widetilde{S} \circ \alpha^{-1}(y) - V \circ \alpha^{-1}(y)] \qquad y \in [\sigma_0, \sigma] \tag{5.23}$$

satisfy system (5.2) *in* Ω *if, and for* $\varphi^{'} \in \mathcal{K}_1$ *only if, the equalities* (3.15) *and* (3.16) *hold good.*

Now, let us consider system (5.2) in the domain Δ.

Evidently, (5.2) is equivalent to

$$\varphi^{'}(x) = (\alpha^{'}(x))^{-1}\psi^{'}\circ\alpha(x) + V(x) = (\beta^{'}(x))^{-1}\psi^{'}\circ\beta(x) + W(x) \quad x \in [0, \tilde{x}_0] \tag{5.24}$$

$$\psi^{'}(y) - b(y)\psi^{'} \circ \tau(y) = F(y) \quad y \in [0, \sigma_0] \tag{5.25}$$

By an argument analogous to that in the proof of Proposition 5.1, one can prove the following

Proposition 5.2. *Equation* (5.25) *has a solution given by the formula*

$$\psi'(y) = \psi'_*(y) := S_*(y) \quad y \in [0, \sigma_0] \tag{5.26}$$

It is the unique solution of (5.25) *in the class* $\mathcal{K}_2$. *The function* ψ' *given by* (5.26) *belongs to* $\mathcal{K}_2$.

Remark 5.2. By (2.4), (3.5) and (4.5) we can assert that if $\psi' \in \mathcal{K}_2$ then (5.24) implies the inequality

$$\|\varphi'(x)\| \leq C^{**}_{\psi'} x^{1+\Theta} \tag{5.27}$$

where $C^{**}_{\psi'}$ is a positive constant depending on $C^{*}_{\psi'}$ (cf. (3.2)).

Corollary 5.2. *In virtue of Proposition* 5.2, *the functions* $\psi'_*(y)$ *and*

$$\varphi'_*(x) = (\beta'(x))^{-1} S_* \circ \beta(x) + W(x) \quad x[0, \tilde{x}_0] \tag{5.28}$$

satisfy system (5.2) *in* Δ *if, and for* $\psi' \in \mathcal{K}_2$ *only if, the equalities*

$$\varphi(x) = \hat{\varphi}_*(x) := \int_0^x [(\beta'(\xi))^{-1} S_* \circ \beta(\xi) + W(\xi)] d\xi + C^3 \tag{5.29}$$

$$\psi(y) = \widehat{\psi}_*(y) := \int_0^y S_*(\xi) d\xi + C^4 \tag{5.30}$$

hold good for $x \in [0, \tilde{x}_0]$ *and* $y \in [0, \sigma_0]$, *where* C^3, C^4 *are arbitrary constants.*

Let u be a solution of problem $(\mathcal{P})$ (cf. (2.9) and (2.10)) with φ' and ψ' belonging to $\mathcal{K}_1$ and $\mathcal{K}_2$ on $[\tilde{x}_0, 1]$ and $[0, \sigma_0]$, respectively.

By Corollaries 5.1 and 5.2 we have

$$\varphi(x) = \begin{cases} \widehat{\varphi}_*(x) & \text{for } x \in [0, \tilde{x}_0] \\ \tilde{\varphi}(x) & \text{for } x \in [\tilde{x}_0, 1] \end{cases} \tag{5.31}$$

and

$$\psi(y) = \begin{cases} \widehat{\psi}_*(y) & \text{for } y \in [0, \sigma_0] \\ \tilde{\psi}(y) & \text{for } y \in [\sigma_0, \sigma] \end{cases} \tag{5.32}$$

and hence φ and ψ are continuous in the intervals $[0,1]$ and $[0,\sigma]$, respectively, if and only if the equalities

$$\tilde{\varphi}(\tilde{x}_0) = \widehat{\varphi}_*(\tilde{x}_0); \quad \tilde{\psi}(\sigma_0) = \widehat{\psi}_*(\sigma_0) \tag{5.33}$$

are satisfied.

Using formulae (3.15), (3.16) and (5.29), (5.30) we get from (5.33) relations (3.17), (3.18) where a and b are given by (3.19), respectively.

Moreover, it is clear that $\varphi^{'}$ and $\psi^{'}$ are continuous in the above-mentioned intervals if and only if

$$\tilde{\varphi}^{'}(\tilde{x}_0) = \varphi^{'}_*(\tilde{x}_0); \quad \tilde{\psi}^{'}(\sigma_0) = \psi^{'}_*(\sigma_0) \tag{5.34}$$

Basing on (3.6)-(3.11), (5.5), (5.23), (5.26) and (5.28) we can assert that relations (5.34) are equivalent to the equalities

$$\begin{aligned} V(\tilde{x}_0) = \sum_{n=0}^{\infty} \{ & -(\alpha^{'}(x_0))^{-1} A_n \circ \alpha(x_0) F \circ \tau^n[\alpha(\tilde{x}_0)] + \\ & + B_n(\tilde{x}_0) \widetilde{F} \circ \mu^n(\tilde{x}_0) \} \end{aligned} \tag{5.35}$$

and

$$V(x_0) = \sum_{n=0}^{\infty} [-(\alpha^{'}(x_0))^{-1} A_n(\sigma_0) F \circ \tau^n(\sigma_0) + B_n(x_0) \widetilde{F} \circ \mu^n(x_0)] \,, \tag{5.36}$$

respectively.

It is easily seen that we still have to fulfil the following requirement: the first of conditions (2.2) must be satisfied on the part of Γ marked on figure 1, that is equation (5.2)(a) should be valid for $x \in [\tilde{x}_0, x_0]$; $y \in [0, \sigma_0]$. This requirement yields, in virtue of (5.2)(a), (5.5), (5.26), (5.31) and (5.32), the condition (3.14) which contains the equalities (5.35) and (5.36).

It follows from the above-performed considerations that condition (3.14) is necessary and sufficient for the existence of a solution u of problem $(\mathcal{P})$ such that the derivatives $\varphi^{'}$ and $\psi^{'}$ of the functions φ and ψ appearing in formula (2.10) are continuous and belong to the classes $\mathcal{K}_1$ and $\mathcal{K}_2$ on $[\tilde{x}_0, 1]$ and $[0, \sigma_0]$, respectively. The said solution is given by formula (2.10) together with relations (3.15)-(3.19).

Thus, the proof of Theorem 3.1 is completed.

6 – Final remarks

Remark 6.1. By Propositions 5.1 and 5.2 we can assert that if $\varphi^{'} : [0,1] \to Y$ and $\psi^{'} : [0,\sigma] \to Y$ are continuous and belong to the classes $\mathcal{K}_1$ and $\mathcal{K}_2$ on $[\tilde{x}_0, 1]$ and $[0, \sigma_0]$, respectively, then the solution $(\varphi^{'}, \psi^{'})$ of system (5.2) is unique. However, the solution u of problem $(\mathcal{P})$ is not uniquely determined since it depends on an arbitrary constant $C := C^1 + C^2$ (cf. (2.10), (3.17), (3.18), (5.5) (5.23), (5.31) and (5.32)), and in order to get the unique solution of problem $(\mathcal{P})$ one should impose on u an additional condition.

For example, if we demand from u to satisfy the equality

$$u(0,0) = 0 \tag{6.1}$$

then (cf. (2.10), (3.17) and (3.18))

$$C = -(a+b) \tag{6.2}$$

and the unique solution of problem $(\mathcal{P})$ is given by the formula

$$u(x,y) = R(x,y) + r(x,y) \tag{6.3}$$

with

$$r(x,y) = \begin{cases} I_3(x) + I_4(y) & \text{for } (x,y) \in \Delta \\ I_1(x) + I_4(y) - a & \text{for } x \in [\tilde{x}_0, 1]; \quad y \in [0, \sigma_0] \\ I_1(x) + I_2(y) - (a+b) & \text{for } (x,y) \in \Omega \\ I_3(x) + I_2(y) - b & \text{for } x \in [0, \tilde{x}_0]; \quad y \in [\sigma_0, \sigma] \end{cases} \tag{6.4}$$

where $I_1(x)$, $I_2(y)$, $I_3(x)$ and $I_4(y)$ denote the integrals appearing in formulae (3.15)-(3.18), respectively.

REMARK 6.2. In this remark we give an example of the relations between the data under which condition (3.14) is satisfied.

Let $c(x,y) \equiv 0$ in $\mathbf{P}$. As a consequence (cf. (3.5), (3.12) and (3.13))

$$\begin{aligned}\widetilde{F}(x) &= (\beta'(x))^{-1}\{\tilde{e}(x)N(x) + [e(z)M(z)]_{z=\mu(x)}\}\\ \widetilde{F}(y) &= -\beta' \circ \beta^{-1}(y)\Big[\frac{\tilde{e}(x)}{\beta'(x)}N(x) + \frac{e(x)}{\alpha'(x)}M(x)\Big]_{x=\beta^{-1}(y)}\end{aligned} \tag{6.5}$$

and hence (3.14) takes the form

$$-\frac{e(x)}{\alpha'(x)}M(x) = \frac{\tilde{e}(x)}{\beta'(x)}N(x) + E_1(x) + E_2(x) \tag{6.6}$$

($x \in [\tilde{x}_0, 1]$) with

$$\begin{aligned}E_1(x) &= (\alpha'(x))^{-1}\beta' \circ \lambda(x)\Big[\frac{\tilde{e}(z)}{\beta'(z)}N(z) + \frac{e(z)}{\alpha'(z)}M(z)\Big]_{z=\lambda(x)}\\ E_2(x) &= (\beta'(x))^{-1}[e(z)M(z)]_{z=\mu(x)}+\\ &\quad + \sum_{n=1}^{\infty}\{-(\alpha'(x))^{-1}A_n \circ \alpha(x)F \circ \tau^n[\alpha(x)]+\\ &\quad + B_n(x)\widetilde{F} \circ \mu^n(x)\}\end{aligned} \tag{6.7}$$

where $\lambda(x) = \beta^{-1} \circ \alpha(x)$ (it follows from Lemma 2 in [2] that $\lambda^n \to 0$ on $[0,\sigma)$ when $n \to \infty$).

We shall use the following relations resulting from (3.3) and the definition of $\lambda(x)$:

$$\begin{aligned}&\alpha(x) \in [\alpha(\tilde{x}_0), \sigma_0];\ \lambda^n(x) \in (0, \tilde{x}_0]\\ &\tau^n \circ \alpha(x) = \alpha \circ \lambda^n(x) \in (0, \sigma_0]; \qquad \beta^{-1} \circ \tau^n[\alpha(x)] \in (0, \tilde{x}_0]\\ &\mu^n(x) \in [\tilde{x}_0, 1]\end{aligned} \tag{6.8}$$

($x \in [\tilde{x}_0, x_0]$; $n \in \mathcal{N}$).

It easily follows from (6.5)-(6.8) that the equalities

$$N(x) = -\Big(\frac{1+\alpha'^2(x)}{1+\beta'^2(x)}\Big)^{\frac{1}{2}} \cdot \frac{\beta'(x)}{\alpha'(x)}M(x) \quad x \in [0,1] \tag{6.9}$$

$$M(x) = 0 \quad x \in [x_0, 1] \tag{6.10}$$

imply the validity of (6.6).

As the choice of $\sigma_0 \in (0, \sigma)$ is arbitrary and $x_0 \to 1$ when $\sigma_0 \to \sigma$ (cf. fig. 1), it is sufficient to assume that there is a number $\delta_* \in (0, 1)$ such that condition (6.10) holds for $x \in [1 - \delta_*, 1]$.

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