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A real Schwarz lemma and some applications

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RIASSUNTO: *Il "Volume minimo" di una varietà differenziabile è stato introdotto da M. Gromov allo scopo di generalizzare (in dimensione qualunque) le disuguaglianze dedotte dalla teoria delle classi caratteristiche di Gauss-Bonnet-Chern-Weil. La generalizzazione è stata ottenuta assegnando una limitazione per il volume minimo in termini del volume simpliciale. Calcolando i volumi simpliciali delle varietà iperboliche, M. Gromov (con W. Thurston) ha reso tale disuguaglianza esplicita (ed ha rivisitato il teorema di rigidità di Mostow). Egli ha formulato la congettura che tale disuguaglianza possa essere precisa, cioè che il volume minimo sia raggiunto con la metrica iperbolica. Noi abbiamo dimostrato questa congettura stabilendo un analogo reale del Lemma di Schwartz: se X , Y sono due varietà tali che la curvatura di X è negativa e minore di quella di Y, allora ogni classe di omotopia di applicazioni da Y a X contiene una applicazione che contrae i volumi. Noi abbiamo dato una costruzione esplicita di questa applicazione che, nelle ipotesi dei teoremi di rigidità di Mostow, risulta una isometria, provvedendo una dimostrazione unificata degli stessi teoremi. Dimostriamo inoltre che l'insieme di tutte le metriche di Einstein su ogni varietà iperbolica 4-dimensionale si riduce ad un singolo elemento. Una versione modificata del Lemma di Schwartz reale dà una disuguaglianza precisa tra le entropie di Y e X (ammesso che X sia localmente simmetrica e che esista una applicazione di grado non banale da Y a X). Ciò dà una risposta alle congetture di M. Gromov e A. Katok riguardo alla entropia minima. Poiché tale disuguaglianza è una uguaglianza se e solo se Y è isometrica a X, essa implica che Y e X hanno flussi geodetici coniugati se e solo se le due varietà sono isometriche. Ciò completa anche la dimostrazione della congettura di Lichnerowicz: ogni varietà localmente armonica, compatta, a curvatura negativa è un quoziente di uno spazio simmetrico di rango 1 (noncompatto).*

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Abstract: *The "Minimal Volume" of a differentiable manifold was introduced by M. Gromov in order to generalize (in any dimension) the inequalities deduced from the Gauss-Bonnet-Chern-Weil theory of characteristic classes. By bounding the minimal volume in terms of the simplicial volume, M. Gromov gave such a generalization. By computing the simplicial volumes of the hyperbolic manifolds, M. Gromov (with W. Thurston) made this inequality explicit (and revisited Mostow's rigidity theorem). He conjectured that this inequality might be sharpen, (i.e. that the minimal volume is attained for the hyperbolic metric). We proved this conjecture by settling a real analogue of the Schwarz's lemma: if X, Y are two manifolds such that the curvature of X is negative and smaller than the one of Y, then any homotopy class of maps from Y to X contains a map which contracts volumes. We give an explicit construction of this application which, under the assumptions of Mostow's rigidity theorems, occurs to be an isometry, providing a unified proof of these theorems. It moreover proves that the set of all Einstein metrics, on any compact 4-dimensional hyperbolic manifold, reduces to a single point. A modified version of the real Schwarz's lemma gives a sharp inequality between the entropies of Y and X (provided that X is locally symmetric and that there exists an application of non trivial degree from Y to X). This answers conjectures of M. Gromov and A. Katok about the minimal entropy. As this inequality is an equality iff Y is isometric to X, it implies that Y and X have conjugate geodesic flows iff they are isometric. This also ends the proof of the Lichnerowicz's conjecture: any negatively curved compact locally harmonic manifold is a quotient of a (noncompact) rank-one-symmetric space.*

1 – The problem of minimal (and maximal) volume

Let M be a compact connected manifold; its "minimal volume" (denoted by $Min\,Vol(M)$) has been defined by M. Gromov [14] as being the infimum of the volumes of all riemannian metrics g on M, whose sectional curvature K_g satisfies $-1 \leq K_g \leq 1$.

Similarly, when the manifold admits some metric with strictly negative sectional curvature, one may define the "maximal volume" of M as the supremum of $Vol(g)$, for all metrics g whose sectional curvature satisfies $K_g \leq -1$.

Example. In dimension 2, one can compute the exact value of the minimal (resp. the maximal) volume of any manifold and, moreover, characterize all the metrics for which this minimum (resp. this maximum) is attained. In fact, the Gauss-Bonnet formula gives

$$\int_M K_g dv_g = 2\pi\chi(M)\,,$$

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where $\chi(M)$ is the Euler characteristic of the surface M. This immediately implies that

$$Min\,Vol(M) = 2\pi|\chi(M)|\,.$$

Similarly, assuming that $\chi(M) < 0$ and $K_g \leq -1$, we get:

$$Max\,Vol(M) = 2\pi|\chi(M)|\,.$$

It is then obvious that the minimal and the maximal volume are both attained for (and only for) metrics with constant sectional curvature ± 1. $\square$

It is thus a natural question to ask whether results of this kind exist in higher dimensions. In the even dimensional case, the analogue of the Gauss-Bonnet formula is the Allendœrfer-Chern-Weil one which writes:

$$\chi(M) = \int_M P(R_g)dv_g\,,$$

where P is the universal polynomial of degree $n/2$ on the exterior algebra, called the Pfaffian polynomial and where R_g is the curvature tensor, viewed as a matrix with coefficients in $\Lambda^2(T^*M)$. An immediate consequence is that, for any metric g on M,

$$|\chi(M)| \leq C_n \|K_g\|_{L^\infty}^{n/2} \cdot\ Vol(M,g)\,,$$

which implies that $Min\,Vol(M) \geq C_n^{-1} \cdot |\chi(M)|$, where C_n is a universal constant (other lower bounds for the minimal volume may be obtained by using other characteristic classes than the Euler class).

Though the above inequality provides a lower bound for the minimal volume in terms of a homology-invariant, this result does not fit our purpose for two reasons, already underlined by M. Gromov:

(1) Except for dimension 2, it can never be sharp, so it cannot help computing the exact value of the minimal volume. Moreover, it gives no information about the fact that this minimal volume may be attained for some metric or not.

(2) The Euler characteristic is a quite rough invariant, which often vanishes, in particular for every odd-dimensional manifold. In these cases, the above inequality is nothing but trivial.

2 – M. Gromov's approach

a) A bound from below for the minimal volume

A first main result is that the above inequality remains valid if one replaces the Euler characteristic by another homology-invariant: the simplicial volume (denoted by *Simpl Vol*). This writes:

THEOREM 2.1 (M. Gromov, [14]). *For any manifold M, one has $Min\,Vol(M) \geq C_n\, Simpl\,Vol(M)$, where C_n is a universal constant (only depending on the dimension).*

As illustrated in Section 1, this inequality may be seen as the most natural generalization of the Gauss-Bonnet inequality that one may expect if one wants it to be valid in any dimension. Notice that the simplicial volume may be non trivial in any dimension (for instance, as we shall see later, it is non trivial for any negatively curved manifold).

Let us recall the definition of the "simplicial volume": let us consider the L^1-norm on the real singular chains, defined by $\|c\|_1 = \sum |\lambda_i|$ when c is the linear combination of simplices $c = \sum \lambda_i \sigma_i$. The associated seminorm on $H_k(M, R)$ is defined by

$$\|\gamma\| = \; Inf\{\|c\|_1 / c \text{ closed chain s.t. } [c] = \gamma\}\,.$$

We may then define the simplicial volume of M as being $\|[M]\|$, where $[M]$ is the fundamental n-class of M. If the coefficients were integers, the simplicial volume could be interpreted as the minimal number of n-simplices in a simplicial triangulation of M. As the coefficients are real (or, equivalently, rational), one accepts triangulations covering the manifold p-times and may see ([14]) the simplicial volume as:

$$Inf_{p\in N} \left(\frac{1}{p}\,(\text{Minimal number of simplices in a triangulation of } p\cdot[M])\right).$$

b) Computation of some simplicial volumes

OBVIOUS PROPERTY 2.2. *Let $f: Y \to X$ be a continuous map between two compact manifolds of the same dimension, then $Simpl\,Vol(Y) \geq |\deg f| \cdot Simpl\,Vol(X)$.*

PROOF. If $[Y]$ is represented by $\sum \lambda_i \sigma_i$, then $[X] = \frac{1}{\deg f} f_*([Y])$ is represented by $\sum \frac{\lambda_i}{\deg f}(f \circ \sigma_i)$. This implies that $\|[X]\| \leq \sum \frac{|\lambda_i|}{|\deg f|}$. ∎

This gives a method for proving that some simplicial volumes are trivial, for example one has the following immediate corollary:

COROLLARY 2.3. *If there exists a continuous map $f : X \to X$ whose degree is different from -1, 0 or $+1$, then* $Simpl\,Vol(X) = 0$.

For instance, this proves that the simplicial volumes of S^n and $\mathbf{T^n}$ are trivial for any n. Notice that the minimal volume of $\mathbf{T^n}$ is also trivial (just write $\mathbf{T^n}$ as $\mathbf{T} \times \mathbf{T^{n-1}}$ and multiply the first factor by ε: the volume goes to zero, but the curvature is zero, and thus bounded); the minimal volume of S^{2p+1} is also trivial (just see it as the total space of the Hopf-fibration with fiber S^1 and multiply the metric of the fiber by ε: the volume goes to zero and the curvature remains bounded; these examples are from M. BERGER, cf. [8 p. 70] and [26]). On the contrary, the Allendœrfer-Chern-Weil formulas (cf. Section 1) show that the minimal volume of S^{2p} is non zero; as its simplicial volume is trivial, the Theorem 2.1 gives nothing in this case.

On the contrary, the simplicial volume is non zero when the (compact) manifold admits a metric with strictly negative curvature. When it admits a hyperbolic metric, one has the following exact computation of the simplicial volume:

THEOREM 2.4 (M. GROMOV and W. Thurston, see [14]). *Let X be a compact riemannian manifold whose sectional curvature is constant and equal to -1, then $Simpl\,Vol(X) = T_n^{-1}\,Vol(X)$, where T_n is the supremum of the volumes of all geodesic n-simplices on the real hyperbolic space-form.*

Let us recall that a "geodesic simplex" is a simplex whose boundary is made of pieces of totally geodesic hypersurfaces.

2.5. One interpretation of the Theorem 2.4 is that all possible volumes of compact hyperbolic manifolds are n-homology invariants. For example, any 4-dimensional compact hyperbolic manifold satisfies

$$Vol(X) = \frac{\chi(X)}{2} \cdot\ Vol(S^4)\,.$$

PROOF. As X and S^4 are Einstein manifolds, the Allendœrfer-Weil formula writes, in this case:

$$8\pi^2\chi(X) = \int_X \|R_X\|^2 ,$$
$$8\pi^2\chi(S^4) = \int_{S^4} \|R_{S^4}\|^2 ,$$

where the curvature tensors R_X and R_{S^4} of the two manifolds X and S^4 satisfy $\|R_X\| = \|R_{S^4}\| = $ Constant, because both manifolds have constant curvature ± 1. This gives $\frac{Vol(X)}{Vol(S^4)} = \frac{\chi(X)}{\chi(S^4)} = \frac{\chi(X)}{2}$, since $\chi(S^4) = 2$. ∎

In the odd-dimensional case, it is more difficult to compute the possible volumes of a compact hyperbolic manifold, for instance, an important open problem is the:

PROBLEM 2.6. *What is the smallest volume of a hyperbolic 3-manifold?*

Another important problem is to compute explicitely T_n. We have first to find what geodesic n-simplices have maximal volume.

Let us, in general, denote by $\tilde{X}$ the universal covering of X , endowed with the pulled-back metric. In the above situation, $\tilde{X}$ is the real hyperbolic space H^n, which may be regarded as the unit ball B^n endowed with the metric g_0 defined (at the point $x \in B^n$) by

$$g_0 = \frac{4}{(1 - \|x\|^2)^2} g_E$$

where g_E is the euclidean metric.

We may thus compactify B^n by adding the "ideal boundary" $S^{n-1} = \partial\tilde{X}$, and endowing $B^n \cup S^{n-1}$ with the obvious topology.

We call "ideal simplices" those geodesic n-simplices of $\tilde{X}$ all of whose vertices lie on $\partial\tilde{X}$. Such a simplex will be called "regular" when every permutation of its vertices may be achieved by an isometry of the hyperbolic space (a regular ideal simplex is a limit of simplices all of whose 1-dimensional edges have same length). We then have the

THEOREM 2.7 (U. HAAGERUP and H. J. MUNKHOLM, [16]). *Ideal regular simplices have maximal volume among all geodesic n-simplices (and thus their volume is equal to T_n).*

In order to make explicit the equality of the Theorem 2.4, it remains to compute the volumes of the ideal simplices as a function $Vol(\theta_0, \theta_1, \ldots, \theta_n)$ of the vertices $\theta_0, \theta_1, \ldots, \theta_n$ (lying on $\partial\tilde{X} = S^{n-1}$).

– In dimension 2, as all ideal triangles have zero angles, the Gauss-Bonnet formula (with boundary) gives that all ideal simplices have volume equal to $\pm\pi$.
– In dimension 3, an explicit formula is known (see for example J. Milnor [19].
– In any dimension $n \geq 3$, we give an expression of $Vol(\theta_0, \theta_1, \ldots, \theta_n)$ as an infinite sum of symmetric polynomials in the θ_i's [7].

c) *A sketch of the proof of the Gromov-Thurston Theorem* 2.4

First step: $Vol(X) \leq T_n \, Simpl\, Vol(X)$.

This proof is attributed to W. Thurston (see [14]). As X is hyperbolic (i. e. with constant curvature -1), it is quite obvious that any simplex σ on X is homotopic to a geodesic simplex $\bar{\sigma}$ having the same vertices (just lift the simplex in the universal covering, which is the real hyperbolic space $\tilde{X} = H^n$, and take the geodesic simplex, with same vertices, that you just push down on X). This process is called "straightening a simplex" and is denoted by $\sigma \to \bar{\sigma}$. W. Thurston proved that one can straighten a (real) simplicial decomposition $\sum \lambda_i \sigma_i$ of $[X]$ in order that the result $\sum \lambda_i \bar{\sigma}_i$ is still a (real) simplicial decomposition of $[X]$. Let ω_0 be the volume form on X associated to the hyperbolic metric, we get

$$Vol(X) = \Big\langle \omega_0, \Big[\sum_i \lambda_i \cdot \bar{\sigma}_i\Big]\Big\rangle = \sum_i \lambda_i \; Vol(\bar{\sigma}_i) \leq T_n \cdot \sum |\lambda_i| \, .$$

We get the desired inequality by taking the infimum in this last inequality. ∎

Second step: $Vol(X) \geq T_n \, Simpl\, Vol(X)$. (M. Gromov [14]).

– *In dimension* 2, any compact hyperbolic 2-manifold (with Euler characteristic χ) may be triangulated by $2(|\chi| + 1)$ geodesic triangles: in fact, such a manifold may be obtained from an hyperbolic k-gon (with $k = 2|\chi|+4$) by gluing the edges together, and such a k-gon is triangulated by $(k - 2)$ triangles.

Let $\pi : X_p \to X$ be a p-sheeted covering of X. As $\chi(X_p) = p \cdot \chi(X)$, X_p is triangulated by $m(p) = 2(p \cdot |\chi(X)| + 1)$ triangles $\sigma_1, \ldots, \sigma_{m(p)}$. We thus get

$$[X] = \frac{1}{p}\pi_*([X_p]) = \sum_{i=1}^{m(p)} \frac{1}{p}(\pi \circ \sigma_i)\,,$$

and deduce that $\|[X]\|$ is bounded from above by $\frac{m(p)}{p}$, which goes to $2|\chi(X)|$ when p goes to $+\infty$, and thus $\|[X]\| \leq 2|\chi(X)|$. Then the Gauss-Bonnet formula gives that

$$Vol(X) = \int_X (-K_{g_X}) dv_{g_X} = 2\pi|\chi(X)| \geq T_2\|[X]\|$$

because, as we have seen in 2.7, $T_2 = \pi$.

Remark 2.8. This (and the previous inequality) in fact proves that, in dimension 2, *Simpl Vol*$(X) = 2|\chi(X)|$ when $\chi(X) < 0$.

– How M. Gromov generalizes this proof in higher dimensions?

Roughly speaking, the idea of M. Gromov ([14], for more explanations see [21]) is to admit chains which are limits of linear combinations of simplices, i.e. chains whose coefficients are measures. Thus, instead of writing a chain $c = \sum_i \lambda_i \cdot \sigma_i$, we shall write it $c = \int_I \lambda(i) \cdot \sigma_i \, d\mu(i)$, where I may be a continuous set of parameters and where μ is a positive measure on this set. One may analogously define

$$\|c\|_1 = \int_I |\lambda(i)| d\mu(i)\,.$$

In the case where X is a hyperbolic manifold, assumed to be compact (resp. with finite volume), $\pi_1(X)$ acts on $\tilde{X} = H^n$ by deck-transformations, which are isometries. We thus consider the set of parameters $I = Isom(H^n)/\pi_1(X)$, which is compact (resp. with finite volume), and the measure μ on I, whose pull-back by the quotient map is the Haar measure of $Isom(H^n)$. We shall denote by $g \to sign(g)$ the function which assigns to each $g \in Isom(H^n)$, the number $+1$ or -1 whether it preserves or changes the orientation.

Let σ_0 be a fixed regular ideal simplex of H^n then, for any $g \in Isom(H^n)$, $g \circ \sigma_0$ is also a regular ideal simplex whose projection $\pi \circ g \circ \sigma_0$ (by the covering map $\pi : \tilde{X} \to X$) only depends on the image $\hat{g}$ of g in $Isom(H^n)/\pi_1(X)$ by the quotient map. We thus define a chain c by

$$c = \int_{Isom(H^n)/\pi_1(X)} sign(g)(\pi \circ g \circ \sigma_0) d\mu(\hat{g})$$

which gives

$$\|c\|_1 = \ Vol\big(Isom(H^n)/\pi_1(X)\big) = \ Vol(I)\,.$$

As the symmetry, with respect to a totally geodesic hypersurface containing one of the $(n-1)$-dimensional faces of σ_0, changes the sign of g without changing the sign of the corresponding face, it comes that c is a closed chain. Let ω_0 be the canonical volume-form associated to the hyperbolic metric, one gets:

$$\big\langle \omega_0, [c]\big\rangle = \ Vol(\sigma_0)\cdot \ Vol(I) = T_n \cdot \ Vol(I)\,.$$

As $\langle \omega_0, [X]\rangle = \ Vol(X)$, this implies that $[X] = \frac{Vol(X)}{T_n\cdot\ Vol(I)}[c]$, and thus that

$$\|[X]\| \leq \frac{Vol(X)}{T_n \cdot \ Vol(I)}\cdot \|c\|_1 = \frac{Vol(X)}{T_n}\,. \tag{2.9}$$

∎

This inequality, together with the previous one, proves that $\|[X]\| = \frac{Vol(X)}{T_n}$, so that the first inequality of (2.9) is in fact an equality and we have the

Remark 2.10. *Simpl* $Vol(X) = \dfrac{Vol(X)}{T_n\cdot \ Vol(I)}\|c\|_1$.

d) A first application: Gromov's proof of Mostow's rigidity theorem

The above ideas led M. Gromov to a new proof of the

Theorem 2.11 (G. D. Mostow). *Let X, Y be two n-dimensional hyperbolic manifolds ($n \geq 3$), which are both compact (or noncompact with same (finite) volume), then any homotopy-equivalence $f : Y \to X$ is homotopic to an isometry.*

Some ideas for the proof (see [21]): Any homotopy equivalence $f : Y \to X$ may be lifted to a quasi-isometry $\tilde f : \tilde Y \to \tilde X$, which may be continuously extended to an homeomorphism $\bar f : \partial\tilde Y \to \partial \tilde X$.

Let σ be any ideal simplex of $\tilde Y$, with vertices $\theta_0, \dots, \theta_n$; we shall denote by $\bar f(\sigma)$ the ideal simplex of $\tilde X$ whose vertices are $\bar f(\theta_0), \dots, \bar f(\theta_n)$.

Let I and μ be as in the previous proof, and $\lambda_0 = \frac{Vol(Y)}{T_n\, Vol(I)}$. We have seen in the previous proof that $[Y]$ may be represented by the chain $c = \lambda_0 \int_I sign(g)(\pi \circ g \circ \sigma_0) d\mu(\hat{g})$; this implies that $[X] = f_*[Y]$ is represented by the chain $f_* c$, which may be straightened as the (homotopic) chain $\lambda_0 \int_I sign(g)(\pi[\bar{f}(g \circ \sigma_0)]) d\mu(\hat{g})$.

Computing $\langle \omega_0, [X] \rangle$ from two different ways, we obtain:

$$\begin{aligned} Vol(X) &\leq \lambda_0 \int_I Vol[\bar{f}(g \circ \sigma_0)] d\mu(\hat{g}) \leq \lambda_0 \cdot T_n \cdot \ Vol(I) = \\ &= \ Vol(Y) = T_n \| [Y] \| = T_n \| [X] \| \, , \end{aligned} \tag{2.12}$$

where the last two equalities come from the Theorem 2.4, from the Property 2.2, and from the fact that f is a homotopy equivalence. By the Theorem 2.4, all inequalities of (2.12) are in fact equalities, this implies that $Vol(X) = Vol(Y)$ and that, for almost every g, the ideal simplex $\bar{f}(g \circ \sigma_0)$ has maximal volume T_n, and thus is also regular. This implies that every regular ideal simplex is sent onto a regular ideal simplex by $\bar{f}$, and thus that $\bar{f}$ is the trace on the ideal boundaries of an isometry from $H^n = \tilde{Y}$ onto $H^n = \tilde{X}$. ∎

e) A bound for the minimal volume of a compact hyperbolic manifold

Let us consider any manifold X which admits a hyperbolic metric (denoted by g_0). Theorems 2.1 and 2.4 imply the

THEOREM 2.13 (M. Gromov, [14]). *$Min\, Vol(X) \geq \frac{C_n}{T_n} Vol(X, g_0)$, where constants C_n and T_n are defined in Theorems* 2.1 *and* 2.4.

Let us remind the two objections that we have made in Section 1 about the Gauss-Bonnet-Allendœrfer-Chern-Weil approach. M. Gromov's Theorem 2.13 answers quite conveniently to the objection (2), for it is valid in any dimension. On the contrary, it does not answer to the objection (1) because it is not sharp ($\frac{C_n}{T_n} < 1$). This led M. GROMOV [14] to make the following conjectures.

f) M. Gromov's conjectures

Except in dimension 2 (cf. Section 1) or when it vanishes, one could never compute the exact value of the minimal volume of a given manifold. That is the reason why M. Gromov was naturally led to ask the following:

QUESTION Q_1. *For a given compact manifold X, what is the exact value of $Min\,Vol(X)$?*

Mimicking the proof which works in dimension 2 (see Section 1), M. Gromov remarked that, in the case where X admits a *hyperbolic metric* g_0 (i.e. with constant sectional curvature $K_{g_0} \equiv -1$), the answer to the question Q_1 would then immediately derive from the solution of the following

CONJECTURE Q_2. *If some compact manifold X admits a hyperbolic metric g_0, then $Min\,Vol(X) = Vol(X, g_0)$.*

These two conjectures were explicitely asked by M. GROMOV [14].

The conjecture Q_2 says that the functional $g \to Vol(g)$, defined on the set of the metrics g on M which satisfy $-1 \leq K_g \leq 1$, attains its minimum at the point g_0. It is thus a natural question to ask if this minimum is unique, this leads to the

CONJECTURE Q_3. *If some compact manifold X admits a metric g_0 with constant sectional curvature $K_{g_0} \equiv -1$, any metric for which the minimal volume is attained is isometric to g_0.*

Notice that Q_1 and Q_2 are already solved in dimension 2, as direct consequences of the Gauss-Bonnet's formula. On the contrary, Q_3 is false in dimension 2, for the minimal volume is attained for any metric lying in the Teichmüller space of hyperbolic metrics, and it is well-known that these metrics are not isometric to a fixed one (here denoted by g_0). The situation must be quite different in dimension $n \geq 3$, because Mostow's rigidity Theorem 2.11 is then valid. By the way, let us notice that Mostow's rigidity theorem would be an immediate consequence of the conjectures Q_2 and Q_3.

3 – Our approach

a) Classical Kählerian Schwarz lemmas

The original Schwarz lemma may be rewritten in the language of the hyperbolic geometry, that is to say on the ball B^2 endowed with the hyperbolic metric $g_0 = \frac{4}{(1-\|x\|^2)^2}\, g_E$, where g_E is the canonical euclidean metric (this rewriting is due to Pick):

SCHWARZ LEMMA 3.1. *Given any holomorphic map $f : B^2 \to B^2$, then f is a contracting map from (B^2, g_0) in (B^2, g_0).*

Considering holomorphic maps between compact Kählerian manifolds, there have been many extensions of the above Schwarz lemma (due in particular to L. Ahlfors, S. T. Yau, N. Mok, and others ...). We shall choose the following one, which may be found for instance in [20] (see [6] Appendix A for a complete proof).

PROPOSITION 3.2. *Let X, Y be compact Kählerian manifolds of the same dimension. If $Ricci_{g_Y} \geq -C^2 \geq Ricci_{g_X}$, then any holomorphic map $F : Y \to X$ satisfies $|Jac\, F| \leq 1$. Moreover, if $|Jac\, F| = 1$ at some point y, then $d_y F$ is isometric.*

Let us recall that $Ricci_g$ is the Ricci curvature tensor of the metric g, and that the assumption $Ricci_g \geq -C^2$ means that $Ricci_g(u,u) \geq -C^2 g(u,u)$ for any tangent vector u.

Let us also remark that, when the sectional curvature of X is negative, there is at most one holomorphic map: $Y \to X$ in each homotopy class ([15]). So, when such a map exists, it realises (in some sense) the "best possible choice" for a map F (in the homotopy class) contracting the volumes. So a natural question is the following one: when the manifolds Y, X are not any more assumed to be complex, or when the homotopy class does not contain any holomorphic map, what is the "best possible choice" for F.

b) A real Schwarz lemma:

THEOREM 3.3 ([4], improved in [5] and [6]). *Let Y^n, X^m be (real) complete riemannian manifolds satisfying $3 \leq \dim(Y) \leq \dim(X)$, let us assume that there exists some constant $C \neq 0$ such that $K_{g_X} \leq -C^2$ and that $Ricci_{g_Y} \geq -(n-1)C^2 \cdot g_Y$. Then any continuous map $f : Y \to X$ may be deformed to a family of C^1 maps $F_\varepsilon (\varepsilon \to 0_+)$ such that $Vol[F_\varepsilon(A)] \leq (1+\varepsilon) Vol(A)$ for any measurable set A in Y. Moreover*

(i) *if Y, X are compact of the same dimension and if $Vol(Y) = |\deg f|$ $Vol(X)$, then Y, X have constant sectional curvature (equal to $-C^2$) and we may choose the F_ε's such that they converge, when $\varepsilon \to 0$, to a riemannian covering F (an isometry when $|\deg f| = 1$).*

(ii) *If Y, X are compact, homotopically equivalent, of the same dimension, and if $K_{g_Y} < 0$, then any homotopy equivalence f may be deformed to a smooth map F such that $Vol[F(A)] \leq Vol(A)$ for any open subset A in Y, the equality beeing attained iff F is an isometry on A.*

Remarks 3.4 and generalization (cf. [6]). The property (ii) of the Theorem 3.3 remains valid when $\dim(Y) < \dim(X)$ and when X is not compact (however, we have to assume that $\pi_1(X)$ acts on the universal covering $\tilde{X}$ in a "convex cocompact" way, i.e. that X retracts to a compact submanifold with convex boundary). *In this case, we still can deform any homotopy equivalence $f : Y \to X$ to some map F such that $|Jac\,F| \leq 1$ and such that $|Jac\,F| \equiv 1$ iff F is isometric and moreover totally geodesic.*

Let us also remark that the maps F_ε and F of the Theorem 3. 3 and of the Remark 3.4 are explicitely built (see [4], [5], [6] and the Section 4 of the present paper). Thus, applied to the case where Y and X are both hyperbolic, this theorem also gives a construction of the isometry which was only proved to exist in the Mostow's rigidity theorem (Theorem 2.11).

Before giving the ideas of the proof of the Theorem 3. 3, let us first settle some applications.

c) Applications to minimal (and maximal) volume:

The following corollary answers to the conjectures Q_1, Q_2 and Q_3.

Corollary 3.5 ([4], improved in [6]). *Let X be a compact manifold with dimension $n \geq 3$. If X admits a hyperbolic metric g_0 (i.e. $K_{g_0} \equiv -1$), then*

(i) *$MinVol(X) = Vol(g_0) = MaxVol(X)$.*

(ii) *A metric g on X (such that $|K_g| \leq 1$) realizes the minimal volume iff it is isometric to g_0.*

(iii) *For any other manifold Y^n and any map $f : Y^n \to X^n$, one has $Min\,Vol(Y) \geq |\deg f| \cdot Min\,Vol(X)$.*

Proof of (i) and (ii). Any metric g satisfying the assumption $|K_g| \leq 1$ obviously verifies $Ricci_g \geq -(n-1)$. As $K_{g_0} \equiv -1$, by the Theorem 3.3, this implies, for every ε, the existence of a map $F_\varepsilon : (X, g) \to (X, g_0)$, homotopic to id_X, such that $|Jac\, F_\varepsilon| \leq 1 + \varepsilon$. We thus get:

$$(1+\varepsilon)Vol(g) \geq \int_X |Jac\, F_\varepsilon| dv_g \geq |\deg F_\varepsilon| \cdot Vol(g_0) = Vol(g_0)\,.$$

This proves the first equality of (i). Moreover, if $Vol(g) = Vol(g_0)$, the equality case in the Theorem 3.3.(i) proves that the F_ε's converge, when $\varepsilon \to 0$, to a riemannian covering of degree 1, i.e. an isometry. This proves (ii).

On the contrary, if $K_g \leq -1$, by the Theorem 3.3 (ii), there exists a map $F : (X, g_0) \to (X, g)$, homotopic to id_X, such that $|Jac\, F| \leq 1$. We thus get:

$$Vol(g_0) \geq \int_X |Jac\, F| dv_{g_0} \geq |\deg F| \cdot Vol(g) = Vol(g)\,.$$

This proves the second equality of (i). □

Remark 3.6. Improving the arguments of the corollary 3.5, A. Sambusetti ([24], Theorem 5.1) recently proved a sharp version of M. Gromov's Theorem 2.1 for any manifold Y^n, when there exists some map f from Y^n to an hyperbolic manifold X^n such that the induced representation $f_* : \pi_1(Y) \to \pi_1(X)$ is an isomorphism (or, more generally, when its kernel has subexponential growth). In fact, he proves that, in these cases, $MinVol(Y) \geq T_n\ SimplVol(Y)$ (let us recall that the equality is attained for real hyperbolic manifolds by the Theorem 2.4).

d) Applications to Einstein manifolds:

On a given manifold, an "Einstein metric" is a Riemannian structure whose Ricci curvature tensor is proportional to the metric (and thus is constant on the unit tangent bundle). In dimension 2 and 3, every Einstein metric has constant sectional curvature. So the fundamental problem of describing the whole moduli space of Einstein metrics on a given manifold only begins at dimension 4.

When the dimension ($denoted\ by\ n$) *is greater than 5*, one knows almost nothing about this problem: for instance, there is no counter-example to the following.

CONJECTURE (see for instance [2]). *Every manifold of dimension* $n \geq 5$ *admits at least one Einstein metric.*

In dimension 4, the following obstructions to the existence of Einstein metrics on a given manifold Y were known:

3.7. *If* $\chi(Y) < 0$ *then* Y *does not admit any Einstein metric* (M. Berger, [2])

3.8. *If* $\chi(Y) - \frac{3}{2}|\tau(Y)| < 0$ *(where* $\tau(Y)$ *is the signature of* Y*), then* Y *does not admit any Einstein metric* (J. Thorpe, [2] p. 210).

3.9. *If* $\chi(Y) < \frac{1}{2592\pi^2} \cdot SimplVol(Y)$*, then* Y *does not admit any Einstein metric* (M. GROMOV, [14], see also [2] Theorem 6.47).

The obstructions 3.7 and 3.8 derive from the Allendœrfer-Chern-Weil formulas (see the Section 1). In fact, the space of quadrilinear forms on a 4-dimensional vector space V, satisfying the same algebraic properties as a curvature tensor, (i.e. the space of symmetric bilinear forms on $\Lambda^2(V)$ which satisfy the first Bianchi identity) splits as the direct sum of 4 subspaces which are irreducibly invariant under the action of SO(4). When $V = T_yY$, let W_g^+, W_g^-, Z_g and U_g be the components of the riemannian curvature tensor R_g with respect to this decomposition. The components W_g^+ and W_g^- both vanish iff the metric g is locally conformally flat and only differ by the fact that the composition by the Hodge operator $*$ acts as $+id$ or $-id$ on each of them. The component Z_g corresponds to the trace-free part of the Ricci curvature and vanishes iff g is Einstein; on the contrary, the component U_g is the canonical symmetric bilinear forms on $\Lambda^2(V)$ associated to g, multiplied by the scalar curvature (see for instance [2], Theorem 1.126, and [3] for more explanations). The Allendœrfer-Chern-Weil theory (see Section 1, [2] p. 161 and [3]) gives the following formulas for the Euler characteristic $\chi(Y)$ and for the signature $\tau(Y)$ of Y:

$$8\pi^2\chi(Y) = \int_Y (\|W_g^+\|^2 + \|W_g^-\|^2 - \|Z_g\|^2 + \|U_g\|^2)dv_g\,,$$
$$12\pi^2\tau(Y) = \int_Y (\|W_g^+\|^2 - \|W_g^-\|^2)dv_g\,.$$

As a consequence, any Einstein metric g on Y is such that $\|U_g\|^2$ is constant and satisfies:

$$8\pi^2\chi(Y) = \int_Y (\|W_g^+\|^2 + \|W_g^-\|^2 + \|U_g\|^2)dv_g \geq \geq \|U_g\|^2 Vol(Y,g)\,, \tag{3.10}$$

$$8\pi^2\Big(\chi(Y) \pm \frac{3}{2}\tau(Y)\Big) = \int_Y (2\|W_g^\pm\|^2 + \|U_g\|^2)dv_g \geq \geq \|U_g\|^2 Vol(Y,g)\,; \tag{3.11}$$

moreover any metric g_0 with constant sectional curvature satisfies $W_{g_0}^+ = W_{g_0}^- = 0$ (for g_0 is then locally conformally flat), and both inequalities (3.10) and (3.11) are equalities in this case.

The obstructions 3.7 and 3.8 immediately follow from the fact that all the terms of the inequalities (3.10) and (3.11) are nonnegative when g is Einstein.

On the other hand, the Theorem 2.1 bounds the volume from below in terms of the simplicial volume when one assumes the sectional curvature to be bounded. In fact, M. Gromov proved a stronger result: the same result is valid when one only assumes the Ricci curvature to be bounded from below ([14] p. 12, [2] result 6.46). For an Einstein manifold (Y, g), it writes

$$|scal_g|^{n/2} Vol(Y,g) \geq A_n\, SimplVol(Y)\,,$$

where $scal_g$ is the (constant) scalar curvature of the metric g and where A_n is a universal constant. In dimension 4, comparing with (3.10), this implies the existence of a universal constant B such that

$$\chi(Y) \geq B|scal_g|^2 Vol(Y,g) \geq BA_4\, Simpl\, Vol(Y)\,.$$

This ends the proof of the obstruction 3.9 by estimating the constants B and A_4 ([14], [2] result 6.46).

Looking at J. Thorpe's Theorem 3.8, one might conjecture that any manifold with $\chi(Y) - \frac{3}{2}|\tau(Y)| > 0$ admits an Einstein metric. M. Gromov's Theorem 3.9 already gave some counter-examples (see for example [2] example 6.48); more recently, A. Sambusetti gave a systematic answer to this question by proving the

Proposition 3.12 (A. Sambusetti, [22], [23]). *For any* 4*-dimensional manifold* Z*, there exists an infinity of* (*non homeomorphic*) 4*-dimensional manifolds* Y_i*, which have the same signature and the same Euler characteristic as* Z*, and which admit no Einstein metric.*

The manifolds Y_i are obtained by gluing to any hyperbolic compact manifold X (such that $\chi(X) > \chi(Z)$) enough copies of CP^2 (with the direct or reverse orientation) and enough copies of $S^2 \times S^2$ or $S^2 \times \mathbf{T}^2$, in order to obtain the right signature and Euler characteristic. Let us notice that $\tau(X) = 0$. As there exists a map $Y_i \to X$ of degree 1, the non-existence of Einstein metrics on Y_i is then a consequence of the

Proposition 3.13 (A. Sambusetti, [22], [23]). *Let* Y *be a compact* 4*-dimensional manifold. Let us assume that there exists a hyperbolic* 4*-manifold* X *and a continuous map* $f : Y \to X$ *satisfying* $|\deg f|(\chi(X) - \frac{3}{2}|\tau(X)|) > \chi(Y) - \frac{3}{2}|\tau(Y)|$*, then* Y *admits no Einstein metric.*

Example. As an illustration of this proposition, let us consider the connected sum $X\#X$ of two copies of a compact hyperbolic 4-manifold. By cellular decomposition, one gets $\chi(X\#X) = 2\chi(X) - 2 < 2\chi(X)$, and there exists an obvious application $f : X\#X \to X$ of degree 2. The Proposition 3.13 then proves that $X\#X$ does not admit any Einstein metric.

On the contrary, when a manifold X is known to admit a very canonical Einstein structure (i.e. a locally symmetric one), the main problem is the

Conjecture. *On a manifold* X *which admits a locally symmetric metric with negative curvature, this metric is* (*modulo homotheties*) *the only Einstein metric.*

This would be a very strong version of the Mostow's rigidity Theorem 2.11. In fact, in this version of Mostow's rigidity theorem, one assumes the sectional curvature to be constant (equal to -1), thus all candidates are quotients of the same hyperbolic space-form H^n (endowed with its canonical hyperbolic metric) by some discrete subgroups Γ of $Isom(H^n)$. So all possible candidates are locally isometric and the problem is to decide if two isomorphic subgroups Γ and Γ' of $Isom(H^n)$ give two globally isometric quotient-spaces H^n/Γ and H^n/Γ' (assumed to be manifolds), in which case the isomorphism $\rho : \Gamma \to \Gamma'$ extends to an isomorphism $\bar{\rho} : Isom(H^n) \to Isom(H^n)$ given by $\bar{\rho}(g) = \tilde{F} \circ g \circ \tilde{F}^{-1}$, where $\tilde{F} : H^n \to H^n$ is the lift of the isometry $F : H^n/\Gamma \to H^n/\Gamma'$.

On the contrary, two Einstein metrics (with constant Ricci curvature, equal to $-(n-1)$ for example) are generally not locally isometric: for example, the real and complex hyperbolic spaces are both Einstein with constant Ricci curvature equal to $-(n-1)$ (if we rescale the complex hyperbolic metric), but they are not locally isometric. Moreover, the possible local models for Einstein manifolds are much more numerous, and generally not locally symmetric.

Thus, there must be some global arguments saying that, even if all these models are suitable from a local point of view, only one is suitable for global reasons. This is certainly the reason why the first (partial) answers to the above conjecture about the uniqueness of the Einstein structure where given in 1994 by two different methods (one uses the real Schwarz lemma, the other the Seiberg-Witten invariants). That is the

THEOREM 3.14 ([4]). *Let X be a compact 4-manifold which admits a real hyperbolic metric, then this metric is (modulo homotheties) the only Einstein metric on X.*

THEOREM 3.15 (C. LEBRUN, [17]). *Let X be a compact 4-manifold which admits a complex hyperbolic metric, then this metric is (modulo homotheties) the only Einstein metric on X.*

REMARK 3.16. A. Sambusetti noticed that the Theorem 3.14 may be generalized in the following way: *Any compact Einstein 4-manifold (Y, g), admitting a continuous map f of non-zero degree onto a hyperbolic 4-manifold (X, g_0) which satisfies $|\deg f|(\chi(X) - \frac{3}{2}|\tau(X)|) = \chi(Y) - \frac{3}{2}|\tau(Y)|$, is hyperbolic. Moreover f is homotopic to a riemannian covering (an isometry if f has degree 1).*

This also gives a complete characterization of the equality case in the inequality of the Theorem 3.13.

PROOFS OF THEOREMS 3.14 AND 3.13 AND OF THE REMARK 3.16. Let (Y, g) be the Einstein manifold and (X, g_0) the real hyperbolic one (in the Theorem 3.14, we have $Y = X$). We may always assume the existence of a map $f : Y \to X$ of nonzero degree (in the Theorem 3.14, $f = id_X$ has degree 1). As the assumptions imply that the Ricci curvature cannot be nonnegative, one may assume (after rescaling) that $Ricci_g = -(n-1)g$.

The real Schwarz lemma (Theorem 3.3 (i)) then implies that

$$Vol(Y, g) \geq |\deg f| Vol(X, g_0) \,. \tag{3.17}$$

On the other hand, as $\|U_g\|^2 = \|U_{g_0}\|^2$ is a constant (here denoted by C^2), the inequality (3.11) and its equality case give:

$$\chi(Y) - \frac{3}{2}|\tau(Y)| \geq \frac{C^2}{8\pi^2} Vol(Y, g) \text{ and } \chi(X) - \frac{3}{2}|\tau(X)| = \frac{C^2}{8\pi^2} Vol(X, g_0)\,,$$

and implies that $\chi(Y) - \frac{3}{2}|\tau(Y)| \geq |\deg f|(\chi(X) - \frac{3}{2}|\tau(X)|)$, in contradiction with the assumption of the Theorem 3.13. In the Remark 3.16 (resp. in the Theorem 3.14), the last inequality is an equality. This implies that the inequality (3.17) is also an equality and thus, applying the equality-case of the Theorem 3.3 (i), that f is homotopic to a riemannian covering (resp. an isometry) $F : (Y, g) \to (X, g_0)$. ∎

Notice that the method used by C. Lebrun also provides obstructions to the existence of Einstein metrics on some 4-manifolds (obtained by blow-up from complex surfaces, cf. [18] and the conference of C. LeBrun in this issue).

For a more complete survey about Einstein manifolds, see [2] and [23] (in this issue).

4 – Sketch of the proof of the real Schwarz lemma (see [4], [5], [6] for a complete proof)

We shall first prove the Theorem 3.3 (ii). Rescaling the metrics g_Y and g_X, we notice that, in order to prove the Theorem 3.3 in the general case, it is sufficient to prove it in the case where the constant $-C^2$ which bounds the curvatures of Y and X is equal to -1. In order to simplify the proof, we shall moreover suppose that $K_{g_X} = -1$ (the proof in the general case $K_{g_X} \leq -1$ is given in [6]).

a) Reduction of the problem

We have already seen how to compactify the riemannian universal covering $(\tilde{X}, \tilde{g}_0)$ of a compact hyperbolic manifold (X, g_0) by adding the ideal boundary $\partial\tilde{X} = S^{n-1}$ (cf. Sections 2.6, 2.7).

It is classical that this construction generalizes to the riemannian universal covering $(\tilde{Y}, \tilde{g}_Y)$ of any negatively curved compact riemannian manifold (Y, g_Y), the ideal boundary beeing defined as the set of geodesics of $(\tilde{Y}, \tilde{g}_Y)$, quotiented by the relation which identifies two geodesics c_1 and c_2 iff $d_{\tilde{Y}}(c_1(t), c_2(t))$ is bounded when $t \to +\infty$ (*where* $d_{\tilde{Y}}$ *denotes the riemannian distance in* $(\tilde{Y}, \tilde{g}_Y)$); the corresponding point of the ideal boundary is denoted by $\theta = c_1(+\infty) = c_2(+\infty)$.

Choosing an origin y_0 in $\tilde{Y}$, one may identify the unit sphere U_{y_0} of $T_{y_0}\tilde{Y}$ with $\partial\tilde{Y}$ by the map $v \to c_v(+\infty)$, where c_v is the geodesic of $(\tilde{Y}, \tilde{g}_Y)$ such that $\dot{c}_v(0) = v$. The topology of $\partial\tilde{Y}$ is defined when deciding that this map is a homeomorphism. Rescaling the time-parameter of the geodesic c_v, one may also identify $\tilde{Y}$ with the unit ball B_{y_0} of $T_{y_0}\tilde{Y}$: this gives the (topological) compactification of $\tilde{Y}$, i e. $\tilde{Y} \cup \partial\tilde{Y} = B_{y_0} \cup U_{y_0}$.

Let us also notice that the action $y \to \gamma \cdot y$ of $\Gamma = \pi_1(Y)$ on $\tilde{Y}$ (by deck-transformations) induces an action $c \to \gamma \circ c$ on geodesics, an thus an action on $\partial\tilde{Y}$ that we shall still denote by $\theta \to \gamma \cdot \theta$.
As we have already seen in the proof of the theorem 2.11, any homotopy equivalence $f : Y \to X$ may be lifted to a quasi-isometry $\tilde{f} : \tilde{Y} \to \tilde{X}$, which may be continuously extended to a homeomorphism $\bar{f} : \partial\tilde{Y} \to \partial\tilde{X}$ (see for instance [9]). Moreover, if $\rho = [f]$ is the induced representation $\pi_1(Y) \to \pi_1(X)$, for any $\gamma \in \pi_1(Y)$, one has the following "equivariance properties":

$$\tilde{f} \circ \gamma = \rho(\gamma) \circ \tilde{f} \qquad \bar{f} \circ \gamma = \rho(\gamma) \circ \bar{f}\,. \tag{4.1}$$

For any topological space Z, we shall denote by $\mathbf{M}(Z)$ the space of positive finite Borel measures on Z. Denoting by $\bar{f}_*\mu$ the push-forward of a measure $\mu \in \mathbf{M}(\partial\tilde{Y})$, one defines a map $\bar{f}_* : \mathbf{M}(\partial\tilde{Y}) \to \mathbf{M}(\partial\tilde{X})$. By (4.1), this map is equivariant, i.e. $\bar{f}_* \circ \gamma_* = \rho(\gamma)_* \circ \bar{f}_*$. In order to build the map $F : Y \to X$ it is thus sufficient to build an equivariant map $\tilde{F} : \tilde{Y} \to \tilde{X}$ that we shall define by

$$\tilde{F}(y) = bar(\bar{f}_*\mu_y)\,, \tag{4.2}$$

where *bar* is the canonical "barycentre map": $\mathbf{M}(\partial\tilde{X}) \to \tilde{X}$, introduced by H. Furstenberg ([12], see also [10]), whose definition will be recalled in Section 4.d, and where $y \to \mu_y$ is the canonical map: $\tilde{Y} \to \mathbf{M}(\partial\tilde{Y})$ called the "Patterson-Sullivan measures", whose definition will be recalled in Section 4.c. The fact that $\tilde{f}$ and $\tilde{F}$ obey to the same equivariance property with respect to the representation ρ implies that f and F induce the same representation $[f] = \rho = [F] : \pi_1(Y) \to \pi_1(X)$. All manifolds of negative curvature beeing $K(\pi, 1)$, this proves that f and F are homotopic.

b) The Busemann function

The Busemann function $B^Y : \tilde{Y} \times \partial\tilde{Y} \to \mathbf{R}$ is defined by

$$B^Y(y,\theta) = \lim_{t\to+\infty}[d_{\tilde{Y}}(c_\theta(t), y) - d_{\tilde{Y}}(c_\theta(t), y_0)] \tag{4.3}$$

where y_0 is the fixed origin in $\tilde{Y}$ and where c_θ is the geodesic ray such that $c_\theta(0) = y_0$ and $c_\theta(+\infty) = \theta$. Roughly speaking, $B^Y(y,\theta)$ measures the (rescaled by the choice of the origin) distance from y to θ. Thus the function $y \to B^Y(y,\theta)$ inherits all the properties of the distance function, in particular its gradient has norm equal to 1; moreover its level sets (called "horospheres") are limits of the spheres of radius $d(c_\theta(t), y)$ centered at $c_\theta(t)$.

c) The Patterson-Sullivan measures $y \to \mu_y$

Let us denote by $\tilde{B}(y, R)$ the geodesic balls of $(\tilde{Y}, \tilde{g}_Y)$ centered at y and of radius R. We can then settle the

Definition 4.4. The entropy h_Y of (Y, g_Y) is the number

$$\lim_{R\to+\infty}[\frac{1}{R}Log(Vol\tilde{B}(y,R)]\,.$$

It is classical that this limit exists (when Y is compact) and does not depend on the particular choice of y.

Let now μ_0 be a finite measure on $\partial\tilde{Y}$, *one defines* μ_y as $e^{-h_Y B^Y(y,\bullet)}\mu_0$.

Remark 4.5. Only some very particular choices of μ_0 are suitable: in fact, in order to obtain the equivariance property for $\tilde{F}$, we want the measures to satisfy $\mu_{\gamma\cdot y} = \gamma_*\mu_y$ for any $\gamma \in \pi_1(Y)$. This may be done by a geometric construction as follows: let μ_y^c be defined by $\mu_y^c = e^{-c d_{\tilde{Y}}(y,\bullet)}dv_{\tilde{g}_Y}$, where $d_{\tilde{Y}}$ and $dv_{\tilde{g}_Y}$ are the riemannian distance and the riemannian measure of $(\tilde{Y}, \tilde{g}_Y)$; μ_y^c is a measure on $\tilde{Y}$ which is finite iff $c > h_Y$. We shall consider μ_y^c as a family of measures on the compact set $\tilde{Y} \tilde{\cup} \partial\tilde{Y}$ and a classical compactness result says that there exists a subsequence $c_n \to h_Y$ such that $\frac{1}{\mu_{y_0}^{c_n}(\tilde{Y})}\mu_y^{c_n}$ weakly converges to a measure μ_y, whose support lies in $\partial\tilde{Y}$. From the equality $\mu_y^c = e^{-c[d_{\tilde{Y}}(y,\bullet) - d_{\tilde{Y}}(y_0,\bullet)]}\mu_{y_0}^c$, one easily deduces that the subsequence c_n may be chosen independant from y and that $\mu_y = e^{-h_Y B^Y(y,\bullet)}\mu_{y_0}$.

The equivariance of $y \to \mu_y$ then derives from that of $y \to \mu_y^c$, which is a consequence of the invariance of $d_{\tilde{Y}}$ and $dv_{\tilde{g}_Y}$ with respect to the isometries $\gamma \in \pi_1(Y)$.

d) The barycentre map

Let B^X be the Busemann function of $(\tilde{X}, \tilde{g}_X)$; given $\mu \in \mathbf{M}(\partial\tilde{X})$, the function

$$\mathbf{D}_\mu(x) = \int_{\partial\tilde{X}} B^X(x,b)\,d\mu(b)$$

may be seen as the mean value of the distance from the point x to $\partial\tilde{X}$. When $K_{g_X} < 0$, the distance function $d_{\tilde{X}}(\bullet, z)$ is convex and thus $x \to B^X(x,b)$ and $\mathbf{D}_\mu$ are also convex when restricted to any geodesic. Moreover, if the measure μ is sufficiently spread, more precisely if every single point $x \in \partial\tilde{X}$ satisfies $\mu(\{x\}) < \frac{1}{2}\mu(\partial\tilde{X})$, then $\mathbf{D}_\mu$ is strictly convex and goes to $+\infty$ at infinity. Thus $\mathbf{D}_\mu$ attains its minimal value at its unique critical point, denoted $bar(\mu)$ and characterized by the implicit equation

$$(d\mathbf{D}_\mu)_{|bar(\mu)} = 0\,. \tag{4.6}$$

e) Implicit formulas for $\tilde{F}$ and $d\tilde{F}$:

Let us define $\mathfrak{R} : \tilde{X} \times \tilde{Y} \to \mathbf{R}$ by

$$\mathfrak{R}(x,y) = \mathbf{D}_{\bar{f}_*\mu_y}(x) = \int_{\partial\tilde{Y}} B^X(x, \bar{f}(\theta))e^{-h_Y B^Y(y,\theta)}d\mu_0(\theta)$$

and let ∂^1 (resp. ∂^2) denote the derivatives with respect to the first (resp. the second) parameter in $\tilde{X}$ (resp. in $\tilde{Y}$). By (4.2) and (4.6), $\tilde{F}$ is defined by the implicit equation: $\partial^1\mathfrak{R}_{|(\tilde{F}(y),y)} = 0$. By derivation, it comes

$$\partial^1\partial^1\mathfrak{R}_{|(\tilde{F}(y),y)}(d\tilde{F}(u), v) = -\partial^2\partial^1\mathfrak{R}_{|(\tilde{F}(y),y)}(u,v)$$

for any $u \in T_y\tilde{Y}$ and $v \in T_{\tilde{F}(y)}\tilde{X}$ This writes

$$
\begin{aligned}
&\int_{\partial\tilde{X}} DdB^X_{|(\tilde{F}(y),\bar{f}(\theta))}(d\tilde{F}(u),v)d\mu_y(\theta) = \\
&\qquad = h_Y \int_{\partial\tilde{Y}} dB^X_{|(\tilde{F}(y),\bar{f}(\theta))}(v)dB^Y_{|(y,\theta)}(u)d\mu_y(\theta) \leq \\
&\qquad \leq h_Y\, \tilde{g}_X(H_y(v),v)^{1/2}\, \tilde{g}_Y(K_y(u),u)^{1/2}\,,
\end{aligned}
\tag{4.7}
$$

where H_y (resp. K_y) is the symmetric endomorphism of $T_{\tilde{F}(y)}\tilde{X}$ (resp. of $T_y\tilde{Y}$) associated to the quadratic form $v \to \int_{\partial\tilde{Y}}(dB^X_{|(\tilde{F}(y),\bar{f}(\theta))}(v))^2 d\mu_y(\theta)$ (resp. to the quadratic form $u \to \int_{\partial\tilde{Y}}(dB^Y_{|(y,\theta)}(u))^2 d\mu_y(\theta)$).

As the gradient of $B^X(\bullet, \bar{f}(\theta))$ at the point x is the unit vector normal to the horosphere of $\tilde{X}$ centered at the point $\bar{f}(\theta)$ and containing the point x, the second fundamental form of this horosphere is equal to $DdB^X|_{(x,\bar{f}(\theta))}(\bullet,\bullet)$. When $(\tilde{X},\tilde{g}_X)$ is the real hyperbolic space, the subgroup of the isotropy group of x which fixes the unit normal vector acts irreducibily on the hyperplane tangent to the horosphere at this point, thus the second fundamental form is diagonal and

$$DdB^X = \tilde{g}_X - dB^X \otimes dB^X\,.$$

Plugging this in (4.7), it gives:

$$\tilde{g}_X((Id - H_y)\circ d_y\tilde{F}(u),v) \leq h_Y\, \tilde{g}_X(H_y(v),v)^{1/2}\, \tilde{g}_Y(K_y(u),u)^{1/2}\,, \tag{4.8}$$

which induces (by a simple argument of linear algebra) the same inequality on determinants, that is

$$\frac{\det(Id - H_y)}{(\det H_y)^{1/2}}|\det(d_y\tilde{F})| \leq h_Y^n\,(\det K_y)^{1/2} \leq h_Y^n\left(\frac{1}{n}Trace\,K_y\right)^{n/2}. \tag{4.9}$$

The fact that $\|dB^Y\| = 1 = \|dB^X\|$ implies that $Trace\,K_y = 1 = Trace(H_y)$; on the other hand, the function $A \to \frac{\det(I-A)}{(\det A)^{1/2}}$ (defined on the set of symmetric positive definite $n\times n$ matrices ($n \geq 3$) whose trace is equal to 1) attains its minimum at the unique point $A_0 = \frac{1}{n}I$. Plugging this in (4.9), it gives:

$$|\det(d_y\tilde{F})| \leq \left(\frac{h_Y}{n-1}\right)^n \leq 1\,, \tag{4.10}$$

the last inequality deriving from the comparison theorem of R. L. Bishop and the assumption $Ricci_{g_Y} \geq -(n-1)$.

When $|\det(d_y\tilde{F})| = 1$, then inequalities (4.10) and (4.9) are equalities and $K_y = \frac{1}{n}I$; moreover $H_y = A_0 = \frac{1}{n}I$. Plugging this in (4.8) and replacing v by $d_y\tilde{F}(u)$, we deduce that $d_y\tilde{F}$ is a contracting map whose determinant is equal to 1, thus it is an isometry. ∎

f) Extensions and generalizations

The general inequality of the Theorem 3.3 may be obtained even more easily: considering the family μ_y^c on $\tilde{Y}$ defined in the Remark 4.5, we define

$$\tilde{F}_c(y) = Bar(\tilde{f}_*\mu_y^c)\,,$$

where we have modified the previous notion of barycentre: in fact, this new barycentre $Bar(\mu)$ of a measure μ on $\tilde{X}$ is now defined as the unique point where the function

$$\Delta_\mu(x) = \int_{\tilde{X}} d^2_{\tilde{X}}(x,z)d\mu(z)$$

attains its minimum (we restrict ourselves to measures such that the above integral is finite).

Replacing $B^Y(\bullet,\theta)$ and $B^X(\bullet,\bar{f}(\theta))$ by $d_{\tilde{Y}}(\bullet,z)$ and $d_{\tilde{X}}(\bullet,z)$, the same proof as in 4.e works and gives:

$$|\det(d_y\tilde{F}_c)| \leq \Big(\frac{c}{n-1}\Big)^n \leq (1+\varepsilon)^n\,, \tag{4.11}$$

for c may be chosen as equal to $h_Y+\varepsilon$ and $h_Y \leq n-1$ by the comparison theorem of R. L. Bishop. ∎

Remark 4.12. In the Theorem 3.3 (at least in the case where the curvature is negative), it is not necessary to assume the existence of some map $f: Y \to X$. In fact, *given any homomorphism $\rho: \pi_1(Y) \to \pi_1(X)$, we can directly build the family of maps $F_\varepsilon : Y \to X$, satisfying $|Jac(F_\varepsilon)| \leq 1+\varepsilon$, such that the induced homomorphism $[F_\varepsilon]:\pi_1(Y) \to \pi_1(X)$ is equal to ρ.*

Proof. Let us fix origins y_0 and x_0 in $\tilde{Y}$ and $\tilde{X}$, and define $\nu_y^c \in \mathbf{M}(\tilde{X})$ as $\sum_{\gamma\in\pi_1(Y)} e^{-c\,d_{\tilde{Y}}(y,\gamma\cdot y_0)}\delta_{\rho(\gamma)\cdot x_0}$, where δ_z is the Dirac measure of the point $z \in \tilde{X}$. We now define $\tilde{F}_c(y)$ as $Bar(\nu_y^c)$. Then the same proof as above gives

$$|\det (d_y \tilde{F}_c)| \leq \Big(\frac{c}{n-1}\Big)^n \leq (1+\varepsilon)^n \,.$$

It is moreover easy to verify the equivariance of $\tilde{F}_c$, for $\nu^c_{\gamma\cdot y} = (\rho(\gamma))_* \nu^c_y$. □

We have just seen that the inequality of the Theorem 3.3 (i) is somewhat easier to settle than the inequality of the Theorem 3.3 (ii). On the contrary, the equality case of the Theorem 3.3 (i) is much more difficult to settle than it was in the Theorem 3.3 (ii): the reason is that, in the Theorem 3.3 (ii), we directly built the good candidate to be the isometry $F : Y \to X$ while, in the Theorem 3.3 (i), we have to prove that the limit F of the $F_c : Y \to X$ (when c goes to h_Y) exists and is an isometry (see [4] Sections 7 and 8 for a proof).

5 – Generalization to locally symmetric manifolds (cf. [4], [5])

We shall now assume that (X, g_X) is a compact n-dimensional locally symmetric manifold with negative curvature, i.e. a compact quotient of the real or complex or quaternionic or Cayley hyperbolic space. The entropy of such a manifold will be denoted by h_X, and is equal to $n+d-2$ (where d is the real dimension of the corresponding real or complex or quaternionic or Cayley field), when the locally symmetric metrics are rescaled in order that the maximum of their sectional curvatures is equal to -1.

a) Main theorem

The following theorem solves conjectures of A. Katok and M. Gromov about "minimal entropy"

Theorem 5.1 ([4], [5], [6]). *Let (X, g_X) be a compact locally symmetric manifold with negative curvature and (Y, g_Y) be any compact riemannian manifold such that* $\dim X = \dim Y \geq 3$, *then any continuous map $f : Y \to X$ may be deformed to a family of C^1 maps $F_\varepsilon (\varepsilon \to 0_+)$ such that $Vol[F_\varepsilon(A)] \leq (\frac{h_Y+\varepsilon}{h_X})^n \, Vol(A)$ for any measurable set A in Y. In particular, one has $(h_Y)^n \, Vol(Y) \geq |\deg f| (h_X)^n Vol(X)$. Moreover,* *if* $(h_Y)^n\, Vol(Y) = |\deg f|(h_X)^n\, Vol(X)$*, then* Y *is also locally symmetric and* f *is homotopic to a riemannian covering* F *(an isometry when* $|\deg f| = 1$*).*

REMARK 5.2. As in the Theorem 3.3 (ii), when (Y, g_Y) has negative curvature and f is a homotopy equivalence, we may build directly and explicitely the limit F of the F_ε's. It satisfies $Vol[F(A)] \leq (\frac{h_Y}{h_X})^n Vol(A)$ for any open subset A in Y and is an isometry in the equality case.

SKETCH OF THE PROOF. The first inequalities of (4.10) and (4.11) (recalling that μ_y^c is a finite measure iff $c > h_Y$) and their equality cases, already proved the Theorem 5.1 and the Remark 5.2 when (X, g_X) is (locally) real hyperbolic. It easily generalizes to the other cases. For example, let us assume that (X, g_X) is (locally) complex hyperbolic, the proof is exactly the same as it was when (X, g_X) was (locally) real hyperbolic (cf. Section 4), except for the fact that DdB^X is now equal to $\tilde{g}_X - dB^X \otimes dB^X + (dB^X \circ J) \otimes (dB^X \circ J)$. This modifies the formula (4.9), but the only new problem is to prove that the function $A \to \frac{\det(I-A-JAJ)}{(\det A)^{1/2}}$ (still defined on the set of symmetric positive definite $n \times n$ real matrices whose trace is equal to 1) still attains its minimum at the unique point $A_0 = \frac{1}{n} I$. This comes from the Log-concavity of the determinant which reduces the problem to the previous one (see [4], Appendix B). ∎

b) Application to the general Mostow's rigidity theorem

A corollary is a unified proof of the following theorem, initially proved by G. D. Mostow:

THEOREM 5.3. *Let* (X, g_X) *and* (Y, g_Y) *be two compact locally symmetric manifolds with negative curvature such that* $\dim X = \dim Y \geq 3$, *then any homotopy-equivalence* $f : Y \to X$ *is homotopic to an isometry.*

PROOF. Let $g : X \to Y$ be such that $g \circ f \approx id_Y$. By the Remark 5.2, there exist deformations F and G of f and g such that

$$Vol[G \circ F(Y)] \leq \Big(\frac{h_X}{h_Y}\Big)^n \Big(\frac{h_Y}{h_X}\Big)^n Vol(Y)\,.$$

As the degree of $G \circ F$ is equal to 1, this inequality is an equality and we are in the equality case of the Remark 5.2, thus F is an isometry. ∎

c) Application to dynamics

Two riemannian manifolds Y and X are said to have the "same dynamics" iff there exists a C^1-diffeomorphism ϕ between their unitary tangent bundles UY and UX which exchanges their geodesic flows, i.e. $\phi(\dot{c}_v(t)) \equiv \dot{c}_{\phi(v)}(t)$ for any unit vector $v \in UY$, where c_v is the geodesic such that $\dot{c}_v(0) = v$. The fundamental question is the following:

Conjecture 5.4 (believed to be by E. Hopf). *Two riemannian manifolds Y and X having the same dynamics are isometric.*

In the absence of any additive assumption, this conjecture is false, because theire exists non isometric manifolds all of whose geodesic are closed with the same period (see [1] for more informations).
C. B. Croke and J. P. Otal proved this conjecture to be true when Y and X are 2-dimensional and negatively curved.

A corollary of the Theorem 5.1 is the following

Theorem 5.5 ([4]). *The conjecture is true in any dimension provided that one of the two manifolds is locally symmetric with negative curvature.*

Proof. As UY and UX are homeomorphic and $n \geq 3$, the manifolds Y and X are homotopically equivalent. It is well known that the volume and the entropy are invariants of the dynamics, thus the fact that Y and X have the same dynamics implies that $h_Y = h_X$ and that $Vol(Y) = Vol(X)$, which means that we are in the equality case of the Theorem 5.1. This proves that Y and X are isometric. □

d) Application to the Lichnerowicz's conjecture

A riemannian manifold is said to be "locally harmonic" when all geodesic spheres of its universal covering have constant mean curvature. It is well known that any locally symmetric manifold of rank one is locally harmonic. A. Lichnerowicz asked the following converse question:

Conjecture 5.6. *Any locally harmonic manifold is locally symmetric of rank one.*

In the case where the universal covering $\tilde{X}$ is compact, the conjecture has been proved by Z. Szabo ([25]), any locally harmonic manifold beeing (in this case) locally symmetric with positive sectional curvature.

In the case where the universal covering $\tilde{X}$ is non compact, it is known ([1]) that the geodesics of X have no conjugate points, and the conjecture is not significantly changed when assuming the sectional curvature to be negative. However, E. Damek and F. Ricci ([11]) proved the conjecture to be false (even for negatively curved locally harmonic manifolds), but we may notice that their counter-examples are such that $\tilde{X}$ and X are both non compact. Thus the only case where the conjecture remained open was the case where $\tilde{X}$ is non compact and where its quotient X is compact, which is answered by the

Corollary 5.7 ([4]). *Any compact negatively curved locally harmonic manifold is locally symmetric of rank one.*

Proof. It was proved by P. Foulon and F. Labourie ([13]) that, under these assumptions, the manifold has the same dynamics as a locally symmetric manifold with negative curvature. We conclude by applying the Theorem 5.5. □

REFERENCES

[1] A. L. Besse: *Manifolds all of whose geodesics are closed*, Ergebnisse der Math., Springer, 1978.

[2] A. L. Besse: *Einstein Manifolds*, Ergebnisse der Math., Springer, 1987.

[3] A. L. Besse: *Géométrie riemannienne en dimension 4*, Séminaire Arthur Besse, Textes Mathématiques, Cedic-Nathan.

[4] G. Besson – G. Courtois – S. Gallot: *Entropies et rigidités des espaces localement symétriques de courbure négative*, Geom. And Funct. Anal., **5** (1995), pp. 731-799.

[5] G. Besson – G. Courtois – S. Gallot: *Minimal Entropy and Mostow's rigidity theorems*, Ergod. Theory Dynam. Syst., **16** (1996), pp. 623-649.

[6] G. Besson – G. Courtois – S. Gallot: *Lemme de Schwarz réel et applications*, preprint.

[7] G. Besson – G. Courtois – S. Gallot: *Sur le volume des simplexes hyperboliques idéaux*, manuscript preprint.

[8] J. Cheeger – D. Ebin: *Comparison Theorems in Riemannian Geometry*, North-Holland (1975).

[9] R. Benedetti – C. Petronio: *Lectures on Hyperbolic Geometry*, Universitext, Springer, 1993.

[10] E. Douady – C. Earle: *Conformally natural extension of homeomorphisms of the circle*, Acta Math., **157** (1986), pp. 23-48.

[11] E. Damek – F. Ricci: *A class of non-symmetric harmonic riemannian spaces*, Bull. Amer. Math Soc., **27** (1992), pp. 139-142.

[12] H. Furstenberg: *A Poisson formula for semi-simple Lie groups*, Annals of Maths, **77** (1963), pp. 335-386.

[13] P. Foulon – F. Labourie: *Sur les variétés compactes asymptotiquement harmoniques*, Invent. Math., **109** (1992), pp. 97-111.

[14] M. Gromov: *Volume and Bounded Cohomology*, Publ. Math. I. H. E. S., **56** (1981), pp. 213-307.

[15] P. Hartman: *On homotopic harmonic maps*, Canad. J. Math., **19** (1967), pp. 673-687.

[16] U. Haagerup – H. J. Munkholm: *Simplices of maximal volume in hyperbolic n-space*, Acta Math., **147** (1981), p.p. 1-11.

[17] C. LeBrun: *Einstein metrics and Mostow rigidity*, Math. Res. Lett., **2** (1995), pp. 1-8.

[18] C. LeBrun: *Four-manifolds without Einstein metrics*, Math. Res. Lett., **2** (1996), pp. 133-147.

[19] J. Milnor: *Hyperbolic geometry: The first 150 years*, Bull. of the A. M. S. (new series) Vol. 6, N° 1 (1982), pp. 9-24.

[20] N. Mok: *Metric rigidity Theorems on Hermitian locally symmetric Manifolds*, Series in pure Maths, **6**, World Scientific.

[21] H. J. Munkholm: *Simplices of maximal volume in hyperbolic space, Gromov's norm and Gromov's proof of Mostow's rigidity theorem* (*following Thurston*), Top. Symp. Siegen 1979, Lecture Notes in Maths, **788** (1980), Springer, p.p. 109-124.

[22] A. Sambusetti: *An obstruction to the existence of Einstein metrics on 4-Manifolds*, preprint ENS. Lyon, to appear in Math. Annalen.

[23] A. Sambusetti: *Obstructions to the existence of Einstein metrics on* 4-*manifolds*, to appear in the same issue (Acta del convegno di Roma Giugno 1996).

[24] A. Sambusetti: *Minimal entropy and simplicial volume*, preprint Università di Roma “La Sapienza”.

[25] Z. Szabo: *The Lichnerowicz conjecture on harmonic manifolds*, J. Diff. Geom., **31** (1990), pp. 1-28.

[26] A. Weinstein: *Distance Sphere in complex projective Spaces*, Proc. Amer. Math. Soc. **39** (1973), 649-650.

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