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Stability for a third order Sine-Gordon equation

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RIASSUNTO: *Preliminarmente si studia un problema di valori iniziali e al contorno per una equazione non lineare di ordine superiore. Successivamente, si discutono diverse proprietà di stabilità usando il secondo metodo di Liapunov. Si forniscono esempi tratti dalla teoria dei superconduttori e dalla meccanica quantistica. Inoltre, si applicano i risultati a un problema di perturbazioni iniziali e al contorno.*

ABSTRACT: *We first analyze an initial-boundary-value problem for a nonlinear higher order equation. Then, we discuss several stability properties basing on the Liapunov second method. We give examples from the Superconductor Theory and Quantum Mechanics. Moreover, we apply the results to a problem of initial and boundary perturbations.*

1 – Introduction and main results

In this paper we give a detailed analysis of some stability properties of the third order nonlinear equation

$$Lu = f(x,t,u,u_x,u_{xx},u_t),\ L = -\varepsilon\partial_{xxt} - c^2\partial_{xx} + \partial_{tt}, \tag{1.1}$$

where the constants ε and c are positive. One of the most interesting

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applications of (1.1) occurs when the nonlinear forcing term reduces to

$$f = -b \sin u - a u_t + \Gamma(x,t), \tag{1.2}$$

with a and b positive constants. In this case, (1.1) represents the *perturbed Sine-Gordon* equations which arises in the Superconductor Theory when one studies the current flow in the Josephson tunnel junctions, see e.g. [6], [15].

Moreover, when in (1.2) it is $b = 0$, equation (1.1) is related to a viscoelastic fluid [18]-[21] or to a linear viscous gas ($a = 0$) [7], [14].

Some stability properties and wave features for equation (1.1) are discussed in [9]-[12], [16], [22]. However, in [8] the *parabolic* aspects of the operator L are emphasized. This seems to agree with the analysis developed in [3] where basic properties of the fundamental solution of the operator L are proved.

In this paper we investigate stability properties for the nonlinear equation (1.1) basing on the Liapunov second method. We first discuss an initial-boundary-value problem by means of Volterra integral equations, Section 2. Then, we consider a Liapunov function depending on a parameter that plays a flexible role in the different problems we deal with. In particular, we obtain some useful inequalities and the derivate along the solution of the strip problem, Section 3.

Under reasonable hypotheses on the forcing term and employing the definitions of *eventual properties* due to YOSHIZAWA [23], we prove a first theorem concerning the eventual uniform boundedness of the solutions and the eventual quasi uniform asymptotic stability in the large of the origin. Specializing the hypotheses on f, a theorem of quasi uniform asymptotic stability in the large is also obtained. Of course, the assumption $(3.13)_1$ of Theorem 3.1 is incompatible with the condition $f = -c^2 u_{xx} + f_1(x,t,u,u_x,u_t)$.

Next, we discuss the case when the forcing term can be expressed as sum of two terms: the first depends only on u and the second is given by u_t times a bounded function. Considering a new Liapunov function, we show the uniform asymptotic stability in the large of the origin.

Finally, the results are applied to some examples: the *perturbed Sine-Gordon* equation, an equation of Quantum Mechanics and a problem of initial and boundary perturbations.

2 – Existence and uniqueness

We consider the following initial-boundary-value problem

$$Lu = f(x,t,u,u_x,u_{xx},u_t), \qquad 0<x<1, \quad 0<t<T \tag{2.1}$$

where $L = -\varepsilon\partial_{xxt} - c^2\partial_{xx} + \partial_{tt}$ and c and ε are positive constants,

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad 0<x<1, \tag{2.2}$$

$$u(0,t) = h_1(t), \qquad u(1,t) = h_2(t), \qquad 0<t<T\,. \tag{2.3}$$

We discuss the above problem by means of an equivalent integro-differential equation. We start from the identity

$$\begin{aligned} &\partial_\xi(c^2uw_\xi - c^2u_\xi w + \varepsilon u_\xi w_\tau - \varepsilon u w_{\xi\tau}) + \partial_\tau(u_\tau w - uw_\tau - \varepsilon u_{\xi\xi}w) = \\ &\qquad = fw - u(\varepsilon w_{\xi\xi\tau} - c^2 w_{\xi\xi} + w_{\tau\tau}), \end{aligned} \tag{2.4}$$

that follows from (2.1), assuming $u(\xi,\tau)$, $0<\xi<1$, $\tau>0$, is a smooth solution of (2.1). Moreover, we take

$$w(x,\xi,t-\tau) = \theta(x-\xi,t-\tau) - \theta(x+\xi,t-\tau), \tag{2.5}$$

with

$$\theta(x,t) = K(|x|,t) + \sum_{m=1}^{\infty}\left[K(|x+2m|,t) + K(|x-2m|,t)\right], \tag{2.6}$$

$$K(|x|,t) = \int_0^t \frac{e^{-c^2\tau/\varepsilon}}{\sqrt{\pi\varepsilon\tau}}\,d\tau \int_0^\infty \frac{x^2(z+1)}{4\varepsilon\tau} e^{-x^2(z+1)^2/4\varepsilon\tau} I_0\!\left(\frac{c}{\varepsilon}2|x|\sqrt{z}\right) dz, \tag{2.7}$$

where I_0 is the modified Bessel function of order 0. The function K represents the fundamental solution of the linear operator L. It has been determined in [3] where its interesting properties have beeen proved. Since $\theta(-x,t) = \theta(x,t)$ and $\theta(x+2m,t) = \theta(x,t), m \in N$, we consider $0<x<2$ and note that θ is continuous together its partial derivatives and satisfies the equation $L\theta = 0$. Moreover, from the analysis of K developed in [3], we can deduce that θ is a positive function that has properties similar to ones of the analogous function θ used for the heat operator, see [1].

For the data we shall assume that:

(2.8) $f(x,t,u,p,q,r)$ is defined and continuous on the set

$$\{(x,t,u,p,q,r) \mid 0 \le x \le 1,\ 0 \le t \le T,\ -\infty < u,p,q,r < \infty,\ \ T > 0\},$$

(2.9) there exists a constant μ such that

$$\begin{aligned}&|f(x,t,u_1,p_1,q_1,r_1) - f(x,t,u_2,p_2,q_2,r_2)| \le \\ &\le \mu\{|u_1-u_2| + |p_1-p_2| + |q_1-q_2| + |r_1-r_2|\},\end{aligned}$$

$$u_0,\ u_0',\ u_0'',\ u_1 \ \text{ continuous on } \ 0 \le x \le 1, \tag{2.10}$$

$$h_i,\ \dot h_i,\ i=1,2, \ \text{ continuous on } \ 0 \le t \le T, \tag{2.11}$$

$$h_1(0) = u_0(0),\ h_2(0) = u_0(1),\ \dot h_1(0) = u_1(0),\ \dot h_2(0) = u_1(1). \tag{2.12}$$

By integrating (2.4) on $\{(\xi,\tau) \mid 0 < \xi < 1, \delta < \tau < t-\delta\}$, $\delta > 0$ and letting $\delta \to 0$, we get the following integral equation for a solution of (2.1)-(2.3)

$$\begin{aligned}u(x,t) = &\int_0^1 w_t(x,\xi,t)u_0(\xi)d\xi + \int_0^1 w(x,\xi,t)[u_1(\xi) - \varepsilon u_0''(\xi)]d\xi + \\ &- 2\int_0^t h_1(\tau)(c^2+\varepsilon\partial_t)\theta_x(x,t-\tau)d\tau + \\ &+ 2\int_0^t h_2(\tau)(c^2+\varepsilon\partial_t)\theta_x(1-x,t-\tau)d\tau + \\ &+ \int_0^t d\tau \int_0^1 w(x,\xi,t-\tau) f(\xi,\tau,u(\xi,\tau),u_\xi(\xi,\tau),u_\tau(\xi,\tau),u_{\xi\xi}(\xi,\tau))d\xi.\end{aligned} \tag{2.13}$$

Viceversa, as $L\theta = 0$, a solution u of (2.13), under the assumptions (2.8)-(2.12), satisfies (2.1) and the initial conditions (2.2), as one can verify basing on (2.5)-(2.7) and using an argument of [3]. Moreover, we have

$$2(c^2+\varepsilon\partial_t)\theta_x(x,t) = 2(c^2+\varepsilon\partial_t)K_x(x,t) + R(x,t), \qquad 0 < x < 1,$$

where, as it easy to verify,

$$\lim_{x\to 0} R(x,t) = 0, \qquad \lim_{x\uparrow 1}(c^2+\varepsilon\partial_t)\theta_x(x,t) = 0.$$

Consequently, since for $0 < x < 1$

$$\lim_{x\downarrow 0}\int_0^t 2(c^2+\varepsilon\partial_t)K_x(x,t-\tau)h_1(\tau)d\tau = -h_1(t),$$

see [4], the first boundary condition $(2.3)_1$ is satisfied. Similarly, $(2.3)_2$ can be verified.

In the linear case, when $f = f(x,t)$, (2.13) gives the unique explicit solution of (2.1)-(2.3). However, in our nonlinear situation, (2.13) is an integro-differential equation that we shall discuss briefly.

We consider the mapping

$$\begin{aligned} \mathcal{T}u(x,t) =& \int_0^1 w(x,\xi,t)u_0(\xi)d\xi + \int_0^1 w(x,\xi,t)[u_1(\xi)-\varepsilon u_0''(\xi)]d\xi + \\ & -2\int_0^t h_1(\tau)(c^2+\varepsilon\partial_t)\theta_x(x,t-\tau)d\tau + \\ & +2\int_0^t h_2(\tau)(c^2+\varepsilon\partial_t)\theta_x(1-x,t-\tau)d\tau + \\ & +\int_0^t d\tau\int_0^1 w(x,\xi,t-\tau)f(\xi,\tau,u(\xi,\tau),u_\xi(\xi,\tau),u_\tau(\xi,\tau),u_{\xi\xi}(\xi,\tau))d\xi \end{aligned} \tag{2.14}$$

that maps the set $\mathcal{B}$, defined by

$$\mathcal{B} = \{u(x,t) \mid u, u_x, u_t, u_{xx} \in C([0,1]\times[0,T])\},$$

into

$$\begin{aligned} \|u\|_T = & \sup_{[0,1]\times[0,T]} |\mathrm{e}^{-\lambda t}u(x,t)| + \sup_{[0,1]\times[0,T]} |\mathrm{e}^{-\lambda t}u_x(x,t)| + \\ & + \sup_{[0,1]\times[0,T]} |\mathrm{e}^{-\lambda t}u_t(x,t)| + \sup_{[0,1]\times[0,T]} |\mathrm{e}^{-\lambda t}u_{xx}(x,t)|, \end{aligned}$$

where, λ is a positive constant such that

$$\lambda > \max\{1, \mu(2+1/c+3/\varepsilon)\}. \tag{2.15}$$

From (2.14) we get

$$|\mathcal{T}u_1(x,t)-\mathcal{T}u_2(x,t)|\mathrm{e}^{-\lambda t} \leq \mu\|u_1-u_2\|_T \int_0^t \mathrm{e}^{-\lambda(t-\tau)}d\tau \int_0^1 |w(x,\xi,t-\tau)|d\xi,$$

where

$$\int_0^1 |w(x,\xi,t-\tau)|d\xi = \int_0^1 |\theta(x-\xi,t-\tau)-\theta(x+\xi,t-\tau)|d\xi \leq t-\tau.$$

From here we have,

$$|\mathcal{T}u_1(x,t)-\mathcal{T}u_2(x,t)|\mathrm{e}^{-\lambda t} \leq \frac{\mu}{\lambda^2}\|u_1-u_2\|_T. \tag{2.16}$$

Morever, making use of the basic properties of K proved in [3], the following estimations can be deduced:

$$\int_0^1 |\theta_x(x-\xi,t-\tau)-\theta_x(x+\xi,t-\tau)|d\xi \leq 1/c,$$
$$\int_0^1 |\theta_t(x-\xi,t-\tau)-\theta_t(x+\xi,t-\tau)|d\xi \leq 1,$$
$$\int_0^1 |(\partial_t-\partial_x^2)\theta(x-\xi,t-\tau)-(\partial_t-\partial_x^2)\theta(x+\xi,t-\tau)|d\xi \leq 1.$$

From here one can get results similar to (2.16) related to the partial derivatives ∂_x, ∂_t, ∂_x^2 of (2.14). Thus, we achieve

$$\|\mathcal{T}u_1(x,t)-\mathcal{T}u_2(x,t)\|_T \leq \mu\left(\frac{1}{\lambda^2}+\frac{1}{c\lambda}+\frac{1}{\lambda}+\frac{3}{\varepsilon\lambda}\right)\|u_1-u_2\|_T. \tag{2.17}$$

Under assumption (2.15), inequality (2.17) shows that $\mathcal{T}$ is a contraction of $\mathcal{B}$ into itself. Thus we can state the following

THEOREM 2.1. *Under hypotheses* (2.8)-(2.12)*, the nonlinear problem* (2.1)-(2.3) *has a unique smooth solution.*

3 – Stability via the Liapunov direct method

Let us define the set

$$X = \{(x,t) \mid x \in]0,1[,\ t > t_0 \in J = [0,\infty[\}$$

and consider the problem

$$(3.1) \qquad \begin{cases} -\varepsilon u_{xxt} + u_{tt} - c^2 u_{xx} = f(x,t,u,u_x,u_{xx},u_t),\ (x,t) \in X, \\ u(0,t) = 0,\ u(1,t) = 0, \end{cases}$$

with the initial conditions

$$(3.2) \qquad u(x,t_0) = u_0(x),\ u_t(x,t_0) = u_1(x).$$

In the space $C_0^2([0,1]) \times C_0([0,1])$ we introduce the distance of an element (φ,ψ) from the origin $O = \{\varphi = 0, \psi = 0\}$ setting

$$(3.3) \qquad d^2(\varphi,\psi) = \int_0^1 (\varphi^2 + \varphi_x^2 + \varphi_{xx}^2 + \psi^2)dx.$$

For the solution $u \in C^2$, $u_t(\cdot,t) \in C$ of (3.1) we shall write $d(u(t),u_t(t))$ instead of $d(u(\cdot,t),u_t(\cdot,t))$; in particular $d(u_0,u_1)$ means the initial value $d(u(t_0),u_t(t_0))$.

3.1 – Definitions

Definition 3.1. The solutions of (3.1) are eventually uniform-bounded (e.u.b.) if for any $\alpha > 0$ there exist an $s(\alpha) \geq 0$ and a $\beta(\alpha) > 0$ such that if $t_0 \geq s(\alpha)$, $d(u_0,u_1) \leq \alpha$, then $d(u(t),u_t(t)) < \beta(\alpha)$ for all $t \geq t_0$. If $s(\alpha) = 0$ the solutions of (3.1) are uniformly bounded (u.b.).

Definition 3.2. The origin O of $C_0^2([0,1]) \times C_0([0,1])$ is eventually quasi-uniform-asymptotically stable in the large (e.q.u.a.s.l.) for the solutions of (3.1) if for any $\varepsilon > 0$ and $\alpha > 0$ there exist an $s(\alpha) \geq 0$ and a $T(\varepsilon,\alpha) > 0$ such that if $t_0 \geq s(\alpha)$, $d(u_0,u_1) \leq \alpha$, then $d(u(t),u_t(t)) < \varepsilon$ for all $t \geq t_0 + T(\varepsilon,\alpha)$. If $s(\alpha) = 0$, O is said to be quasi-uniform-asymptotically stable in the large (q.u.a.s.l.) for the solutions of (3.1).

Definition 3.3. The origin O of $C_0^2([0,1])\times C_0([0,1])$ is exponential-asymptotically stable in the large (ex.a.s.l.) if for any $\alpha>0$ there are two positive constants $D(\alpha), C(\alpha)$ such that if $d(u_0,u_1)\le\alpha$, then

$$d(u(t),u_t(t))\le D(\alpha)\exp\left[-C(\alpha)(t-t_0)\right]d(u_0,u_1),\qquad \forall t\ge t_0.$$

3.2 – Preliminary results

Consider the functional

$$V(\varphi,\psi)=\frac{1}{2}\int_0^1\{(\varepsilon\varphi_{xx}-\psi)^2+\gamma\psi^2+c^2(1+\gamma)\varphi_x^2\}dx, \tag{3.4}$$

where γ is an arbitrary positive constant. It results

$$V\le\frac{1}{2}\int_0^1\{\varepsilon^2\varphi_{xx}^2+\psi^2+\varepsilon\varphi_{xx}^2+\varepsilon\psi^2+\gamma\psi^2+c^2(1+\gamma)\varphi_x^2\}dx;$$

setting

$$c_2^2=\max\{c^2(1+\gamma)/2,\varepsilon(1+\varepsilon)/2,(1+\varepsilon+\gamma)/2\}, \tag{3.5}$$

we derive

$$V(\varphi,\psi)\le c_2^2d^2(\varphi,\psi). \tag{3.6}$$

On the other hand, it is known that

$$\varphi(0)=0\implies\int_0^1\varphi_x^2(x)dx\ge\int_0^1\varphi^2(x)dx, \tag{3.7}$$

and [5]

$$\varphi(0)=0,\ \varphi(1)=0\implies\int_0^1\varphi_{xx}^2(x)dx\ge\int_0^1\varphi_x^2(x)dx. \tag{3.8}$$

As it is also

$$\begin{aligned}V(\varphi,\psi)=\frac{1}{2}\int_0^1\{&(\varepsilon\varphi_{xx}/2-\psi)^2+(\varepsilon\varphi_{xx}-\psi)^2/2+\varepsilon^2\varphi_{xx}^2/4+\\&+(\gamma-1/2)\psi^2+c^2(1+\gamma)\varphi_x^2\}dx,\end{aligned}$$

from (3.7),(3.8) we get

$$V(\varphi,\psi) \geq \frac{1}{2}\int_0^1 \left\{ \frac{\varepsilon^2}{8}\varphi_{xx}^2 + c^2(1+\gamma)\varphi_x^2 + \frac{\varepsilon^2}{8}\varphi^2 + \left(\gamma - \frac{1}{2}\right)\psi^2 \right\} dx$$

and hence, if

$$c_1^2 = \min\{\varepsilon^2/16, c^2(1+\gamma)/2, (\gamma - 1/2)/2\},\ (\gamma > 1/2), \tag{3.9}$$

it results

$$V(\varphi,\psi) \geq c_1^2 d^2(\varphi,\psi). \tag{3.10}$$

Along a solution of (3.1)

$$\dot V(u,u_t) = \int_0^1 \{(\varepsilon u_{xxt} - u_{tt})(\varepsilon u_{xx} - u_t) + \gamma u_t u_{tt} + c^2(1+\gamma)u_x u_{xt}\}dx;$$

integrating by parts and considering (3.1), it results

$$\dot V = \int_0^1 \{-\varepsilon c^2 u_{xx}^2 - \varepsilon\gamma u_{xt}^2 + (1+\gamma)f u_t - \varepsilon f u_{xx}\}dx. \tag{3.11}$$

3.3 – Stability results for a general forcing term

Now we are able to give some stability properties concerning problem (3.1).

Theorem 3.1. *Suppose that for the function f of* (3.1) *there exists a positive constant M, and two continuous functions $g_i(t,\eta) \geq 0$, $i = 1,2$, $t \in J$, $\eta > 0$, such that*

$$\begin{aligned} d(\varphi,\psi) \leq \eta \Longrightarrow f^2(x,t,\varphi,\varphi_x,\varphi_{xx},\psi) \leq \\ \leq M(\varphi^2 + \varphi_x^2 + \varphi_{xx}^2 + \psi^2) + \\ + [(\varepsilon/2c^2) + 2/\varepsilon]^{-1}[g_1(t,\eta) + g_2(t,\eta)]. \end{aligned} \tag{3.12}$$

Moreover, admit

$$\frac{\varepsilon c^2}{6} - M\left(\frac{\varepsilon}{2c^2} + \frac{2}{\varepsilon}\right) > 0\ ,\ \frac{\varepsilon}{2} - M\left(\frac{\varepsilon}{2c^2} + \frac{2}{\varepsilon}\right) > 0\ , \tag{3.13}$$

$$g_1(t,\eta) \to 0 \ \ as \ \ t \to +\infty,\ \int_0^\infty g_2(t,\eta)dt < +\infty,\ \eta > 0. \tag{3.14}$$

Then the solution of (3.1) *are e.u.b. and the origin O is e.q.u.a.s.l.*

PROOF. Choose in (3.4) $\gamma = 1$. From (3.11) we get

$$\dot{V} = \int_0^1 \Big\{ -\frac{\varepsilon c^2}{2} u_{xx}^2 - \varepsilon u_{xt}^2 + \frac{\varepsilon}{2} u_t^2 - \frac{\varepsilon}{2} (c u_{xx} + f/c)^2 + -\frac{\varepsilon}{2} (u_t - 2f/\varepsilon)^2 + \Big(\frac{\varepsilon}{2c^2} + \frac{2}{\varepsilon} \Big) f^2 \Big\} dx$$

and hence, owing to (3.7), (3.8), (3.12), if $d(u, u_t) \leq \eta$ then

$$\dot{V}(u, u_t) \leq -\int_0^1 \Big\{ \Big[\frac{\varepsilon c^2}{6} - \Big(\frac{\varepsilon}{2c^2} + \frac{2}{\varepsilon} \Big) M \Big] (u^2 + u_x^2 + u_{xx}^2) + + \Big[\frac{\varepsilon}{2} - \Big(\frac{\varepsilon}{2c^2} + \frac{2}{\varepsilon} \Big) M \Big] u_t^2 \Big\} dx + g_1(t, \eta) + g_2(t, \eta).$$

Considering (3.13), we obtain the positive constant

$$c_3^2 = \min \Big\{ \frac{\varepsilon c^2}{6} - \Big(\frac{\varepsilon}{2c^2} + \frac{2}{\varepsilon} \Big) M, \;\; \frac{\varepsilon}{2} - \Big(\frac{\varepsilon}{2c^2} + \frac{2}{\varepsilon} \Big) M \Big\}, \tag{3.15}$$

and consequently the previous inequality becomes

$$\dot{V}(u, u_t) \leq -c_3^2 d^2(u, u_t) + g_1(t, \eta) + g_2(t, \eta). \tag{3.16}$$

Exploiting (3.16) and (3.6), considered for $\gamma = 1$, we finally obtain that, if $d(u, u_t) \leq \eta$ then

$$\dot{V}(u, u_t) \leq -(c_3/c_2)^2 V(u, u_t) + g_1(t, \eta) + g_2(t, \eta). \tag{3.17}$$

For each $\alpha > 0$ we choose $\beta(\alpha) = \sqrt{3}\alpha c_2/c_1$, where c_2 and c_1 are defined by (3.5) and (3.9) when $\gamma = 1$. Now, employing an argument of [23], we consider the solution $u(x, t)$ of (3.1) satisfying initial conditions (3.2) such that $d(u_0, u_1) \leq \alpha$. As long as $d(u(t), u_t(t)) \leq \beta(\alpha)$ we draw from (3.17)

$$\dot{V}(u(t), u_t(t)) \leq -hV(u(t), u_t(t)) + g_1(t, \beta) + g_2(t, \beta), \;\; h = (c_3/c_2)^2. \tag{3.18}$$

Let us indicate by $\omega(t)$ the solution of

$$\dot{\omega}(t) = -h\omega(t) + g_1(t, \beta) + g_2(t, \beta), \;\; \omega(t_0) = V(u_0, u_1); \tag{3.19}$$

it results

$$\omega(t) = V(u_0, u_1)e^{-h(t-t_0)} + e^{-ht}\int_{t_0}^{t} e^{hs}[g_1(s,\beta) + g_2(s,\beta)]ds.$$

By (3.6) we derive

$$V(u_0, u_1) \leq c_2^2\alpha^2.$$

From $(3.14)_1$ we obtain a $T_1(\alpha) > 0$ such that for $t \geq t_0 \geq T_1(\alpha)$ it is $g_1(t,\beta) < c_3^2\alpha^2$; therefore,

$$e^{-ht}\int_{t_0}^{t} e^{hs} g_1(s,\beta)ds < c_2^2\alpha^2.$$

From $(3.14)_2$ we have a $T_2(\alpha) > 0$ such that

$$\int_{t_0}^{\infty} e^{-h(t-s)} g_2(s,\beta)ds < c_2^2\alpha^2, \quad \text{for} \quad t_0 \geq T_2(\alpha).$$

Thus, for the solution $\omega(t)$ we get

$$\omega(t) < 3c_2^2\alpha^2.$$

On the other hand, employing the comparison principle and (3.10) when $\gamma = 1$, the condition

$$d(u(t), u_t(t)) < \beta(\alpha)$$

for

$$t \geq t_0 \geq s(\alpha) = \max\{T_1(\alpha),\ T_2(\alpha)\}$$

is proved. So the solutions of (3.1) are e.u.b.

Now (3.18) can be considered for $t \geq t_0 \geq s(\alpha)$. Therefore, applying Lemma 24.3 of [23] to the solution ω of (3.19), for every $\rho > 0$ and $\alpha > 0$ there is a $T(\rho, \alpha) > 0$ such that if $d(u_0, u_1) \leq \alpha$, then $d(u(t), u_t(t)) < \rho$ for all $t \geq t_0 + T$ and hence the set $\{u = 0, u_t = 0\}$ is e.q.u.a.s.l.

As a particular case of the previous theorem we obtain

Theorem3.2. *Under the assumptions of Theorem 3.1 then:*

A) *if in* (3.12) *the functions* $g_i = g_i(t)$ $\forall t \in J$ *and* $i = 1, 2$ *are indipendent of* η, *the solutions of* (3.1) *are e.u.b. and moreover the set* $\{u = 0, u_t = 0\}$ *is q.u.a.s.l.;*

B) *if in* (3.12) *the functions* $g_i(t) \equiv 0$ $\forall t \in J$ *and* $i = 1, 2$, *the origin* O *is ex.a.s.l.*

PROOF. If A) holds, formula (3.17) becomes

$$\dot{V}(u, u_t) \leq -hV(u, u_t) + g_1(t) + g_2(t).$$

for each value of $d(u, u_t)$. Therefore, the solutions of (3.1) are e.u.b.; moreover equation (3.19) holds for every $t_0 \in J$ and it is independent of $\beta(\alpha)$. Thus, Lemma 24.3 of [23] implies the set $\{u = 0, u_t = 0\}$ is q.u.a.s.l.

If B) is satisfied the result is straitght-forward.

3.4 – Stability for a special f

Now we specialize the function f of (3.1) as $F(u)-a(x,t,u,u_x,u_t,u_{xx})u_t$, where $F \in C(\mathsf{R})$ and $a \in C(]0,1[\times \overset{\circ}{\mathsf{J}} \times \mathsf{R}^4)$, and examine the particular problem

$$\left\{ \begin{array}{l} Lu = F(u) - a(x,t,u,u_x,u_t,u_{xx})u_t,\ (x,t) \in X, \\ u(0,t) = 0,\ u(1,t) = 0, \end{array} \right. \tag{3.20}$$

with initial conditions (3.2).

We add a new term to the functional defined by (3.4) changing V into

$$\begin{aligned} W(\varphi, \psi) = \frac{1}{2} \int_0^1 \{(\varepsilon\varphi_{xx} - \psi)^2 + \gamma\psi^2 + c^2(1+\gamma)\varphi_x^2\}dx + \\ - \int_0^1 \left\{(1+\gamma) \int_0^\varphi F(z)dz\right\} dx \end{aligned} \tag{3.21}$$

and consequently exploiting (3.11) we obtain immediately

$$\dot{W}(u, u_t) = -\int_0^1 \{c^2\varepsilon u_{xx}^2 + \varepsilon\gamma u_{xt}^2 + a(1+\gamma)u_t^2 + \varepsilon F(u)u_{xx} - \varepsilon a u_{xx} u_t\}dx.$$

From this one considering inequalities (3.7), (3.8) it follows

$$\begin{aligned} \dot{W}(u, u_t) \leq -\int_0^1 \{(3/4)c^2\varepsilon u_{xx}^2 + [\varepsilon\gamma + a(1+\gamma-\varepsilon a/c^2)]u_t^2 + \varepsilon F u_{xx} + \\ + \varepsilon[(c/2)u_{xx} - (a/c)u_t]^2\}dx. \end{aligned} \tag{3.22}$$

Now we are able to show the following

THEOREM 3.3. *Suppose that $F(u) \in C^1(\mathsf{R})$, $F(0) = 0$, and there exists a positive constant K such that*

$$F_u \le K < 3c^2/4. \tag{3.23}$$

Moreover, admit the function a satisfies

$$\inf a > -\varepsilon, \quad \sup a < +\infty. \tag{3.24}$$

Then the set $\{u = 0, u_t = 0\}$ is u.a.s.l.

PROOF. Condition $F(0) = 0$ considered together with (3.23) implies

$$\int_0^\varphi F(z)dz \le 3c^2\varphi^2/8$$

and therefore, employing (3.7) too, from (3.21) we get

$$\begin{aligned} W(\varphi, \psi) \ge \frac{1}{2}\int_0^1 \{&(\varepsilon\varphi_{xx} - 2\psi)^2/4 + (\varepsilon\varphi_{xx} - \psi)^2/2 + (\gamma - 1/2)\psi^2 + \\ &+ c^2(1+\gamma)\varphi^2/4 + \varepsilon^2(\varphi_{xx}^2 + \varphi_x^2)/8\}dx \end{aligned}$$

and hence if

$$k_1^2 = \min\{\varepsilon^2/16,\ c^2(1+\gamma)/8,\ (2\gamma - 1)/4\},\ \gamma > 1/2, \tag{3.25}$$

it results

$$W(\varphi, \psi) \ge k_1^2 d^2(\varphi, \psi). \tag{3.26}$$

Now, using an argument employed in [2] and exploiting condition $F(0) = 0$, we obtain

$$\left|\int_0^\varphi F(z)dz\right| = \left|\int_0^\varphi F_\zeta(\zeta)(\varphi - \zeta)d\zeta\right|.$$

Consequently, setting

$$m(|\varphi|) = \max\{|F_\zeta(\zeta)| \ :\ |\zeta| \le |\varphi|\}$$

we have

$$\left|\int_0^{\varphi} F(z)dz\right| \le m(|\varphi|)\varphi^2/2. \tag{3.27}$$

Therefore, considering (3.21) and using (3.5) and (3.27), it results

$$W(\varphi,\psi) \le c_2^2 d^2(\varphi,\psi) + (1+\gamma)/2\int_0^1 m(|\varphi|)\varphi^2 dx. \tag{3.28}$$

Finally, because $\varphi(0) = 0$ implies $\varphi^2 \le d^2(\varphi,\psi)$, (3.28) gives

$$W(\varphi,\psi) \le c_2^2[1 + m(d(\varphi,\psi))]d^2(\varphi,\psi). \tag{3.29}$$

Now, we refer to (3.22) integrating by parts the third term. Exploiting hypotheses $F(0) = 0$ and (3.23), inequality (3.8) and boundary conditions $(3.20)_2$, we deduce

$$\begin{aligned}\dot W(u(t),u_t(t)) \le -\int_0^1 \{&(3/4)\varepsilon c^2(1-\lambda)u_{xx}^2 + \varepsilon(3c^2\lambda/4 - K)u_x^2 + \\ &+ [\varepsilon\gamma + a(1+\gamma-\varepsilon a/c^2)]u_t^2\}dx,\end{aligned}$$

where $\lambda \in]0,1[$ is a constant choosed in such a way that $3c^2\lambda/4 - K > 0$. Owing to (3.24) we assume

$$\gamma = [1 + \sup|a(a\varepsilon/c^2 - 1)|]/(\varepsilon + \inf a) + \frac{1}{2}$$

and observe

$$\dot W(u,u_t) \le -\int_0^1 \{(3/4)\varepsilon c^2(1-\lambda)u_{xx}^2 + \varepsilon(3c^2\lambda/4 - K)u_x^2 + u_t^2\}dx.$$

Finally, considering (3.7) and (3.8), it results

$$\dot W(u(t),u_t(t)) \le -k_3^2 d^2(u(t),u_t(t)) \tag{3.30}$$

where

$$k_3^2 = \min\{3\varepsilon c^2(1-\lambda)8,\ \varepsilon(3c^2\lambda/4 - K),\ 1\}. \tag{3.31}$$

Using (3.26), (3.29), (3.30) we obtain

$$d(u(t), u_t(t)) < (c_2/k_1)[1 + m(d(u_0, u_1))]d(u_0, u_1). \tag{3.32}$$

Moreover, if α is a positive constant, we set

$$\beta(\alpha) = (c_2/k_1)[1 + m(\alpha)]\alpha$$

and hence from (3.32) we derive

$$d(u(t), u_t(t)) < \beta(\alpha) \quad \text{for} \quad d(u_0, u_1) \leq \alpha \quad \text{and} \quad t \geq t_0 \tag{3.33}$$

and therefore the uniform boundness of the solutions of problem (3.20) is shown.

Referring to (3.29), for the solutions of (3.20) satisfying (3.33), we get

$$W(u(t), u_t(t)) \leq c_2^2[1 + m(\beta(\alpha))]d^2(u(t), u_t(t))$$

and therefore (3.30) implies

$$\dot{W}(u(t), u_t(t)) < -C(\alpha)W(u(t), u_t(t)) \tag{3.34}$$

where $C(\alpha) = k_3^2/c_2^2[1 + m(\beta(\alpha))]$. Considering (3.34) jointly to (3.29) we check Definition 2.3 is fulfilled.

4 – Examples

4.1 – We consider equation $(3.20)_1$ when

$$F(u) = -b \sin u, \ b = \text{constant}, \ a = 0.$$

If $|b| < 3c^2/4$, all the hypotheses of Theorem 3.3 are fulfilled and therefore the set $\{u = 0, u_t = 0\}$ is u.a.s.l. with respect to the metric d.

4.2 – We can employ Theorem 3.3 to the equation $(3.20)_1$ when

$$F(u) = -K|u|^p u, \ p = \text{constant} > 0, \ a = 0.$$

If K is a positive constant, we get the u.a.s.l. of the set $\{u = 0, u_t = 0\}$.

4.3 – Now we examine the initial-boundary-value problem

$$\begin{cases} Lv = F(v),\ (x,t) \in X, \\ v(0,t) = h_1(t),\ v(1,t) = h_2(t),\ t > t_0 \in J, \\ v(x,t_0) = v_0(x),\ v_t(x,t_0) = v_1(x),\ x \in]0,1[, \end{cases} \tag{4.1}$$

where $v_0 \in C^2([0,1])$, $v_1 \in C([0,1])$, $h_i \in C^2([0,1])$, $i = 1,2$, with the compatibility conditions

$$\begin{cases} v_0(0) = h_1(t_0),\ v_0(1) = h_2(t_0) \\ v_1(0) = \dot{h}_1(t_0),\ v_1(1) = \dot{h}_2(t_0). \end{cases} \tag{4.2}$$

Setting

$$p(x,t) = (1-x)h_1(t) + xh_2(t),\ x \in]0,1[,\ t \in J, \tag{4.3}$$

we consider the function

$$v(x,t) = u(x,t) + p(x,t). \tag{4.4}$$

Using (4.4) we map the problem (4.1)-(4.2) into the problem (3.1)-(3.2) where

$$f = F(u+p) - p_{tt}. \tag{4.5}$$

Assume the function F satisfies

$$|F(v)| \leq K|v| \tag{4.6}$$

for some positive constant K. Therefore

$$f^2 \leq Mu^2 + G_1(t) + G_2(t)$$

where

$$M = 4K^2,\ G_1(t) = 8K^2\left(h_1^2(t) + h_2^2(t)\right),\ G_2(t) = 4\left(\ddot{h}_1^2(t) + \ddot{h}_2^2(t)\right).$$

Thus, if

$$\frac{\varepsilon c^2}{6} - 4K^2\left(\frac{\varepsilon}{2c^2}+\frac{2}{\varepsilon}\right) > 0, \quad \frac{\varepsilon}{2} - 4K^2\left(\frac{\varepsilon}{2c^2}+\frac{2}{\varepsilon}\right) > 0, \tag{4.7}$$

and for $i = 1, 2$

$$h_i(t) \to 0 \ \text{ as } \ t \to +\infty, \tag{4.8}$$

$$\ddot{h}_i(t) \to 0 \quad \text{as} \quad t \to +\infty \quad \text{or} \quad \ddot{h}_i \in L^2(J), \tag{4.9}$$

the results of Theorem 3.2 hold.

As regard to the problem (4.1)-(4.2) if in addition to (4.6)-(4.9) it is

$$\dot{h}_i(t) \to 0 \ \text{ as } \ t \to +\infty, \ \ i = 1, 2,$$

from (4.3)-(4.4) we have the solutions of (4.1)-(4.2) are bounded and moreover $d(v(t), v_t(t)) \to 0$ as $t \to +\infty$. These results are verified in a uniform sense both with respect to the initial and boundary conditions if we choose the boundary perturbations $h_i(t)$ satisfying

$$\sum_i \left(|h_i(t)| + |\dot{h}_i(t)|\right) \le h(t) \to 0, \quad \text{as} \quad t \to +\infty,$$

$$\sum_i |\ddot{h}_i(t)| \le H(t)$$

and

$$\sum_i |\ddot{h}_i(t)| \le H(t) \to 0, \ \ \text{as} \ \ t \to +\infty,$$

or

$$\int_0^\infty \sum_i |\ddot{h}_i(t)|^2 dt \le \int_0^\infty H(t)^2 dt < +\infty.$$

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