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A short course on the Pirogov – Sinai theory

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PRESENTAZIONE: *Quali sono le caratteristiche delle interazioni molecolari che influenzano le transizioni di fase dando luogo a fenomeni di coesistenza di fase? La domanda, che ha grande interesse teorico e applicativo, è ampiamente dibattuta in Meccanica Statistica, dove ha trovato significative risposte nella teoria iniziata da Pirogov e Sinai negli anni 70 sulla struttura dei sistemi magnetici a basse temperature e che è tuttora oggetto di studio ed estensioni.*

La formula di Gibbs, su cui si fonda la teoria moderna della Meccanica Statistica dell'equilibrio, è il punto di partenza che stabilisce la connessione tra stati termodinamici ed interazioni microscopiche. Lo stato termodinamico è in questa teoria descritto da una misura di probabilità, la probabilità di una configurazione di energia E risultando proporzionale a $\exp\{-E/kT\}$, essendo T la temperatura assoluta e k la costante di Boltzmann. In un limite in cui $T \to 0$, la probabilità si concentra sui minimi dell'energia ed il caso di interesse per le transizioni di fase, è quando vi sia più di un minimizzante. La teoria di Pirogov-Sinai, oggetto del presente articolo, studia la struttura delle misure di Gibbs a temperature prossime allo 0 assoluto e dimostra, sotto opportune ipotesi sulla struttura del sistema, che le fasi termodinamiche pure sono in corrispondenza con i minimizzanti dell'energia, da cui differiscono per isolate e piccole fluttuazioni. La teoria è molto potente e permette di descrivere accuratamente le eccitazioni energetiche rappresentate in termini di "contorni" la cui forma e localizzazione sono l'oggetto principale della teoria.

Le nozioni e le metodologie sviluppate nell'ambito della teoria di Pirogov-Sinai sono a tutt'oggi fondamentali per intraprendere uno studio matematicamente rigoroso della Meccanica Statistica di sistemi a basse temperature.

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– Introduction

In this series of lectures I will explain in some detail the Pirogov-Sinai theory for spin systems. The theory applies to general models of equilibrium statistical mechanics at low temperatures and is one of the most powerful methods developed from the pioneering and fundamental work by Peierls, [36]. Indeed the Peierls approach still plays a leading role in the rigorous theory of Statistical Mechanics and his idea of a *contour* is the main keyword of the Pirogov-Sinai theory.

One should however remark that notions like the "Peierls approach" and the "Peierls argument" have also a broader meaning. A possibility to estimate the probability of a "contour" does not mean yet that we are necessarily in the realm of the Pirogov-Sinai theory. The other main ingredient of the Pirogov-Sinai theory is its ability to establish also the *cluster expansion of the partition function* – which gives a sharp control of its dependence on the region and on the boundary conditions.

Some historical notes: After the appearence of the Peierls paper in the late thirties, there was almost no additional progress till the mid sixties – when Dobrushin and Griffiths started the new, rigorous investigation of the subject. An important milestone in its development was then the work of Minlos and Sinai, [4], with their introduction (and the systematic investigation) of "contour models", which played a crucial role in the later development.

These papers, [4], refer to the Ising model and focus on the problem of *phase transition*. At that time, namely in the early days of the

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mathematically rigorous theory of statistical mechanics, the *coexistence of phases* was in fact the most interesting and new (in its mathematical description) phenomenon. The restriction to the Ising model, on the other hand, allowed to exploit the spin flip symmetry present in the system for zero magnetic field, which greatly simplifies the whole analysis. (This does not mean that there is no nontrivial information in [4] on the Ising model also with nonzero magnetic field, where the spin flip symmetry is broken.)

After [4], it was then just a question of time to generalize the notion of contour models to other systems offering adequate symmetries of their Hamiltonian. Gertzik, [5], formulated quite a general version of what is now commonly called the Peierls condition, working on systems whose Hamiltonians have such symmetries, but the method of Pirogov and Sinai, [1], which appeared at the same time, has proved far more reaching. Quite importantly, it applies also to the *nonsymmetric case*. The Pirogov-Sinai concept of a *contour model* (more generally, of a contour model with parameter) is an ingenious generalization of the original [4] notion, and their method turned out to be a powerful and general tool for the rigorous study of many lattice spin models at low temperatures.

The original version of the theory is described in [1] and also in the book [2].

Since from the beginning this theory was by many "outsiders" viewed as a "complicated tool to be used only when all the other, usual methods (reflection positivity, correlation inequalities, ...) fail". Such an attitude has been slowly changing in the last 20 years, but the feeling that the Pirogov-Sinai theory is *a standard tool* (as, for instance, the high temperature expansions) is not yet widespread in the rigorous statistical mechanics community(1).

Of course I can neither describe the theory in all its aspects nor mention all its various applications which have appeared in the literature. I will instead focus in these lectures on a version developed in Prague starting from the paper [7] and which was then applied, for example, to the study of interface problems. (These aspects, which will not be treated here in detail, were very important to us in understanding the need for a more flexible and powerful version of the theory.) After some

(1)There is hope that the Pirogov-Sinai theory may be also extremely useful in *nonrigorous studies*, but unfortunately, very little has been done so far in this direction.

years and with the experience of several works where the Pirogov-Sinai theory, in the version described in [7], had been applied to various situations, we recently derived in [9] (which exists as a preprint in mp_arc, Texas) a new, simpler and, at the same time, more powerful version of the Pirogov-Sinai theory. The method of [9] originates from our studies of "stratified phases" with one or more interfaces, but, as it turned out, it has wider applications. Even when applied to problems that can be solved by previous versions of the theory it gives a new, simpler and more detailed description. I thus apologize for not describing in detail the original Pirogov-Sinai theory here, but rather the working version of that theory which we are using at the present time.

Of course, the important ideas of the original Pirogov-Sinai approach (and some additional simplifying ideas, like that of the "metastable ensemble", which originally appeared in [7]) are not much changed in the new approach. I simply believe that these ideas can be now expressed in a simpler and stronger form than before. In particular, instead of stressing the importance of the *estimates* (from above and below) of various (so called) diluted partition functions appearing in the theory, we now supplement these estimates by more accurate (and more informative) *expansions*, as we have been able to replace bounds for partition functions by expansion formulas practically everywhere. Inequalities now appear in our text (with few exceptions like (4.40)) only when formulating the Peierls condition or in statements that the terms of some series converge quickly to zero.

The plan of my lectures is the following. In Sections 1 and 2, after introducing some typical examples where the theory usefully applies, I will present the core of the original Pirogov-Sinai theory in the special case when there is a *maximal number of coexisting phases*. I will explain the simplifying role that the *symmetry* has in these considerations, but the emphasis will be always on the nonsymmetric case.

In fact the strength of the Pirogov-Sinai theory is in its ability to treat *nonsymmetric* systems, also when "not all the candidates for the translationally invariant phases either survive the thermodynamic limit or a small change of the parameters of the model".

NOTE. It is erroneous, I think, to consider the Pirogov-Sinai theory just as mere "means to construct the phase diagram". It is far more important that the theory gives really a *detailed control* on the behaviour of all the "phases" in all possible situations (when they become either "stable" or "unstable") in infinite but also (and not less importantly) in any *finite* volume(2).

There will be very few assumptions like "if the volume goes to infinity" resp. "if the temperature goes to zero"; the assumptions on the size of a volume resp. on the smallness of the temperature (for which the statement is valid) will be usually given explicitly.

In conclusion, Sections 1 and 2 cover the simplest applications of the theory, namely Hamiltonians with a finite number of "local" ground states which fulfill the (so called) Peierls condition. We will find the conditions on the Hamiltonian (resp. how to adjust its parameters) for which a maximal number of phases (each phase corresponding to a particular local ground state) would coexist. We will also introduce the fundamental notions of Pirogov-Sinai *contour functional*, and of *contour model*. The latter is mentioned, however, mainly for "historical" reasons. In fact in our successive analysis (in Sections 3 and 4) we will deviate from the concept of a contour model, but the change will not affect seriously the special situation that we investigate in Section 2 (after an overview of the general models was presented in Section 1).

Section 3 is a general course on the Pirogov-Sinai theory in its present "Prague" form. This version follows essentially the approach in [7] with some important recent simplifications and strengthenings from [9]. We abandon here completely the notion of contour models and the behaviour of the "unstable phases" is studied in a more concise way. Instead of the notion of a contour model and the estimates of its partition functions we work with *expansions* of partition functions.

NOTE. Cluster expansions were always an important tool in the Pirogov-Sinai theory. However, in the previous versions of this theory, the expansions were viewed merely as an "auxiliary" technique which was

(2) The question about what happens in a finite volume (under given boundary conditions) is nontrivial, even in the (typical!) situation where only *one* infinite volume phase exists. The case when several phases coexist is just marginal from such a general point of view, and of course it is desirable to be able to treat such a case together with the previous one. This is something which the Pirogov-Sinai theory does!

applied to the study of the contour models. One might conceive that the cluster expansion will be possibly replaced by some other method which gives comparably nice expressions of the partition functions. Namely, the possibility to work with precise *decompositions of the partition functions* into "bulk" and "boundary" (surface tension) terms has always been a characteristics of the Pirogov-Sinai theory, yet one could hope that such a precise information may be obtained by other methods than cluster expansion.

This is not so in the approach I present here, where the concept of an expansion permeates even the basic notions and the basic ideology of the theory. It can be really said that the Pirogov-Sinai theory, in the version I present, is *just the method of organizing the expansions of low temperature partition functions* and the Pirogov-Sinai contour functional F (more specifically, $\exp(-F)$) is the most important quantity entering in these expansions. Our exposition of the expansion method presented in these lectures is selfcontained, and no reference to the literature on cluster expansions is absolutely necessary. However, some previous knowledge of the cluster expansion theory will be useful for the reader, and we remind the (extensive) literature on the subject, see (we restrict only to references having direct influence on this text) [22], [23], [24], [25], ...

The expansions are studied in detail in Section 4 – where also some additional, "topological" investigations of the structure of systems of "large" contours (appearing under "unstable" boundary conditions) are made. This concludes the development of the Pirogov-Sinai theory in its basic form.

1 – General Setting

1.1 – Hamiltonians, Gibbs States

We study the Gibbs states on a lattice $\mathbb{Z}^\nu$ in dimensions $\nu \geq 2$. We consider some norm on $\mathbb{Z}^\nu$ e.g. the norm

$$|t| = \max_i |t_i|$$

and say that a subset $\Lambda \subset \mathbb{Z}^\nu$ is connected if for any $t, s \in \Lambda$ there is a connecting path $\{t_i\}, i = 1, 2, \ldots, n$ such that $t_0 = t$, $t_n = s$ and $|t_i - t_{i-1}| = 1$ for each $i = 1, 2, \ldots, n$.

Our basic set of *configurations* will be the set

$$\mathbb{X} = S^{\mathbb{Z}^\nu}$$

where S is some (finite) set of "spins". One usually takes the σ – algebra $\mathcal{B}$ generated by the set of all "cylindrical" events, i.e. events measurable in terms of some finite projection

$$x \in \mathbb{X} \to x_\Lambda \in S^\Lambda$$

where Λ is some finite subset of $\mathbb{Z}^\nu$.

By a *state* we mean a probability measure on $(\mathbb{X}, \mathcal{B})$.

A *Hamiltonian* on $\mathbb{X}$ will be usually given by some family of *interactions* i.e. functions Φ_A defined on S^A, $A \subset \mathbb{Z}^\nu$. These interactions will usually satisfy some further requirements like the *translation invariance* – if Φ_A commute with all the shifts of the sets A (sometimes only the invariance with respect to some subgroup of $\mathbb{Z}^\nu$, possibly of lower dimension, will be assumed) and the *finite range* – namely if $\Phi_A = 0$ whenever the diameter of A is bigger than some integer r. A Hamiltonian of a configuration x_Λ in a finite volume $\Lambda \subset \mathbb{Z}^\nu$, under a boundary condition x_{Λ^c} will be defined as

$$H(x_\Lambda | x_{\Lambda^c}) = \sum_{A \not\subset \Lambda^c} \Phi_A(x_A).$$

Given a state P on $(\mathbb{X}, \mathcal{B})$ and boundary condition x_{Λ^c} given on a set Λ^c, or at least on the set

$$\partial\Lambda^c = \{t \in \Lambda^c : \mathrm{dist}(t, \Lambda) \leq r\}$$

for a finite Λ, consider the conditional probability

$$P((\cdot)|x_{\Lambda^c})$$

which is defined uniquely for P almost all x_{Λ^c}. We say that P is a *Gibbs state* with respect to the Hamiltonian H if this conditioned probability satisfies, for almost all x_{Λ^c}, the condition

$$P(x_\Lambda | x_{\Lambda^c}) = Z^{-1}(\Lambda, x_{\Lambda^c}) \exp(-H(x_\Lambda | x_{\Lambda^c}))$$

where

$$Z(\Lambda, x_{\Lambda^c}) = \sum \exp(-H(x_\Lambda | x_{\Lambda^c}))$$

the sum being over all $x_\Lambda \in S^\Lambda$.

NOTE. We always incorporate the inverse temperature $\frac{1}{T}$ into our Hamiltonian. In other words, the temperature will be just one of the parameters in the Hamiltonian H (usually not to be explicitly mentioned below). The fact that we are actually studying the *low temperature case* will be formulated later, by the Peierls condition.

A Gibbs state P is said to be an *extremal* one if it is moreover indecomposable in the sense that there are no other Gibbs states $\tilde{P}, \tilde{P}'$ such that P would be a nontrivial convex combination of $\tilde{P}, \tilde{P}'$. In the following we will have to work almost exclusively with the extremal Gibbs states. Namely, these will appear as the limits of finite volume Gibbs states $P((\cdot), x_{\Lambda^c})$ under a special choice of (constant) boundary conditions (yielding an extremal Gibbs state, as we will see).

We will usually omit the adjective "extremal" in the following, and call these Gibbs states also "pure phases" or simply "phases" (especially if one has in mind the *translation invariant* Gibbs states). Later we will call them occasionally also as the "stable phases" – as opposed to the "unstable phases" which appear only in volumes of a limited size. The change of the parameters in the Hamiltonian affects usually the "stability" of the considered "phases". This is the main question to be clarified by the Pirogov-Sinai theory.

One of the basic keywords of the Pirogov-Sinai theory is the notion of a (local) ground state. This is used for the configurations x satisfying the property that whenever we change it "locally" – in a set not exceeding a prescribed size – then its energy increases i.e.

$$H(\tilde{x}) - H(x) = \sum_A (\Phi_A(\tilde{x}_A) - \Phi_A(x_A)) > 0. \tag{0.0}$$

NOTES. 1. Roughly speaking, such a "local" ground state x usually turns out to be the true ground state of a suitable "original", "unperturbed" Hamiltonian $\tilde{H}$ (whose slight perturbation the given "perturbed" Hamiltonian H is).

2. This is noted here just for an intuition; we will *not* use the formal notion of a local ground state below! Even the notion of a ground state –

appearing when we drop any requirements on the size of the perturbation (except of its finiteness) of $\tilde{x}$ with respect to x – will not be employed below (on the formal level). We notice that all the formal requirements to be later used in connection with the idea of a (local) ground state will be *contained in the formulation of the Peierls condition* below.

To avoid misunderstanding: Of course, the idea to construct Gibbs states around some "local ground states of the given model" (and these "local" ground states are often the true ground states of some "unperturbed" Hamiltonian) lies in the very heart of the Pirogov-Sinai theory. However, the very notion of a ground state is *not* used by the theory[(3)].

3. By a *degeneracy* of a (local) ground state one usually has in mind the fact that *several* (local) ground states exist for a given Hamiltonian. This is the interesting case; Namely, in the regions of phase uniqueness, well developed methods based essentially on the Dobrushin' s uniqueness theorem (and later complete analyticity investigations by [15]) are available.

On the other hand, in the regions where phase coexistence is expected apparently no alternative to the Pirogov-Sinai theory is developed attaining a comparable level of generality and universality of its applications.

The idea behind the Pirogov-Sinai theory is the following one: under the condition that there are "sufficiently strong energetical barriers" between different local ground states (formally, such a requirement will be formulated as the Peierls condition – see below) one should construct the expected (generally "unstable") "phases" as some perturbations of the corresponding local ground states. Moreover, one should acquire a full control on how these "unstable" phases turn out to be "stable"(i.e.

[(3)]In practice, one usually starts with a given Hamiltonian (the "unperturbed" one, often having some additional symmetries) whose ground states are to be found – because one expects that no other (even local) ground states will appear even for the slightly *perturbed* Hamiltonian. One moreover expects to be able to construct Gibbs states around *some*, at least, of these ground states, also for the perturbed Hamiltonians. Of course, the determination of all the ground states of a given "unperturbed" Hamiltonian is a very important and often nontrivial task. We are *not* discussing this question here (see e.g. [14]) – because this is an investigation which should *precede* the application of the Pirogov-Sinai theory (giving proper candidates for the elements of the "reference set" Q introduced below & showing that the Peierls condition will be valid). However, such an investigation should not be considered, strictly speaking, as a part of this theory; it just opens a way to its application!

giving rise to the true, infinite volume Gibbs states) under the change of the parameters in the Hamiltonian.

The specification of the "stable phases" thus constructed (for any values of the parameters in the Hamiltonian) is what is usually called the *phase diagram* of the given model.

1.2 – Contours

The central notion of the Pirogov-Sinai theory is the notion of a *contour* (see below). The idea is to consider contours as some "barriers" (more precisely connected components of these barriers) separating the regions occupied by various local ground states. Though very important, such a notion cannot obviously have some "canonical" definition. On the contrary, the notion of a contour in various situation can differ, and one could tailor it with respect to the peculiarities of the considered model. For example, in the Ising model, contours are traditionally defined as some connected paths separating the regions occupied by $+$ or $-$ spins. Having this in mind (the fact that the definition of a contour can be adapted to particular features of the model or to the precise formulation of the problem, which has to be solved) we present here a general definition of a ("thick") contour suitable for all the situations outlined above. This general definition appeared first in [1]. To begin, we have to specify the collection of the local ground states of the general model above. Let us assume that we have already found this collection. Denote their elements as $\{x_q, q \in Q\}$ where $Q \subset S$ and x_q denotes the constant configuration $\{x_t = q, t \in \mathbb{Z}^\nu\}$.

NOTES. 1. This setting is sufficiently general to cover also the case of *periodical* local ground states. Namely, if there are also some periodical local ground states of the given model then it can be assumed that they have the same period G; G is understood here as a subgroup of $\mathbb{Z}^\nu$ such that the factor group $D = \mathbb{Z}^\nu/G$ is finite. By defining the blocks of original spins $x_{D'}$ (D being identified with a suitable subset of $\mathbb{Z}^\nu$, $D' = D + t$ where $t \in G$) as the spins of the new (blocked) model (with the spin space S^D), the original model is converted to a new one having only constant local ground states.

2. We recall that the concept of a local ground state is not defined precisely. The decision whether a given constant (or periodical) configuration

is a local ground state is therefore sometimes a little bit arbitrary. This should not cause substantial problems: namely the "phases" constructed around such "suspect" local ground states will be so "highly unstable" that even the very notion of an unstable "phase" will lose a reasonable sense here.

3. The nontrivial task of finding all the local ground states (of the given model) is completely left out here. The fact is that the Pirogov-Sinai theory actually only *starts* at the moment when the collection of all local ground states is already given. The specification of the family x_q is of course a very important (in spite of some arbitrariness contained in the choice of x_q) but *preliminary* step of the investigation of the given model. We just call by the Pirogov-Sinai theory everything which *follows* the finding of the local ground states of the model.

Now let us define the central notion of the Pirogov-Sinai theory:

Definition of a contour. Say that a point $t \in \mathbb{Z}^\nu$ is a q – correct point of a configuration $x \in \mathbb{X}$ if for all the points $s \in \mathbb{Z}^\nu$ from the r – neighbourhood of t (recall that r denotes the range of interactions of the given model), $x_s = q$. A point which is correct for *no* $q \in Q$ will be called incorrect. The collection, denoted by $B(x)$, of all incorrect points of $x \in \mathbb{X}$ will be splitted into connected components. Having such a component $B \subset B(x)$, the restriction of x to B will be called a *contour* of x. Contours will be denoted by symbols $\mathbf{\Gamma}$.

The above mentioned set B will be called the support $B = \operatorname{supp} \mathbf{\Gamma}$ of the corresponding contour $\mathbf{\Gamma}$.

Diluted configurations. A configuration x will be called diluted, more precisely q – diluted, if all the contours of x have finite supports and moreover, if we denote by ext the infinite component of the complement of $\cup \operatorname{supp} \mathbf{\Gamma}$ then such a component is unique and all the components of the set ext^c are also finite and we have $x = q$ on ext.

Any collection $\{\mathbf{\Gamma}_i\}$ of contours which is the collection of all contours of some diluted configuration will be called an *admissible* collection of contours. On the other hand, any admissible collection determines uniquely some diluted configuration i.e. we can use the notions of an admissible system of contours and of a configuration from $\mathbb{X}$ as synonyma.

In the following text, we will work almost exclusively with finite volumes, which moreover will be often simply connected. In such a case we will consider diluted configurations, which are equal to some $q \in Q$ outside of the given finite volume. More specifically, if $\Lambda \subset \mathbb{Z}^\nu$ is given and x is a q – diluted configuration satisfying the condition that the supports of all its contours have a distance at least 2 from Λ^c and moreover $\Lambda^c \subset$ ext we will say that x is q – diluted *in* Λ. Denoting by $\mathcal{C}(x)$ the union of all supports of contours of x we will write the above condition as $\mathcal{C}(X) \subset\subset \Lambda$.

Diluted partition functions. The partition function over all configurations x_Λ which are q – diluted in Λ (compare (2.2) below for the motivation)

$$Z^q(\Lambda) = \sum_{x_\Lambda:\ \mathcal{C}(x)\subset\subset\Lambda} \exp(-H(x_\Lambda|x^q_{\Lambda^c})) \tag{1.0}$$

will be called the diluted partition function.

This will be our *main object of study* in what follows. We will see that the study of the usual partition functions given for general boundary conditions can be, after all, reduced to the study of diluted partition functions: in particular one can take the diluted partition functions when defining the free energy as the thermodynamic limit of $|\Lambda|^{-1} \log Z^q(\Lambda)$. The convergence will be especially "nice" for some special boundary conditions q, called *stable* in the later text.

Note. The reader maybe expects some "telescopic", recurrent relations, connecting the diluted partition functions in various volumes – in a way analogous to the classical DLR equations. We will minimize the use of such relations in the sequel. See (2.9), (2.10) – however even these relations will be avoided in Section 3 and we will never study directly the relations between diluted partition functions in two different volumes[(4)].

In what follows, the volume Λ will be *fixed* for most of our exposition (better speaking, all possible shifts of Λ will be considered at once)

[(4)] DLR equations will be used only at the very end of our exposition. Having achieved the control over the "external behaviour" of considered configurations (and *this* is the main theme of the Pirogov-Sinai theory) we can supplement the additional information (on what happens inside of the external contours) from the usual DLR equations.

and only at the very end of our investigation the limit $\Lambda \to \mathbb{Z}^\nu$ will be considered. Thus, most of our effort will be devoted to the study of situations appearing in a given finite volume (simultaneously with respect to all its shifts) and it will be quite sensible even not to think about the infinite volume limit which should be (possibly) taken at the very end, after finishing the development of all the important constructions. Just imagine that our "universe" Λ in (1.0) has the cardinality of 10^{27} or so. The philosophy of the Pirogov-Sinai theory is that it should give full answers to all reasonable questions concerning the behaviour of the system in any given finite volume (and not only in the infinite volume limit)!

1.3 – Reformulation of the Hamiltonian

As we already noted the idea is to consider the contours as some "barriers" separating regions occupied by various local ground states. We will now define the *energy* of such a barrier, relating it also to the "ground energy" of the coresponding "underlying configuration" x_q. Define first the second notion:

DENSITY OF ENERGY OF x_q. Put

$$e_q = \sum_{A:\ A \ni 0} |A|^{-1} \Phi_A(x_A) \quad ; \quad e = \min\{e_q\}. \tag{1.1}$$

NOTE. There is some arbitrariness in the precise form of this formula. For example the following quantity has the same value as before:

$$e_q = \sum_{A} \Phi_A(x_A) \tag{1.1'}$$

where the sum is over A having 0 as its "selected" point, in some fixed translation invariant selection rule, given on all finite subsets of $\mathbb{Z}^\nu$. Let us take, for concreteness, the lexicographical order on $\mathbb{Z}^\nu$ and the selection of the first point of A.

In the following we will say that $\mathbf{\Gamma}$ has an "external colour q" if $\mathbf{\Gamma}$ is the only contour of a suitable q – diluted configuration; this configuration (it is uniquely determined by $\mathbf{\Gamma}$) will be denoted by $x_{\mathbf{\Gamma}}$ in the sequel, and we will write $\mathbf{\Gamma} = \mathbf{\Gamma}^q$ in such a case.

Contour energy. Put

$$\Phi(\mathbf{\Gamma}) = \sum_{A: A\cap \operatorname{supp}\mathbf{\Gamma}\neq\emptyset} \Phi_A(x_A)|\operatorname{supp}\mathbf{\Gamma}\cap A|\ |A|^{-1}. \tag{1.2}$$

The following quantities obtained by subtracting, from $\Phi(\mathbf{\Gamma})$, the "ground energy" $e|\operatorname{supp}\mathbf{\Gamma}|$ will be more relevant later when formulating the Peierls condition: Put

$$E(\mathbf{\Gamma}) = \Phi(\mathbf{\Gamma}) - e|\operatorname{supp}\mathbf{\Gamma}| \quad , \quad E_q(\mathbf{\Gamma}^q) = \Phi(\mathbf{\Gamma}^q) - e_q|\operatorname{supp}\mathbf{\Gamma}^q|. \tag{1.3}$$

The latter quantity will be considered only for a contour $\mathbf{\Gamma} = \mathbf{\Gamma}^q$. Notice that then we have the relation $E_q(\mathbf{\Gamma}) = E(\mathbf{\Gamma}) - (e_q - e)|\operatorname{supp}\mathbf{\Gamma}|$.

With these notations one has the following expression of the Hamiltonian, which will play a fundamental role in the following.

Theorem 1. *Let x be a configuration which is q – diluted in Λ*[5]*. Then*

$$\begin{aligned} H(x_\Lambda|x^q_{\Lambda^c}) &= \sum_{q'\in Q}\sum_{t\in\Lambda_{q'}} e_{q'} + \sum_{\mathbf{\Gamma}} (E(\mathbf{\Gamma}) + e|\operatorname{supp}\mathbf{\Gamma}|) + C(q,\Lambda) = \\ &= \sum_{q'\in Q}\sum_{t\in\Lambda_{q'}} e_{q'} + \sum_{\mathbf{\Gamma}}\sum_{t\in\operatorname{supp}\mathbf{\Gamma}} e + \sum_{\mathbf{\Gamma}} E(\mathbf{\Gamma}) + C(q,\Lambda) \quad \textit{i.e.} \end{aligned} \tag{1.4}$$

$$H(x_\Lambda|x^q_{\Lambda^c}) = \sum_{q'} \Big(\sum_{t\in\Lambda_{q'}\cup(\cup_{\mathbf{\Gamma}^{q'}}\operatorname{supp}\mathbf{\Gamma}^{q'})} e_{q'} + \sum_{\mathbf{\Gamma}^{q'}} E_{q'}(\mathbf{\Gamma}^{q'})\Big) + C(q,\Lambda) \tag{1.4$'$}$$

where $\Lambda_{q'}$ denotes the collection of all points of Λ which are q' – correct. The first sum on the right hand side of (1.4) *resp. of* (1.4$'$) *is over all $q' \in Q$ (including q), and the second sum is over all contours $\mathbf{\Gamma}$ of x resp. over all q' contours $\mathbf{\Gamma}^{q'}$ of x. The constant is equal to $C(q,\Lambda) = H(x^q_\Lambda|x^q_{\Lambda^c}) - e_q|\Lambda|$.*

[5] Or, more generally, such that all its contours have supports in Λ.

1.4 – The Abstract Pirogov-Sinai Model

Let us rewrite once again (1.4), from now on without the constant term $C(q,\Lambda)$ (which has no effect on the corresponding Gibbs measure):

$$H(x_\Lambda|x^q_{\Lambda^c}) = \sum_{q'} \sum_{t\in\Lambda_{q'}} e_{q'} + \sum_{\mathbf{\Gamma}} (E(\mathbf{\Gamma}) + e|\operatorname{supp}\mathbf{\Gamma}|). \tag{1.5}$$

Recall that $\Lambda_{q'}$ denotes the collection of all points of Λ which are q' – correct and the sum is over all $q' \in Q$ including the "external colour" q of x_Λ.

Forget now the way how contours were constructed and apply the following, more general approach to the problem of studying the "diluted Gibbs measures" P^q_Λ corresponding to the Hamiltonian (1.5), with the partition function over all q – diluted configurations in Λ:

$$Z^q(\Lambda) = \sum_{x_\Lambda} \exp(-H(x_\Lambda|x^q_{\Lambda^c})). \tag{1.6}$$

Namely, imagine that contours are some abstract, "connected" objects (the exact meaning of the word "connected" can be specified for any particular model; at the moment we may assume that it means the usual connectedness of the supports of contours; as above) which are "coloured" on their boundary by colours from Q, such that any component of the boundary of the set $\operatorname{supp}\mathbf{\Gamma}$ has a *constant* colour $q' \in Q$.

Admissible family of contours. By an admissible family of contours we will mean a family $\{\mathbf{\Gamma}_i\}$ of contours which has the following two properties: 1) Contours of the system do not mutually "touch" (e.g. in the sense that $\operatorname{dist}(\operatorname{supp}\mathbf{\Gamma}_i, \operatorname{supp}\mathbf{\Gamma}_j) \geq 2$ if $i \neq j$) and 2) The prescriptions of the colours outside $\cup\operatorname{supp}\mathbf{\Gamma}_i$ are *not in conflict* i.e. there is a mapping from $(\cup\operatorname{supp}\mathbf{\Gamma}_i)^c$ to Q which is constant on each component of $(\cup\operatorname{supp}\mathbf{\Gamma}_i)^c$ and which is also in accordance with the "colour" of the given component, induced by the neighbouring contours of the system.

Abstract Pirogov-Sinai model. Having defined a family of "colours" Q, a family $\mathcal{G} = \{\mathbf{\Gamma}\}$ of allowable contours of the model, some quantities $\{e_q, q \in Q\}$ and $\{E(\mathbf{\Gamma})\}$ we consider, in any volume Λ, a model whose configurations space is the collection of all admissible systems of contours in Λ and whose Hamiltonian is given by (1.5). This will be called

the abstract Pirogov-Sinai model – corresponding to the given choice of $\mathcal{G}$ and the given quantities $\{e_q, q \in Q\}$ and $\{E(\mathbf{\Gamma})\}$.

If $\mathcal{G}$ and also all the quantities e_q and $E(\mathbf{\Gamma})$ are translation invariant then we will speak about the translation invariant abstract Pirogov-Sinai model. This will be mainly the case considered in these lectures. In fact the assumption of translation invariance can be relaxed both for the quantities $E(\mathbf{\Gamma})$ as well as for e_q[6].

NOTE. We are proposing here the following ideology: do *not* think about the problem in the usual language of Hamiltonians (0.0), DLR equations etc. *Forget* how contours were defined (they could be possibly defined also in another way than before, tailored better to the particular situation) and work with them as with some abstract objects.[7] Of course, one needs some assumptions about the behaviour of the quantities e_q and $E(\mathbf{\Gamma})$. These assumptions can be formulated in a concise way: Namely, the following is really the *only assumption* which is needed to apply the Pirogov-Sinai machinery (developed below) on a given abstract Pirogov-Sinai model.

1.5 – The Peierls Condition

ASSUMPTION. Assume that there is some $\tau \gg 1$ such that for any contour $\mathbf{\Gamma} \in \mathcal{G}$ we have the inequality[8]

$$E(\mathbf{\Gamma}) > \tau |\operatorname{supp} \mathbf{\Gamma}|. \tag{1.8}$$

NOTES. 1. Recall that, whenever we are in the case of a reformulated spin model, we are including the inverse temperature into the Hamiltonian. Typically, $E(\mathbf{\Gamma})/|\operatorname{supp} \mathbf{\Gamma}|$ is then of the order J/T where J denotes

[6] The case of nonconstant e_q requires some modifications of notations in the formulas below. The quantities e_q are usually the leading ones, "more important" than $E(\mathbf{\Gamma})$ because the latter appear usually in the exponential form $\exp(-E(\mathbf{\Gamma}))$; however we have to control the same kind of sums in both cases. See [12] and [13] for the discussion of the problems thus arising. Detailed versions of these papers are in preparation.

[7] Once again, the ideology we propose is: Forget the notion of a spin & of a configuration from $\mathbb{X}$ and replace these primitive notions by other primitive notions of a contour & of an admissible system of contours!

[8] In the literature, such a condition is often formulated only with respect to all true ground states of the original, "unperturbed" (and often more symmetric) Hamiltonian, requiring a lower bound CN, N being the cardinality of the union of all contours of $\tilde{x}$, for the right hand side of (0.0). However, the formulation (1.8) based on the notions of e_q and $E(\mathbf{\Gamma})$ is more adequate for a general Hamiltonian.

the "strength of the interactions" and T is the temperature of the original spin model. See the examples in the forthcoming section.

2. If the "connectedness" of $\mathbf{\Gamma}$ has some "less standard" meaning (than above) then the right hand side of (1.8) should be replaced by another quantity, having the meaning of a "minimal number of points needed to make the set $\operatorname{supp}\mathbf{\Gamma}$ connected".

3. Actually, what will be really needed is not (1.8) but the inequality, with another $\tilde{\tau} \gg 1$ (in the former case we actually have $\tilde{\tau} = \tau - \log|S|$)

$$\sum_{\mathbf{\Gamma}:\operatorname{supp}\mathbf{\Gamma}=G} \exp(-E(\mathbf{\Gamma})) \leq \exp(-\tilde{\tau}|G|). \tag{1.9}$$

It is apparent that for contours constructed from the spin model as above, this means (for $|S|$ not too big and $\tilde{\tau}$ slightly smaller than τ) practically the same as (1.8). However, for large $|S|$ (or for contours defined in some more exotic way) it is sometimes advisable to work directly with (1.9).

4. The exact meaning of the relation $\tau \gg 1$ depends on the dimension of $\mathbb{Z}^\nu$. For $\nu = 3$ this means something like $\tau > 100$. See below.

5. The important observation is that the Peierls condition *remains to be valid* for all sufficiently *small perturbations* of the given model (assuming that for that model this condition was already established).

6. Peierls condition sounds quite natural and there was even a hypothesis for some time that such a condition *always holds* in sufficiently low temperatures (to make τ sufficiently big). The counterexample is due toPECHERSKI [6]. However, in practise, the nonvalidity of the Peierls condition usually means that we just *failed* to find some local ground states of the given model! In other words, our choice of Q was inappropriate. Either too small or, possibly, too big. To illustrate the former case (the latter one can also appear – see the discussion below – but is less important in practise) imagine the Ising model at a small temperature, with an external field whose intensity is even smaller (than the temperature). It is a completely erroneous idea to construct the abstract Pirogov-Sinai model only around the true ground state in such a situation! The other, "slightly unstable" local ground state *must* be also included into Q; otherwise the established constant in the Peierls condition (1.8) would be too poor (namely for the contours marking large droplets of the slightly insta-

ble "forgotten" state) to allow, except of extremely small temperatures, the applicability of the Pirogov-Sinai method.

7. The collection Q should not be also too big. Namely, it could happen that with a too big reference set Q one also would have difficulties with establishing (1.8). Here, we do not mean the possibility that we would not be able to check (1.8) even for those contours $\mathbf{\Gamma}^q$ with e_q roughly equal to e (this would be a real catastrophe; see the point 6 above) but we have in mind the case $e_q - e \gg 1$. Then the remedy is easy (if the contour energy $E(\mathbf{\Gamma}^q)$ is not *very low*): Just remove these q from Q and include also the collection of all q – correct points (for such q) of any configuration x to the set $B(x)$ of its incorrect points. An example of such a situation is the Ising model with a very strong magnetic field, or Blume Capel model with strong one site potentials. Let us look more closely on some of these examples now:

1.6 – Some Examples

ISING MODEL. This is the model where the notion of a contour was invented (for the ferromagnetic case; by Peierls, 1936). Notice that the usual definition of an Ising contour in the ferromagnetic case (as a path in the dual lattice) is *not* identical to the definition of a contour given above, when applied to the Ising model ($r = 1$). However, the interpretation of the usual "contour model" (constructed for the Ising model) as an abstract Pirogov-Sinai model is of course possible, if the notions like "support of a contour" resp. the property "contours do not touch each other" are defined properly[(9)]. It is probably well known to any reader of this text which was already exposed to some presentation of the classical Peierls argument that for the Ising model, the Peierls condition is valid for sufficiently small temperatures. In fact, the energy $E(\mathbf{\Gamma})$ is proportional to the *length* of the contour $\mathbf{\Gamma}$ in this case[(10)] and the quantities e_+, e_- are equal to the $\pm$ intensity of the magnetic field. (It is a good idea to imagine that all e_q are roughly equal to zero when trying to grasp the meaning of (1.5).)

Notice that any finite range *perturbation* of the ordinary Ising model can be transcripted to a suitable abstract Pirogov-Sinai model. If the

[(9)] There are several ways how to do that; for example the connectivity "over the corners" may be or may be not considered etc.

[(10)] Of course, the *antiferromagnetic* case can be also treated by contour methods.

perturbation is sufficiently weak then the validity of the Peierls condition will not be destroyed.

Blume Capel Models. This is a canonical example for the Pirogov-Sinai theory (together with the more general multiple well potential mentioned below). It consists of a variety of models whose properties depend on the particular choice of the numerical constants below. We consider the three spin case, the spin space being denoted by $S = \{-, 0, +\}$. The interactions are again the nearest neighbour ones, and they are given as follows:

$$\Phi_{\{t,s\}}(\pm, \pm) = \Phi_{\{t,s\}}(0, 0) = 0,$$
$$\Phi_{\{t,s\}}(\pm, \mp) = \alpha > 0,$$
$$\Phi_{\{t,s\}}(\pm, 0) = \alpha_{\pm} > 0.$$

The one spin interaction ("external field") is

$$\Phi_t(q) = a_q \quad \text{where } q \in S.$$

i) For $a_{\pm} = 0$ and $a_0 > 0$, this is a simplest discrete version of the "double well model" (which appeared first in the euclidean field theory): the reasonable choice of the set Q is then (for a_0 not too small and for sufficiently small temperatures) $Q = \{+, -\}$[11]. We have

$$e_+ = e_- = 0; \quad (\& \text{ possibly } \quad e_0 = a_0)$$

and (the reader is kindly asked to check this inequality)

$$E(\mathbf{\Gamma}) > C|\operatorname{supp}\mathbf{\Gamma}|$$

where $C = \min\{a_0, \frac{\alpha}{2}, \frac{\alpha_{\pm}}{2}\}$ (resp. $C = \min\{\frac{\alpha}{2}, \frac{\alpha_{\pm}}{2}\}$ if 0 is not included to Q).

ii) Rather trivial case is obtained for $a_0 = 0$ if $a_{\pm} > 0$ are large. Taking $Q = \{0\}$ we obtain an object called a "polymer model", where compatibility of contours means just the compatibility of supports and no "outside colours" of the contours have to be discussed[12].

[11] If a_0 is small then it may be advisable to include 0 into Q. See the point iii).

[12] Similar situation is obtained for the Ising model with a strong magnetic field h. Then it is advisable to take the one element set $Q = \{\text{sign } h\}$.

iii) The case when all the quantities a_0, a_+, a_- are small (compared to $\alpha_\pm$, α) is the most interesting one. It was first treated by Bricmont and Slawny [14]. Then it is natural to take $Q = S$ and for any sufficiently large $\alpha, \alpha_\pm$ (sufficiently large is meant after being divided by the temperature; this remark should be applied everywhere in the following examples!) one can adjust $a_\pm, a$ near zero such that the coexistence of *all three phases* takes place.

Consider, to be more specific, the case $1 \ll \alpha_+ = \alpha_- \ll \alpha$. Then the 0 –th phase has obviously "more freedom" to make the smallest (one point) perturbations. More specifically these perturbations (from 0 to $\pm$) "cost the same" as 0 perturbations of the + resp. − regime[13] but (because of the double possibility of the choice $\pm$ of the sign) the entropy of these perturbations in the 0 regime is almost twice as big as the entropy of the 0 perturbations of the + resp. − regime.

This, of course, should be compensated (to keep the coexistence of all three phases) by requiring that a_0 is suitably (slightly!) greater than $a_+ = a_-$.

Finally, in the very special case when all the quantities $\alpha, \alpha_\pm$ are the same then also the quantities $a, a_\pm$ should be the same to obtain the phase coexistence. The model is symmetric in that case.

Polymer models. The class of so called polymer models is a very special case of the abstract Pirogov-Sinai model, for Q containing *one* element only. Of course there are no phase transitions in such a model – but the question of constructing the *cluster expansions* for these models is very important for us – as we will see below[14].

Below, in Section 2, we will assume some familiarity with the cluster expansion theory. However, later in Section 3 we will present another, more general, selfcontained approach to the Pirogov-Sinai theory which *incorporates* a variant of the usual cluster construction methodology (presented here in a spirit of the recent reference [25]). When applied to polymer models, our method of Section 3 will be just a variant of such a cluster expansion method.

[13] We omit, in this approximate discussion, the energetically more expensive perturbations going from + to − and vice versa.

[14] We ignore here completely the fact that polymer models are important also in the study of *high temperature* situations.

Other examples. Among many other examples (see the literature on the subject) of models where it is advisable to use the Pirogov-Sinai theory we mention here the *Potts model* [49] and, perhaps most typically, the *"double well potential"* models of the quantum field theory (on a discrete spin space[15]).

Our next strategy. The exposition given so far in Section 1 will be in the sequel developed in *two separate* (and more or less independent) *ways*. The first one (presented in the forthcoming Section 2) will give a brief explanation of the essence of the *original Pirogov-Sinai approach* in the situation where "all the possible phases coexist". By the last statement one means the situation where for each $q \in Q$ there exists a "q – like phase" characterized by the following property: Almost all its configurations have the following structure: their spins have "mostly the value q" and the "islands" formed by contours of the given configuration are relatively rare (though they are distributed with a uniform density throughout the whole lattice).

Namely, this is the case where the original Pirogov-Sinai idea of constructing the so called contour functional (and the contour model) is seen in a *most simple and most characteristic* situation.

We will not follow here the other original Pirogov-Sinai constructions like the parametric contour models. Instead, we will rather apply, *starting from Section 3*, quite a *different* (and arguably, a stronger one) approach based on [7] and [8].

2 – The coexistence of all phases. The contour model

Notations. Given a contour $\mathbf{\Gamma}$ (more generally, an admissible system) with a support $\operatorname{supp}\mathbf{\Gamma}$ denote by $\operatorname{ext}(\mathbf{\Gamma})$ resp. $\operatorname{int}(\mathbf{\Gamma})$ the only infinite component of the set $(\operatorname{supp}\mathbf{\Gamma})^c$ resp. the collection of all finite components of $(\operatorname{supp}\mathbf{\Gamma})^c$. Denote by

$$V(\mathbf{\Gamma}) = (\operatorname{ext}\mathbf{\Gamma})^c = \operatorname{supp}\mathbf{\Gamma} \cup \operatorname{int}\mathbf{\Gamma}. \tag{2.1}$$

Contour partition functions. Denote by $\mathbb{X}_{\mathbf{\Gamma}}$ the collections of all configurations $y \in \mathbb{X}$ which satisfy the property that $\mathbf{\Gamma}$ belongs to the

[15] The continuous spin case is more technical ([8]) and it will be treated in a continuation of these lectures [50].

collection of all their contours and moreover all the other contours of y are *inside* $V(\mathbf{\Gamma})$. Let q be the external colour of $\mathbf{\Gamma} = \mathbf{\Gamma}^q$. For any $x \in \mathbb{X}_{\mathbf{\Gamma}}$ and any volume $V \supset V(\mathbf{\Gamma})$ take the Hamiltonian $H(x_V|x^q_{V^c})$ where q denotes the external colour of $\mathbf{\Gamma} = \mathbf{\Gamma}^q$ and the Hamiltonian is given, as always in the following, by (1.5). Notice that we have the value $e|\operatorname{supp}\mathbf{\Gamma}|$ "below" $\operatorname{supp}\mathbf{\Gamma}$ (in addition to $E(\mathbf{\Gamma})$). Take the corresponding partition function

$$Z(\mathbf{\Gamma}) = Z^q(\mathbf{\Gamma}) = \sum_{x \in \mathbb{X}_{\mathbf{\Gamma}}} \exp(-H(x_V|x^q_{V^c})). \tag{2.2}$$

This partition function is called "crystallic" in the original Pirogov-Sinai notation.

The following is really the *central notion of the Pirogov-Sinai theory*. However, the definition given below causes no difficulties only in the "coexistence regime" we are studying below; other aspects of this notion will be discussed later, in Section 3.

The contour functional. Define the quantity, called the contour functional[(16)] of $\mathbf{\Gamma} = \mathbf{\Gamma}^q$

$$F(\mathbf{\Gamma}) = \log Z^q(\operatorname{int}\mathbf{\Gamma}) - \log Z^q(\mathbf{\Gamma}) - e_q|\operatorname{supp}\mathbf{\Gamma}|. \tag{2.3}$$

The following statement follows immediately from the definition of $F(\mathbf{\Gamma})$ and from the *additivity* of the Hamiltonian (1.5) as a function of a volume Λ:

Proposition. *The quantity $F(\mathbf{\Gamma})$ can be expressed as follows (recall (1.4)):*

$$F(\mathbf{\Gamma}) = E_q(\mathbf{\Gamma}) + \log Z^q(\operatorname{int}\mathbf{\Gamma}) - \sum_{q' \in Q} \log Z^{q'}(\operatorname{int}_{q'}\mathbf{\Gamma}) \tag{2.4}$$

where $\operatorname{int}_{q'}\mathbf{\Gamma}$ denotes the union of the components of $\operatorname{int}\mathbf{\Gamma}$ which "have the colour q'" and the last sum is over all $q' \in Q$ (including q).

[(16)] $F(\mathbf{\Gamma})$ is chosen just to fit the equation (2.9) below!

NOTES. 0. The value(s) of q' for which $\mathrm{int}_{q'}\,\mathbf{\Gamma}$ is nonvoid will be called the interior colour(s) of $\mathbf{\Gamma}$. Typically, $\mathrm{int}_{q'}\,\mathbf{\Gamma}$ is nonvoid for only *one* $q' \in Q$. If, in such a case, $q' = q$ then we will have $F(\mathbf{\Gamma}) = E_q(\mathbf{\Gamma})$ because the logarithms cancel each other in (2.4).

1. In general, a good first approximation (neglecting all the possible contours inside of $\mathbf{\Gamma}$) to the quantity $F(\mathbf{\Gamma})$, $\mathbf{\Gamma} = \mathbf{\Gamma}^q$ is the quantity

$$\tilde{F}(\mathbf{\Gamma}) = E_q(\mathbf{\Gamma}) + \sum_{q' \in Q : q' \neq q} (e_{q'} - e_q) |\,\mathrm{int}_{q'}\,\mathbf{\Gamma}|. \tag{2.5}$$

In fact, this is quite an accurate approximation of $F(\mathbf{\Gamma})$ for typical[17], not too big contours; it is even *exact* for contours not exceeding some limited size (such that there is *no* place for other contours in the volume $\mathrm{int}\,\mathbf{\Gamma}$).

2. One could call the quantity $F(\mathbf{\Gamma})$ as the "work needed to install the contour $\mathbf{\Gamma}$". Notice that "more stable" colour inside $\mathbf{\Gamma}$ (roughly: the negativity of the quantity $(e_{q'} - e_q)$) means that $F(\mathbf{\Gamma}) < E(\mathbf{\Gamma})$ and vice versa

THE SYMMETRIC CASE. Assume that we have some group of transformations G acting on Q in a transitive way i.e. such that any element of Q can be mapped by a suitable transformation from G to a selected element of Q. Assume that G acts also, in some way (which is in accordance to the already determined images of the outside and inside colours of $\mathbf{\Gamma}$), on the family of all contours of the model. (Notice that we formulate here the symmetry of our Hamiltonian in a more general setting, in terms of the abstract Pirogov-Sinai model.) Assume that both e_q and $E(\mathbf{\Gamma})$ are *invariant* with respect to the action of G. Then also the partition functions $Z^q(\Lambda)$ do not change after applying the group actions from G to them and we have again (compare Note 0 above) the simple relation

$$F(\mathbf{\Gamma}) = E_q(\mathbf{\Gamma}) = E(\mathbf{\Gamma}). \tag{2.4 S}$$

[17] By a "typical" contour we mean here a contour which is "not dangerously large" – in the sense that the volume term in the right hand side of (2.4) could not erode $E_q(\mathbf{\Gamma})$ substantially. We will see that for the contours which *are* large in such a sense (these will "mark the jumps to the droplets of other, more stable phases"), the quantities $F(\mathbf{\Gamma})$ will *not* be introduced at all (or at least not as straightforwardly as above). These "residual contours" must be treated in a different way.

This includes the Ising ferromagnetic case with zero magnetic field and also some of the other popular examples – like the symmetric Blume Capel model (the case where $+, -, 0$ play exactly the same role), low temperature Potts model etc. Of course, in such special cases the whole Pirogov-Sinai theory is reduced just to an ordinary "Peierls argument" which can be applied whenever the *Peierls condition* holds, with a sufficiently large τ. Our emphasis will *not* be on such (trivial from the point of view of development of the general Pirogov-Sinai theory) cases.

THE CONTOUR MODEL. Given $q \in Q$ consider the polymer model formed by the contours $\mathbf{\Gamma} = \mathbf{\Gamma}^q$ (having the external colour q) and the weights

$$w_{\mathbf{\Gamma}} = \exp(-F(\mathbf{\Gamma})) \tag{2.6}$$

where F is some "abstract functional of contours" (not necessarily always defined *exactly* by (2.3), see below).

We recall that configurations of such a polymer model are defined as arbitrary collections of contours $\{\mathbf{\Gamma}_i\}$ such that the sets $\operatorname{supp}\mathbf{\Gamma}_i$ do not touch each other i.e. the relation $\operatorname{dist}(\operatorname{supp}\mathbf{\Gamma}_i, \operatorname{supp}\mathbf{\Gamma}_j) > 1$ is fulfilled for any pair $i \neq j$, and the weight of any polymer configuration $\{\mathbf{\Gamma}_i\}$ is defined as the product

$$\prod_i w_{\mathbf{\Gamma}_i}. \tag{2.7}$$

We notice that we are studying here and everywhere (most notable exception will be the section dealing with general polymer models) mostly the translation invariant weights $w_{\mathbf{\Gamma}}$ i.e. the situation when the weights (2.6) are invariant with respect to shifts in $\mathbb{Z}^\nu$.

The polymer model with the weights (2.6), defined for all possible configurations of q–contours will be called below the q –th *contour model* corresponding to the Hamiltonian (1.5). (This is in accordance with the terminology of [2]; at least in the coexistence regime.)

NOTE. This is, more or less, a formal algebraic object up to now. However, to obtain some genuine control over the behaviour of such models (in particular, to obtain some really useful information from the algebraic relations (2.10) below) one needs some information about the

behaviour of the weights (2.6) – which will be available only later. Thus, one should have in mind that the following relations hold *universally* (they actually form the basis of the notion of a *contour model* in the traditional Pirogov-Sinai theory!) –but a really useful information can be extracted from them only in special cases studied below.

PROPOSITION (Equivalence of ensembles). *Denote by* Z^q_Λ *the partition function*

$$Z^q_\Lambda = \sum_{\{\mathbf{\Gamma}_i\}} \prod_i w_{\mathbf{\Gamma}^q_i} \tag{2.8}$$

where the sum is over all admissible collections of polymer (i.e. q–contours; even the nonexternal ones !) $\mathbf{\Gamma}^q_i$ *such that* $\cup_i \mathbf{\Gamma}^q_i \subset\subset \Lambda$ *and the right hand side is from* (2.6). $F(\mathbf{\Gamma})$ *is given by* (2.3). *Then*

$$Z^q(\Lambda) = \exp(-e_q|\Lambda|) Z^q_\Lambda. \tag{2.9}$$

The *proof* is done by the induction over the *size* of Λ: assuming already the validity[18] of (2.9) for the interiors of contours appearing in Λ one can write the partition function $Z^q(\Lambda)$ as the sum (over all collections of *external* contours $\{\mathbf{\Gamma}_i\}$) of the products:

$$Z^q(\Lambda) = \sum_{\{\mathbf{\Gamma}_i\}} \exp(-e_q|\operatorname{ext}|) \prod_i Z(\mathbf{\Gamma}_i)$$

(see (2.2)) which is equal to (the products below are over all $\mathbf{\Gamma}_i$ and all $q' \in Q$)

$$\sum_{\{\mathbf{\Gamma}_i\}} \exp(-e_q|\operatorname{ext}|) \prod_i \exp(-E_q(\mathbf{\Gamma}_i) - e_q|\operatorname{supp}\mathbf{\Gamma}_i|) \prod_{q'} \prod_i Z^{q'}(\operatorname{int}_{q'}\mathbf{\Gamma}_i)$$

and this last expresion is equal (after inserting (2.4)) to

$$\begin{aligned} &\sum_{\{\mathbf{\Gamma}_i\}} \exp(-e_q|\operatorname{ext}|) \prod_i \exp(-F(\mathbf{\Gamma}_i) - e_q|\operatorname{supp}\mathbf{\Gamma}_i|) Z^q(\operatorname{int}\mathbf{\Gamma}_i) = \\ &\qquad = \exp(-e_q|\Lambda|) Z^q_\Lambda \end{aligned} \tag{2.10}$$

[18] The validity of (2.9) for very small volumes (not containing any contour) is trivial.

by the induction assumption for $Z^q(\operatorname{int}\mathbf{\Gamma}_i)$.

COROLLARY (Probabilities of external events). *The probability of any event of the type "* $\mathbf{\Gamma}$ *is an external contour of a configuration in* Λ*" is the same both in the diluted ensemble with the partition function* $Z^q(\Lambda)$ *as well as in the contour ensemble with the partition function* Z^q_Λ.

NOTE. This concept, namely the description of an "external behaviour of a configuration" is quite characteristic for the Pirogov-Sinai theory. The information about the behaviour of the given model (1.5) *inside* of the contours is, of course, *not* directly available from the corresponding contour model. However, knowing the probabilities of external contours one can compute the probabilities of the events inside just by computing the conditional Gibbs distributions.

Now, the question is what contour models offer a reasonable (for $\Lambda \to \mathbb{Z}^\nu$) "external behaviour". This, of course, depends on the values $F(\mathbf{\Gamma})$[(19)].

A standard condition assuring the "stability of q" is the following Peierls type condition for the contour functional. In fact, the stability of q *requires* such a condition; this will be clarified later.

PEIERLS TYPE ASSUMPTION FOR F. Below we are studying only the situations where there is a $\tilde\tau \gg 1$ such that for *any* contour $\mathbf{\Gamma}$ which can appear in the considered ensembles,

$$F(\mathbf{\Gamma}) > \tilde\tau |\operatorname{supp}\mathbf{\Gamma}|. \tag{2.11}$$

Namely, we will see that $F(\mathbf{\Gamma}) = F(\mathbf{\Gamma}^q)$ may *drop almost to zero* if q is "unstable" i.e. if the corresponding "q – th phase" does not exist. However, this is not the behaviour to be studied here, in Section 2:

[(19)] In Section 2, we are interested only in the behaviour which will be later called the "stability of all $q \in Q$"; in other words we assume that for any $q \in Q$ the corresponding "q –th phase" will exist; and, in any volume, a typical configuration of this phase will look like a "sea" of the values q with relatively rare "islands of perturbations", marked by the external contours of the given configuration.

2.1 – The strategy of solving (2.4) in the coexistence case

The original Pirogov-Sinai strategy of solving this particular situation (all q "stable") can be now formulated as follows: If the Hamiltonian $H = H(\lambda)$ in (1.5) depends on some n parameters $\lambda \in \mathbb{R}^n$ (in general one needs at least $n \geq |Q| - 1$ parameters to formulate the theorem below), adjust the parameters λ such that all q will become "stable" (in the above mentioned sense).

This will lead to some nonlinear integral equations for $F(\mathbf{\Gamma})$ – and we will look for the parameters λ for which these equations have a solution satisfying (2.11).

NOTE. Below, we are using some rudiments of the cluster expansion theory. The reader not acquainted with this theory can find more details in Section 4, where an independent exposition of the elements of this theory will be given.

In what follows (including Polymer Lemma below) we mean by a contour $\mathbf{\Gamma}$ a contour $\mathbf{\Gamma}^q$ with a *fixed* "colour" q.

DEFINITION OF A CLUSTER (of contours). A cluster $\mathcal{T}$ of contours is a function $\phi : \mathcal{G} \mapsto \mathbb{N}$ (with a connected support; see below) defined on the collection $\mathcal{G}$ of all contours satisfying the following condition: Denote by $\operatorname{Supp}\mathcal{T}$ resp. $\operatorname{supp}\mathcal{T}$ the collection resp. union of the supports of all contours of $\mathcal{G}$ for which $\phi(\mathbf{\Gamma}) > 0$. The *collection of sets* $\operatorname{Supp}\mathcal{T}$ will be called the "Support" of $\mathcal{T}$. By saying that $\mathcal{T}$ is a cluster we will mean that the support $\operatorname{supp}\mathcal{T}$ is *connected*.

The notion of a cluster and of its support can be defined also *recursively* as follows: i) any contour is a cluster ii) if $\mathcal{T}_i\ ,\ i > 0$ are clusters and $\mathbf{\Gamma}$ is a contour such that $\operatorname{dist}(\operatorname{supp}\mathbf{\Gamma}, \operatorname{supp}\mathcal{T}_i) \leq 1$ then the collection $\mathcal{T} = \mathbf{\Gamma} + \{\mathcal{T}_i\}$ is again a cluster; the + operation should be understood in the sense of the indicators ϕ above. (If we replace contours by their *supports* we obtain above a recursive definition of the collection $\operatorname{Supp}\mathcal{T}$.)

The *cardinality* $|\mathcal{T}|$ of a cluster $\mathcal{T}$ represented by a function ϕ will now be defined as

$$|\mathcal{T}| = \sum_{\mathbf{\Gamma} \in \mathcal{T}} \phi(\mathbf{\Gamma}) |\operatorname{supp}\mathbf{\Gamma}| \tag{2.12}$$

(or, recursively, as $|\operatorname{supp}\mathbf{\Gamma}| + \sum_i |\mathcal{T}_i|$).

Thus, one can interpret clusters as some "connected conglomerates of contours" i.e. some collections of contours[20] which are *indecomposable* into two subcollections – such that any contour of one subcollection would be distant from any contour of the other subcollection.

POLYMER LEMMA. *If the weights of the polymer model (* "polymers" *are just contours here and we denote them by symbols* $\mathbf{\Gamma}$*) satisfy assumptions of the type (with small* ε*)*

$$|\sum_{\mathbf{\Gamma}:\operatorname{supp}\mathbf{\Gamma}=T} w_{\mathbf{\Gamma}}| \le \varepsilon^{|T|} \tag{2.13}$$

then the polymer partition function $Z_\Lambda = 1 + \sum_{\{\mathbf{\Gamma}_i\}:\cup_i \operatorname{supp}\mathbf{\Gamma}_i \subset\subset \Lambda} \prod_i w_{\mathbf{\Gamma}_i}$ *can be expanded as the sum over clusters* $\mathcal{T} \subset\subset \Lambda$ [21]

$$\log Z_\Lambda = \sum_{\operatorname{supp}\mathcal{T}\subset\subset\Lambda} \alpha_{\mathcal{T}} w_{\mathcal{T}} \tag{2.14}$$

where $\alpha_{\mathcal{T}}$ *are some "combinatorial coefficients" satisfying a bound*

$$|\alpha_{\mathcal{T}}| \le C^{|\mathcal{T}|} \tag{2.15}$$

for a suitable constant C *depending only on the dimension* ν*. The quantities* $w_{\mathcal{T}}$ *are given by formulas*

$$w_{\mathcal{T}} = w_{\mathbf{\Gamma}} \prod_i w_{\mathcal{T}_i} \quad \textit{resp.} \quad = \prod_{\mathbf{\Gamma}:\phi(\mathbf{\Gamma})>0} w_{\mathbf{\Gamma}}^{\phi(\mathbf{\Gamma})}$$

and therefore satisfy the estimates

$$|\sum w_{\mathcal{T}}| \le (\varepsilon)^{|\mathcal{T}|} \tag{2.16}$$

where the summation is over all clusters with the same collection $\operatorname{Supp}\mathcal{T}$ *(of the supports of elements of* $\mathcal{T}$*). The sums in* (2.14) *are quickly convergent:*

$$\sum_{\mathcal{T}:\ \operatorname{supp}\mathcal{T}\supset S} |\alpha_{\mathcal{T}} w_{\mathcal{T}}| \le (\varepsilon')^{|S|} \tag{2.16 S}$$

[20] Recall once again that contours in the cluster are not required to be mutually different; ϕ is in general an integer valued function having possibly also values > 1 i.e. multiple copies of the same $\mathbf{\Gamma}$ are allowed.

[21] Recall that this means $\operatorname{dist}(\operatorname{supp}\mathcal{T}, \Lambda^c) \ge 2$.

with another small ε'. Define the quantity s_t (the "density of free energy of the polymer model at the point t") as

$$s_t = \sum_{\mathcal{T}:\ t\in \operatorname{supp}\mathcal{T}} |\operatorname{supp}\mathcal{T}|^{-1}\alpha_{\mathcal{T}} w_{\mathcal{T}}. \tag{2.17}$$

Then the following important approximation for $\log Z_\Lambda \approx \sum_{t\in\Lambda} s_t$ is obtained. Denote by $\Delta(\Lambda)$ the quantity defined by the equation

$$\Delta(\Lambda) = \log Z_\Lambda - \sum_{\{t\}\subset\subset\Lambda} s_t. \tag{2.18}$$

Then $\Delta(\Lambda) \approx 0$ more precisely it satisfies the bound, with another small ε''

$$|\Delta(\Lambda)| \le \varepsilon'' |\partial\Lambda|. \tag{2.19}$$

If we moreover write the weights $w_{\mathbf{\Gamma}}$ as $w_{\mathbf{\Gamma}} = \exp(-F(\mathbf{\Gamma}))$ and define the norm

$$||F|| = \sup_{\mathbf{\Gamma}} |F(\mathbf{\Gamma})(V(\mathbf{\Gamma}))^{-1}|$$

then all the quantities $s_t = s_t(F)$ and $\Delta(\Lambda) = \Delta(\Lambda, F)$ are Lipschitz functions of F:

$$|s_t(F+F') - s_t(F)| \le \varepsilon ||F'|| \ ;\ |\Delta(\Lambda, F) - \Delta(\Lambda, F+F')| \le \varepsilon' ||F'|| \tag{2.20}$$

assuming that both F and $F+F'$ remain in the regime (2.11). *Finally, if all the weights $w_{\mathbf{\Gamma}}$ depend analytically on a complex parameter λ from some open set $\mathcal{O} \subset \mathbb{C}$ such that* (2.13) *remains valid for all these λ then all the functions s_t are analytical in $\lambda \in \mathcal{O}$. The same is true for the quantities $\Delta(\lambda)$ and they moreover fulfill* (2.19) *for all $\lambda \in \mathcal{O}$.*

Results of the type (2.14) – (2.16) are proven in almost any text on cluster expansions. (We will give an independent proof later, in Section 4.) The relation (2.19) is an easy consequence of the smallness of the terms $w_{\mathcal{T}}$ & of the quick convergence of the series (2.16). An analogous argument can be used for the sums of the *derivatives* of $w_{\mathcal{T}}$ with respect to F; this yields (2.20). (Take the segment $\{tF + (1-t)F'\}$ and estimate the derivative with respect to t.)

Notice that for a translation invariant model, all the quantities s_t and Δ are also translation invariant. For a q – contour model, we will write $s_t \equiv s$ more precisely as s_q in the following.

THE INVESTIGATION OF (2.4) CONTINUED. Let us conclude our investigation of the equation (2.4): From now on we are again considering only the translation invariant case (the setting of Polymer Lemma was more general, also for translation noninvariant models!), and we are here also *assuming (2.11) to hold* all the time. Put

$$h_q = e_q - s_q. \tag{2.21}$$

We will se later that this is equal to the limit (in the van Hove sense) $\lim(|\Lambda|^{-1} \log Z^q(\Lambda))$. Write (2.4) as follows: Using (2.9) for the expression of the partition functions on the right hand side of (2.4), using the expression (2.18) from the above lemma, and *forgetting the constant terms* $(h_q - h_{q'})\,|\operatorname{int}_{q'} \mathbf{\Gamma}|$ we can rewrite the equation (2.4) as follows:

$$F_{\text{formal}}(\mathbf{\Gamma}) = E_q(\mathbf{\Gamma}) + \tilde{\Delta}(\mathbf{\Gamma}) \tag{2.22}$$

where, for $\mathbf{\Gamma} = \mathbf{\Gamma}^q$, $\tilde{\Delta}(\mathbf{\Gamma}) = \tilde{\Delta}(\mathbf{\Gamma}, F_{\text{formal}})$ is given as

$$\tilde{\Delta}(\mathbf{\Gamma}) = \Delta_q(\operatorname{int} \mathbf{\Gamma}) - \sum_{q'} \Delta_{q'}(\operatorname{int}_{q'} \mathbf{\Gamma}) \tag{2.23}$$

and where $\Delta_q(\operatorname{int} \mathbf{\Gamma})$ is from (2.18). We consider F, written here more precisely as F_{formal}, as an "independent variable" not defined by (2.3),(2.4) but instead of it satisfying the integral equation (2.22). The quantity $\Delta(\mathbf{\Gamma})$ is then computed by (2.23)) from F_{formal}.

We stress that our forgetting of the (generally, of course, the most important!) terms $(h_q - h_{q'})\,|\operatorname{int}_{q'} \mathbf{\Gamma}|$ above means that we are willing to give the interpretation to the results obtained below *only* in the case when *all* $h_q, q \in Q$ *are the same*.

Then, (2.22) can be really tracted not only as an integral equation for the "unknown functional" F_{formal} – which can be solved (iteratively) by the Banach fixed point theorem – but also as an equation from which the "physical value" of F, determined by (2.3) and written more precisely as $F = F_{\text{phys}}$ can be computed: Namely, we solve for any Hamiltonian $H =$

$H(\lambda)$ (from (1.5)) the equation (2.22); then we *compute the quantities* $\{h_q\}$ from the values F_{formal} and finally we *solve* the equation

$$h_q(\lambda) = h_{q'}(\lambda) \text{ whenever } q \neq q' \tag{2.24}$$

where the quantities h_q depend on λ through the functional F_{formal}.

Only those solutions of (2.22) which satisfy (2.24)[22] have a physical sense; these solutions correspond to the situations where all the q – like phases, $q \in Q$ coexist.

Let us conclude this discussion by some theorem. It can be proven by using the implicit function theorem and by establishing suitable differentiability properties of the mapping $\{H \mapsto \{h_q\}\}$. (In particular, we stress the fact that s_q is a *slowly changing* function of the variable F in the norm $||F||$.)

THEOREM. *Let the Hamiltonian $H = H^{(\lambda,\mu)}$ in* (1.5) *depend on some vector parameters (λ, μ) (one may or may not include the temperature into these parameters) where $\lambda \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^m$. Assume that H is continuously differentiable (resp. infinitely smooth, analytic) in these parameters around the value $\lambda = 0, \mu = 0$ and assume that the Hamiltonian $H^{(0,0)}$ has exactly $n+1$ ground states. Let the matrix of the partial derivatives of $\{e_q^{(\lambda,\mu)}\}$ around $\lambda = 0, \mu = 0$ satisfy the property that when completed by the column "1" (all entries in the column are* 1*), it has the rank $n+1$. Then, for a sufficiently small temperature, there is a continuously differentiable (resp. infinitely smooth, analytic) mapping*

$$\{\ \mu \mapsto \lambda(\mu)\ \} \tag{2.25}$$

("for a prescribed μ one finds a suitable λ") such that for each μ from some neighbourhood of zero and for each $q \in Q$, the values $h_q = h_q(F_{\text{formal}})$ are all the same. Therefore, $F_{\text{formal}} = F_{\text{phys}}$ and for each $q \in Q$ there is a "q – like" Gibbs state $P_q^{(\lambda(\mu),\mu)}$ of the Hamiltonian $H^{(\lambda(\mu),\mu)}$, having the support in the set X^q of all q – diluted configurations.

[22] It is rather straightforward to see (by induction over the volume) that the solution F_{formal} of (2.22) satisfying (2.24) must be equal also to the "physical" value of the functional F_{formal} given by (2.3) or (2.4). This would be, of course, no more valid if (2.24) were violated.

3 – The General Phase Picture

This lecture explains the core of the *general* Pirogov-Sinai theory. We will use essentially the approach of [7] (however, with some important new modifications, made in the spirit of the paper [9]).

NOTE We should warn the reader that no phase diagrams will be explicitly constructed and *no* analogy of the theorem above will be formulated here. This *can* be done, of course, similarly as the final theorem of the preceding lecture was more or less direct consequence of the construction of the quantities h_q . So, our emphasis will be again on the construction of (suitable variants of) the quantities h_q, however we are willing to do it now in the *general* case where any phase picture can appear (which will be clarified only at the very last moment of our investigation)[(23)].

We will see that from these quantities, more specifically from the mapping

$$\{\ H(\lambda) \quad \mapsto \quad \{h_q(\lambda)\}\ \}$$

(λ denotes the parameters on which the given Hamiltonian H depends) everything important about the thermodynamics of the model can be computed, after all. The phase diagram of the model will be then just the specification (for any choice of the parameters λ in the Hamiltonian) of all the $q \in Q$ for which h_q is *minimal* possible.

Concerning the "methodological" aspects of our approach, it will not be of much importance for us whether we are in a regime where *several* phases coexist or in the uniqueness regime. Maybe it sounds unusual to some readers (who may identify the Pirogov-Sinai theory as some strange method of construction of the phase diagram), but the essence of the Pirogov-Sinai theory is conserved even in the case when we are studying a *fixed Hamiltonian*, even in the uniqueness regime.

While in Section 2 we have used essentially the original technique of [1] (adapted to a nontrivial special case of the coexistence of the maximal number of phases; this case also "historically" arose as the first one in the course of the development of the paper [1]) here we *depart* from it (and depart also from the more general notion of a parametric contour model

[(23)] These quantities will have the meaning of the Gibbs free energy of some "metastable model".

– see [2]) and adapt the point of [7] with some improvements adapted from [9]. We abandon here the notion of a contour model completely and replace it by another, more "physical" notion of a "metastable ensemble". What will *not* be abandoned is the crucial notion of a *contour functional* F. The latter notion will have the same meaning as before ($F_{\text{phys}} = F_{\text{formal}}$) in the regime of the coexistence of all phases; otherwise its meaning will be changed compared to [1] (remaining, however, closer to its previous value F_{phys} from (2.3) than to the more formal contour functionals used in connection with the original Pirogov-Sinai parametric contour models).

Some intuitive background. We are considering the abstract Pirogov-Sinai model everywhere in the following. That is, we have a general Hamiltonian of the type (1.5), with the Peierls condition (1.8) resp. (1.9) being satisfied.

Recall for a moment the definition of a contour model from Section 2. The basic problem which appears in the study of such a model – and which we simply *avoided* in lecture 2 by studying only the regime of the coexistence of the maximal possible number of phases – is that the contour functional $F = F_{\text{phys}}$ from (2.4) does *not* in general satisfy the Peierls condition $F(\mathbf{\Gamma}) \geq \tilde{\tau}|\operatorname{supp}\mathbf{\Gamma}|$ from (2.11).

In fact, we will see that $F_{\text{phys}}(\mathbf{\Gamma})$ may drop almost to zero in the cases when the contour $\mathbf{\Gamma}$ marks some "jump to the more stable phase inside". (This can happen only if the volume $V(\mathbf{\Gamma})$ is sufficiently large). This observation will lead us to the *decomposition* of the class of all contours into two subclasses[(24)] with very *different behaviour*: The contours from the first class (of "small contours") will contribute to the *entropy* of the corresponding (stable or metastable) "phase" while the other contours (called "large" in the following) will mark possible large droplets of the "more favorable" phases. These droplets should be *so large*, that the jump into the (more favorable) phase inside (which requires an excessive energy spending around the boundary of the droplet) is either "recommended" or at least "disputable" – if the balance of the total free energy loss/gain around $\mathbf{\Gamma}$ is measured.[(25)]

[(24)] Notice however, that the exact borderline between these two subfamilies will be defined in somehow arbitrary manner.

[(25)] Namely, the loss of the energy $E(\mathbf{\Gamma})$ around the boundary of the droplet may be

There is no *entropy* gain from the contours of the second type as we will see; more precisely the possible entropy gain of these contours would be much smaller than the energy excess due to the dwelling in the "less favorable regime" outside of these large contours. Thus, these large contours tend to be (if they appear at all) *as large as possible* and there will be typically *one* such large contour (at most) in a "normally looking" volume Λ (e.g. in a cube Λ) if the outside boundary condition is unstable.

On the other hand, the *small* contours will be relatively "rare" but nevertheless they *will* appear with a regular (nonzero) density throughout the whole $\mathbb{Z}^\nu$, in any phase. So the contribution, to the free density, of these small contours will be nonzero. This contribution thus may play a decisive role in the result of the "energy entropy fighting of various possible phases of the given model" which determines the behaviour of the given system. Its detailed computation really forms the core of the Pirogov-Sinai theory.

Expansion ("recoloring", "sweeping out") of contours. The basic ideas. To outline the technical constructions used below and to understand better their meaning we will explain first the very idea of a *partial expansion* (and the meaning of the notion of "recolorability" used in it) on a *simplified caricature* of the models considered by us. Imagine, for example, a model where for any $q \in Q$ only *one possible shape* of a contour $\mathbf{\Gamma}^q$ is permitted i.e. all allowed contours are of the type

$$\mathbf{\Gamma}^q + t \ , \ t \in \mathbb{Z}^\nu$$

where $\mathbf{\Gamma}^q$ is a fixed contour. Assume for brevity that $\operatorname{int} \mathbf{\Gamma}^q$ has only one component, having a colour $q' \in Q$ inside. Expand the partition function Z^q for such a case: Write

$$Z^q(\Lambda) = \sum_{\{t_i\}} \exp(-e_q |\operatorname{exxt}|) \prod_i \exp(-E_q(\mathbf{\Gamma}_i)) \exp(-e_{q'} |\operatorname{int} \mathbf{\Gamma}_i|)$$

where the sum is over all collections of points $\{t_i\}$ such that the contours $\mathbf{\Gamma}_i = \mathbf{\Gamma}^q + t_i$ mutually do not touch and where we denote by exxt =

compensated by the free energy gain resulting from the more favorable regime *inside* of $\mathbf{\Gamma}$.

$\Lambda \setminus (\cup_i \operatorname{int} \mathbf{\Gamma}_i)$. This can be expressed as

$$Z^q(\Lambda) = \sum_{\{t_i\}} \exp(-e_q|\Lambda|) \prod_i \exp(-F(\mathbf{\Gamma}_i))$$

where, of course, the contour functional of $\mathbf{\Gamma} = \mathbf{\Gamma}^q$ is defined here simply as

$$F(\mathbf{\Gamma}^q) = E_q(\mathbf{\Gamma}^q) - (e_q - e_{q'})|\operatorname{int} \mathbf{\Gamma}^q|.$$

Assuming that F satisfies a Peierls type bound (compare Polymer Lemma, Section 2!)

$$\exp(-F(\mathbf{\Gamma})) \leq \exp(-\tilde{\tau}|\operatorname{supp} \mathbf{\Gamma}|)$$

we know already from this lemma that the above partition function can be written as

$$Z^q(\Lambda) = \exp(-e_q|\Lambda| + \sum_T k_T^q) \tag{3.0}$$

with quickly decaying cluster terms k_T^q. Thus, already in this simplified case we may foresee the main problems which we will have to tackle in the following:

1) A kind of Peierls condition for functionals $F(\mathbf{\Gamma})$ is strongly desirable here. Contours satisfying such a condition will be called "recolorable" in the following text (roughly speaking) and the expression of the partition functions of the original model by the formula of the type (3.0) (which we presented above in an extremely simplified situation; of course) will be called the *recoloring* of (all the shifts of) $\mathbf{\Gamma}$[(26)].

2) Our desire will be then to *repeat* the expansion leading to formulas (3.0) *as many times as possible*. It is useful to introduce here a kind of a "generalized Pirogov-Sinai model" *whose structure will not be changed* after applying such a "recoloring" procedure[(27)]. So we introduce, in the

[(26)] Notice that the transition from $E_q(\mathbf{\Gamma})$ to $F(\mathbf{\Gamma})$ can be really visualized as the "change of the colour (putting q instead of q') inside of the contour $\mathbf{\Gamma}$".

[(27)] Notice that we are able to "recolor" only the internal (collections of) contours of a configuration. An attempt to "recolor everything at once" would lead to much more difficult control of the behaviour of $F(\mathbf{\Gamma})$ due to the fact that also other contours inside of $\mathbf{\Gamma}$ should be considered in such a case. Our inductive construction of the mixed model solves this difficulty (which is typical in the Pirogov-Sinai theory) in quite a convenient way.

following, the important general notion of a "mixed model" which will be defined as an abstract Pirogov-Sinai model having an "additional cluster external field" $\{k_T\}$ i.e. its Hamiltonian will be defined by the following analogy of (1.4):

$$H(x_\Lambda|x^q_{\Lambda^c}) = \sum_{\mathbf{\Gamma}}(E(\mathbf{\Gamma}) + e|\operatorname{supp}\mathbf{\Gamma}|) + \sum_q \sum_{t\in\Lambda_q} e_q - \sum_q \sum_{T\subset\Lambda} k^q_T$$

where the last sum is over all "clusters" T whose support belongs to the set Λ^q denoting the collection of q –correct points of the given configuration $x_\Lambda = \{\mathbf{\Gamma}_i\}$.

3) Having such a notion, being "invariant" with respect to the procedure outlined in 2) the question arises whether it is possible to apply the recoloring procedure suggested above so many times such that it becomes finally *inactive* in the sense that there is "nothing to recolor" in the final mixed model obtained in such a way. In other words, the question is whether "nothing to recolor" really means that there are no contours in the final mixed model; the latter having the meaning that a *total expansion* of the partition functions of the model was obtained.

The answer to the last question will be "yes, at least for some q" (these will be called stable q). Then no contours $\mathbf{\Gamma}^q$ will remain in the final mixed model and so a complete control over its behaviour under boundary condition q will be obtained (through expansions of $Z^q(\Lambda)$, as we outline below in (3.12)). The meaning of our main theorem will be, more precisely, the following. We will find a simple constructive (at least in principle) criterion on how to determine whether given q is "stable". Namely, looking at the quantities

$$h_q = e_q - \sum_{T:\ 0\in T} k^q_T|\operatorname{supp}T|^{-1}$$

constructed for the final ("most expanded") mixed model we will have the statement that q is stable (in the sense suggested above) if and only if h_q has its *smallest possible* value!

To prove this, we will have to investigate in more detail the relation between the notion of a "recolorability" of $\mathbf{\Gamma}$ and another (more transparent!) new notion of a "smallness" of $\mathbf{\Gamma}$: A contour $\mathbf{\Gamma}$ will be called

small if we are "absolutely sure" that the difference

$$\sum_{q' \in Q} (\log Z^{q'}(\mathrm{int}_{q'} \mathbf{\Gamma}) - \log Z^{q}(\mathrm{int}_{q'} \mathbf{\Gamma}))$$

of partition functions on the right hand side of (2.3) "cannot substantially erode the contour energy $E(\mathbf{\Gamma})$". It is now quite a delicate technical task to define this on a *technical level*.

The way chosen below by us turns out to be quite passable: Denote by

$$a^q = h_q - h \ , \ h = \min_q h_q.$$

We will see that whenever $\square$ is a cube such that, say

$$a^q |\square| \leq \tau \, \mathrm{diam} \, \square$$

(and such a condition will hold for squares $\square$ of *any* size if q is stable i.e. if $a_q = 0$!) then no substantial erosion of the type above can happen for contours $\mathbf{\Gamma}^q$ *inside* of $\square$. These cubes will be called *small*.

4) Notice that the notion of a smallness (and recolorability) would not work well if restricted to *single contours only*. Namely, the very idea of recoloring requires its application only to *interior* contours (see next page) or systems of the model:

Imagine the following concentric system of two contours: let the external contour of the system mark a jump of the configuration from a stable phase outside to a "very instable" one residing in the "middle belt", and let the interior contour "jumps again to same stable phase (e.g. the same as outside) in the center". Clearly, it will be impossible to recolor the interior contour itself because its contour functional may violate the Peierls condition. However, it is obviously possible to recolor both contours *at once* (assuming that no other contours are present in the belt we recolor).

This example becomes even more instructive if we generalize it in such a way that *several* interior (& mutually external) contours are inside of a given external contour. It is clear that to speak about a "connectedness" of such a system requires some care.

5) Having established the need for a "simultaneous recoloring" of all contours in some admissible interior subsystem (like above) we want to

claim now that *all* small objects (not only single contours but also their admissible collections) $\mathbf{\Gamma}$ are recolorable.

To see the importance of such a statement notice that for q stable it says that *all* systems $\mathbf{\Gamma}^q$ would be recolorable i.e. *nonexistent* in the final mixed model! This is quite obvious for *single* contours but requires some supplementary argumentation if general small (admissible, interior) *systems* of contours are considered.

This problem is essentially "topological" in its nature, as we will see later (in the subsection "Tight sets").[(28)]

6) Of course, the quantities h_q used in the argumentation below cannot be defined so simply in a general "recoloring step" (which was outlined above only in its simplest version). A useful idea here is to assume that h_q are defined, in general, as free energies of some "metastable models" where "dangerous" ("nonsmall", "nonrecolorable")[(29)] contours are simply *excluded*.

Now we develop the ideas outlined above in a rigorous way: We start with a precise definition what the metastable model should be:

THE NOTION OF A SUBMODEL. Throughout the rest of Section 3, we will consider always the Hamiltonian (1.5). The corresponding diluted partition functions $Z^q(\Lambda)$ will be defined by (1.6).

In the following we will use a concept of a *submodel* of the model (1.5) (on the configuration space $\mathbb{X}$). This is quite a general concept: the determination of the submodel will be done below simply by specifying the corresponding "subset of allowed configurations"(have in mind that in volumes small enough, everything will be allowed)

$$\mathcal{M} \subset \mathbb{X} \tag{3.1}$$

[(28)] Our very formulation of the "recoloring procedure" also for the *systems of contours* will free us from the necessity to combine both the lower and *upper* bounds for for considered partition functions

$$\exp(-h_q|\Lambda| - \varepsilon|\partial\Lambda|) \le Z^q(\Lambda) \le \exp(-h|\Lambda| + \varepsilon|\partial\Lambda|).$$

These bounds were very important in some previous versions of the Pirogov-Sinai theory. Compare "Main Lemma" in [7]. We are not using these bounds in the development of the theory now. However, we will formulate them later (in a stronger form!) as a *corollary* of our Main Theorem.

[(29)] The difference between the choices of these adjectives is not crucial for the intuitive understanding of this notion. Notice also that the precise value of h_q will depend on these conventions for "nonstable" q.

of the given submodel.

The submodel residing on such a configuration space $\mathcal{M}$ will be denoted often also by the same symbol $\mathcal{M}$.

NOTE. We continue to use the symbol $\mathbb{X}$ for the collection of *all* admissible (finite or infinite) collections of contours in $\mathbb{Z}^\nu$. By a "submodel" we will, however, mean in the following usually something *more specific* than what was mentioned above: We will consider below some special submodels called "metastable" ones. They will be indexed by elements $q \in Q$. Each of these submodels will be defined in terms of exclusion, from the family of all q diluted configurations, of configurations where some "dangerous systems of large contours" (namely systems marking large droplets of stable phases appearing inside of the given unstable regime) appear.

The diluted partition functions $Z^q_{\mathcal{M}}(\Lambda)$ *of the submodel* $\mathcal{M}$ will be defined analogously as in (1.6) but with the additional requirement that $x_\Lambda \in \mathcal{M}$.

Extend now also the notion of a contour functional F to a submodel $\mathcal{M}$ just by putting (compare (2.3) and (2.4); for $\mathcal{D}$ small enough, the restriction $\mathcal{M}$ will mean *no* restriction)

$$F_{\mathcal{M}}(\mathcal{D}) = \log Z^q_{\mathcal{M}}(\operatorname{int}\mathcal{D}) - \log Z^q_{\mathcal{M}}(\mathcal{D}) - e_q|\operatorname{supp}\mathcal{D}| \quad \text{i.e.}$$

$$F_{\mathcal{M}}(\mathcal{D}) = E_q(\mathcal{D}) + \log Z^q_{\mathcal{M}}(\operatorname{int}\mathcal{D}) - \sum_{q'\in Q} \log Z^{q'}_{\mathcal{M}}(\operatorname{int}_{q'}\mathcal{D}). \tag{3.2}$$

where $\operatorname{int}\mathcal{D}$ denotes again the union of all nonexternal components of $(\operatorname{supp}\mathcal{D})^c$.

It will be technically very important to generalize this notion below, by the same prescription as in (3.2), to the case when we have, instead of a single contour $\mathcal{D} = \mathbf{\Gamma}$, a *general admissible system* $\mathcal{D}$ *of contours*.[30] (This is the reason why we already used the different symbol $\mathcal{D}$ – instead of $\mathbf{\Gamma}$ – here.) Working with such a general system of contours we will need a notion describing "how much connected the system is ":

[30] Notice that the notion of an interior can be extended in a natural way to any admissible system $\mathcal{D}$.

DEFINITION. For any set $T \subset \mathbb{Z}^\nu$ denote by $\operatorname{con} T$ the minimal possible cardinality of a connected set $\tilde{T}$ such that $\tilde{T} \supset T$. Write $\operatorname{con} \mathcal{D}$ instead of $\operatorname{con} \operatorname{supp} \mathcal{D}$ (where, for $\mathcal{D} = \{\mathbf{\Gamma}_i\}$, we take $\operatorname{supp} \mathcal{D} = \cup_i \operatorname{supp} \mathbf{\Gamma}_i$).

The following three mutually related notions are *crucial* in our approach. The notion of metastability appeared first in [7] and here it keeps its intuitive meaning. However, technically we define it now in a different way, based after all (this we will see below) also on the idea of a partial expansion (of a metastable model). Such a partial expansion will be based here always on an algorithm called the "recoloring of a contour".

All these notions (and also the partial expansion of the model) will be constructed now rigorously, in an *inductive* way:

RECOLORABILITY, RESIDUALITY, METASTABILITY. Let us start with another two auxiliary topological notions: Say that an admissible subsystem $\mathbf{\Gamma}$ of an admissible system $\mathcal{D}$ is an *interior* one if there are no other contours "inside of $\mathbf{\Gamma}$" i.e. if

$$V(\mathbf{\Gamma}) \cap (\mathcal{D} \setminus \mathbf{\Gamma}) = \emptyset.$$

Say that an admissible subsystem $\mathbf{\Gamma}$ of an admissible system $\mathcal{D}$ is an *exterior* one if there is another admissible subsystem $\tilde{\mathbf{\Gamma}} \subset \mathcal{D}$ such that

$$\operatorname{dist}(V(\mathbf{\Gamma}), V(\tilde{\mathbf{\Gamma}})) \geq 2$$

and moreover the remainder $\mathcal{D} \setminus (\mathbf{\Gamma} \cup \tilde{\mathbf{\Gamma}})$ consists, if nonempty, only of (one or more) interior subsystems (denote them by $\mathbf{\Gamma}_i$) of $\mathcal{D}$ which are *inside* of $\mathbf{\Gamma}$ i.e. which satisfy the condition $V(\mathbf{\Gamma}_i) \subset V(\mathbf{\Gamma})$.

NOTE. One should not interpret here the word "exterior" in a too narrow sense: Even a "concentric" system $\mathbf{\Gamma}$ of several contours can be an exterior one in the above sense, if the volumes $V(\mathbf{\Gamma}_i)$ of the interior subsystems do not intersect $\operatorname{supp} \mathbf{\Gamma}$.

DEFINITION.
i) Say that an admissible (nonempty) system $\mathcal{D}$ of contours is *residual* if it has no recolorable (see the point iii) of this definition) interior subsystems.
ii) Say that an admissible system $\mathcal{D}$ is *metastable* if no residual (see the point i) of this definition) exterior subsystems of $\mathcal{D}$ exist.

iii) Say that $\mathcal{D}$ is *recolorable* if the quantity $F_{\mathcal{M}}(\mathcal{D})$ (defined for the *metastable* model; see below for more information) satisfies the bound

$$F_{\mathcal{M}}(\mathcal{D}) \geq \frac{\tau}{12\nu} \operatorname{con} \mathcal{D} \tag{3.4}$$

and moreover this holds also for any other $\tilde{\mathcal{D}}$ with the same external colour and the same support: $\operatorname{supp} \tilde{\mathcal{D}} = \operatorname{supp} \mathcal{D}$[31]

Here, $\mathcal{M}$ denotes the *metastable* model defined as the subset of the original model consisting of all metastable (see the point ii) of this definition) configurations. Do not care now about the particular choice of the constant namely the value $\tau/12\nu$ on the right hand side of (3.4). We will see later why namely this choice is convenient here. The reason why we replaced here the (already "well established") quantity $F(\mathcal{D})$ by another one is that $F_{\mathcal{M}}(\mathcal{D})$ will be easy to expand. Both quantities are the same for those $\mathcal{D}$ which are not "very" large.

One should emphasise the *inductive* (with respect to the growing volumes) nature of the definitions i), ii), iii) above! There is no "cycle" in these definitions, it is rather a "spiral".

AGREEMENT. In what follows, the subscript $\mathcal{M}$ at the quantity $F_{\mathcal{M}}$ will denote the metastable model mentioned above. We will also denote the value of $F_{\mathcal{M}}(\mathbf{\Gamma})$ as $F_{\mathrm{meta}}(\mathbf{\Gamma})$. We emphasise again that for $\mathbf{\Gamma}$ small enough, $F_{\mathrm{meta}}(\mathbf{\Gamma}) = F(\mathbf{\Gamma})$.

The following theorem contains the core of Section 3 (and the core of our present approach to the Pirogov-Sinai theory in its simplest application to the models with a finite number $|Q|$ of constant local ground states, satisfying the Peierls condition). We precede it by introducing another important notions, closely related to the notion of a recolorability:

[31] Thus, recolorability od $\mathcal{D}$ is, in our approach, rather the property of the *set* $\operatorname{supp} \mathcal{D}$ (and $\operatorname{int} \mathcal{D}$) and also of the *external colour* of $\mathcal{D}$, not the property of a particular contour resp. admissible system $\mathcal{D}$. The reader may find unaesthetical the fact that even some contours with the *same* colour outside and inside may be nonrecolorable (they surely could be expanded if we wished to do so!) but the advantage will be that recolorable objects can now be treated simply as *sets* in $\mathbb{Z}^{\nu}$.

DENSITIES OF FREE ENERGY. THE CONCEPT OF A "SMALL" VOLUME. Denote by h_q the free energy[32]

$$h_q = -\lim_{\Lambda\to\mathbb{Z}^\nu}(|\Lambda|)^{-1}\log Z^q_{\mathrm{meta}}(\Lambda)$$

(the limit is taken in Van Hove sense, i.e. such that any point of $\mathbb{Z}^\nu$ finally falls into Λ and such that $|\partial\Lambda||\Lambda|^{-1}\to 0$) of the metastable model $\mathcal{M}_q$ consisting of all q – diluted metastable configurations. The existence of the limit will be obvious only below (starting from (3.16)) and more precise discussion of the notion of a smallness will be given only in the last part of the proof of Main Theorem. Put

$$h = \min_{q\in Q} h_q \ \ ;\ a_q = h_q - h. \tag{3.5}$$

Say that $q\in Q$ is *stable* if $a_q = 0$. Say that a cube $\square\subset\mathbb{Z}^\nu$ is q –small if

$$a_q|\square| < C\tau\,\mathrm{diam}\,\square. \tag{3.6}$$ [33]

The constant τ is from (2.11) and we can take here $C = 1$ or (better) a slightly bigger $C > 1$ (see (4.27)) for example[34]. Say that an admissible system $\mathcal{D} = \mathcal{D}^q$ with the external colour q is small if it can be "packed by some q – small cube $\square$" namely if $\square\supset\mathrm{supp}\,\mathcal{D}$.

The forthcoming definition recalls the definition of a cluster, given already in Section 2, in a slightly more general context needed here: Namely, the primitive objects from which clusters will be formed now will not be always single contours but more generally some special ("recolorable", see below) admissible *systems* of contours.

CLUSTERS OF SETS RESP. OF SYSTEMS OF CONTOURS. Say that the two collections $\mathcal{T} = \{\mathbf{\Gamma}_i\}$, $\mathcal{T}' = \{\mathbf{\Gamma}_j\}$ of contours resp. of admissible systems of contours (multiple copies of the same contour are allowed!) are compatible if any two pairs $\mathbf{\Gamma}_i, \mathbf{\Gamma}_j$ are compatible in the sense that their supports *do not touch*. Say that a collection $\{\mathbf{\Gamma}_i\}$ of contours resp.

[32] Gibbs potential, or pressure are the other frequent names for this quantity.
[33] Thus, for q stable, *any* cube will be small!
[34] Do not inquire about the particular choice of the constant $C\approx 1$ here. Any constant C such that $C\tau\gg 1$ would do the job.

of admissible systems is (in)decomposable if it is (im)possible to split it into two compatible parts. Indecomposable collections of contours or of admissible systems of contours will be called *clusters* and denoted by symbols $\mathcal{T}$[35].

MAIN THEOREM.

(1) *The quantity h is the* free energy *(in the Van Hove sense) of the model* (1.5)*: For any $q \in Q$ we have*

$$\lim_{\Lambda \to \mathbb{Z}^\nu} |\Lambda|^{-1} \log Z^q(\Lambda) = -h. \tag{3.7}$$

(*Recall that for the metastable partition functions,* $\lim |\Lambda|^{-1} \log Z^q_{\mathrm{meta}}(\Lambda) = -h_q$.)

(2) *Small subsystems $\mathcal{D} = \mathcal{D}^q$ can not be residual. In particular there are no residual systems $\mathcal{D}^q$ for q stable. In other words, for stable q we have $\mathcal{M}^q = \mathbb{X}^q$ where $\mathbb{X}^q$ denotes the collection of all q-diluted configurations from $\mathbb{X}$.*

(3) *The metastable partition functions $Z^q(\Lambda)$ can be more precisely expressed as*

$$\log Z^q_{\mathcal{M}}(\Lambda) = -e_q|\Lambda| + \sum_{\mathcal{T} \subset\subset \Lambda} \alpha_{\mathcal{T}} w^q_{\mathcal{T}} \tag{3.8}$$

where $w^q_{\mathcal{T}}$ are products, over the contours (*resp. admissible systems*) *$\mathbf{\Gamma}^q_i$ which are elements of $\mathcal{T}$, of the values* $\exp(-F(\mathbf{\Gamma}_i))$ *and $\alpha_{\mathcal{T}}$ are some combinatorial coefficients depending only on the "topology" of the cluster $\mathcal{T}$, such that*

$$\alpha_{\mathcal{T}} \le C^{\sum_{\mathbf{\Gamma} \in \mathcal{T}} \operatorname{con} \mathbf{\Gamma}} \tag{3.9}$$

for a suitable constant C depending only on the dimension ν and also on the meaning of the statement "contours $\mathbf{\Gamma}$ and $\mathbf{\Gamma}'$ do not touch"[36].

[35] This is closely related to the slightly less general notion of a cluster already introduced in Section 2.

[36] If the latter is meant in the usual sense $\operatorname{dist}(\mathbf{\Gamma}, \mathbf{\Gamma}') \ge 2$ then C can be taken something like $C = 2\nu$.

The cluster series (3.8) *quickly converge, like in* (2.16S). *In analogy to* (2.17), *the quantities* h_q *can be computed also from the formulas*

$$h_q = e_q - \sum_{\mathcal{T}:\ 0\in\operatorname{supp}\mathcal{T}} |\operatorname{supp}\mathcal{T}|^{-1}\alpha_{\mathcal{T}} w_{\mathcal{T}}^q\,. \tag{3.10}$$

(4) *Assume that the Hamiltonian* $H = H^{(\lambda,\mu)}$ *in* (1.5) *depends on some vector parameters* (λ,μ) *where* $\lambda\in\mathbb{R}^n$ *and* $\mu\in\mathbb{R}^m$. *Assume that* H *is differentiable* (*resp. infinitely smooth, analytic*) *in these parameters around the value* $\lambda = 0, \mu = 0$ *and assume that the Hamiltonian* $H^{(0,0)}$ *has stable values of* h_q *for* $q\in\tilde{Q}$ *where* $\tilde{Q}$ *has the cardinality* $n+1$. *Let the matrix of the partial derivatives of* $\{h_q^{(\lambda,\mu)}\}$ *around* $\lambda = 0, \mu = 0$ *satisfy the property that when completed by the column "*1*"* (*all entries in the column are* 1), *it has the rank* $n+1$. *Then there is a differentiable* (*resp. infinitely smooth, analytic*) *mapping*

$$\{\ \mu\mapsto\lambda(\mu)\ \} \tag{3.11}$$

(*"for a prescribed* μ *one finds a suitable* λ*"*) *such that for each* μ *from some neighbourhood of zero and for each* $q\in\tilde{Q}$, *the values* $h_q(\lambda,\mu), q\in\tilde{Q}$ *are all the same, they depend differentiably* (*smoothly, analytically*) *on* μ *and* $h_q > h_{q'}$ *whenever* $q'\in Q\setminus\tilde{Q},\ q\in\tilde{Q}$.

Corollary. *The Hamiltonian* (1.5) *restricted to* $\mathcal{M}^q = \mathbb{X}^q$ *gives a probability measure* $P_{\mathcal{M}}^q$ *which can be interpreted, by taking the inclusion* $\mathcal{M}^q\subset\mathbb{X}$, *as the "*$q$ *– th" Gibbs state on* $\mathbb{X}$ *if* q *is stable.*

Notes. (See Corollary at the end of Section 4 for additional information and interpretation.)

1. Of course, the support of the injection of $P_{\mathcal{M}}$ into $\mathbb{X}$ has the support in $\mathbb{X}^q$ and thus the measures $P_{\mathcal{M}}^q$ are really disjoint[(37)]. Therefore it has a reasonable sense to say that almost any configuration of $P_{\mathcal{M}}^q$ is "externally equal to q". More specifically, take the expansion (3.8) and express the probability of the following event: "a given point $t\in\Lambda$ is outside of all contours of x in the volume Λ under boundary condition

[(37)] A support of a probability is a Borel set whose complement has measure zero.

q". More precisely consider the event "$\mathbf{\Gamma} \subset\subset \Lambda \setminus t$ for *any* contour $\mathbf{\Gamma}$ of x". The probability $P^q_\Lambda(t)$ of such an event is clearly equal (just subtract the expansions of $\log Z^q(\Lambda)$ and $\exp(-e_q)Z(\Lambda \setminus t)$!) to

$$P^q_\Lambda(t) = \exp\Big(\sum_{\mathcal{T}\subset\subset\Lambda}^{\operatorname{dist}(\operatorname{supp}\mathcal{T},\ t)\le 1} \alpha_{\mathcal{T}} w^q_{\mathcal{T}} \Big) \quad \ge 1-\varepsilon \tag{3.12}$$

for some small ε. This can really be interpreted by saying that typical configuration in Λ, under the stable boundary condition q, looks like a "sea" of q with some rare, typically small islands of perturbations.

2. The main message of our theorem is that the properties of the *phase diagram* of the model can be extracted from the mapping

$$\{\ H \quad \mapsto \quad \{h_q\}\ \},$$

more specifically from the mapping (3.11). One should have in mind that while for stable q the quantity $h_q = h$ has the *unique possible meaning*, for unstable q there *is* some arbitrariness of the definition of h_q (steming from some arbitrariness in the definition of a recolorable subsystem $\mathcal{D}$, and from the related arbitrariness in the notion of a metastability). Thus (for example), if we study the differentiability properties of the phase diagram i.e. if the smoothness (as good as possible) of the mapping (3.12) is required than it may be advisable to modify somehow (to "smoothen" as possible) the notion of $F(\mathbf{\Gamma})$ – or to work with a variant of the notion of a recolorability which is *fixed* throughout the given (small) range of parameters. Both these devices guarantee the nonexistence of "jumps" in the formulas for h_q (which would be otherwise caused by the fact that some $\mathbf{\Gamma}$ may cease to be recolorable during a change of the parameter in H).

3. Thus, if we keep the stable boundary condition q outside a set Λ and take the thermodynamic[(38)] limit $\Lambda \to \mathbb{Z}^\nu$ then the "q –th regime" inside will survive the limit. Such a statement is *not* true for unstable q; in that case one expects, for enough large volumes (of the size at least $1/a_q$), the formation of a droplet of a stable phase inside of Λ. We do not study

[(38)] The limit can be taken in a very general sense, requiring that any point of $\mathbb{Z}^\nu$ finally falls inside Λ.

in detail this phenomenon here; see however the Corollary at the end of Section 4 for some preparatory information in this direction.

THE PROOF OF MAIN THEOREM. INTRODUCTION. An earlier variant of the theorem was proven in [7], using still some variant of the original Pirogov-Sinai concept of a contour model (applied there, however, to the study of the corresponding metastable ensemble). Here, we will apply a more recent approach of [9] – which seems to be more powerful and at the same time (at least conceptionally) more simple. Namely, the *bounds* for partition functions will be now replaced by the exact *expansions*, whenever possible. (In fact, *no* bounds for partition functions will now be employed below.)

Let us make some "philosophical" remark about the role of contour models in the Pirogov-Sinai theory: Analyzing the simplest possible answer to the question "why it is necessary to introduce the contour models (i.e. some auxiliary gases of contours)?" the possible answer could be the following[(39)] one: Contour models are *good for expansions*!

Asking further "why expansions are so needed here, in Pirogov-Sinai theory"? the reasonable answer could be that there are, at present, *no* other means enabling to prove the bounds (absolutely crucial for the Pirogov-Sinai theory; this we have seen already in Section 1) of the type (for stable q)

$$|\log Z^q_{\mathcal{M}}(\Lambda) + h_q|\Lambda|| \leq \varepsilon|\partial\Lambda|.$$

The essence of our present approach is the following. In the sequence of reasonings
(metastable) partition functions → their equivalent expression by contour models → expansions of these partition functions → extraction of useful *estimates* from them[(40)] we will now omit the middle term, namely the very *construction of the contour model*. Instead of the notion of a contour model, the keywords of our approach will be the following:

1) The notion of a *partial expansion* of the model. This will be called as

[(39)] There are, of course, also relevant "historical" arguments explaining how the notion of a general contour model emerged: The actual line of development of these ideas was as follows: Peierls argument → Minlos – Sinai contour models → Pirogov-Sinai contour models (in the maximal coexistence regime) → general Pirogov-Sinai theory.

[(40)] This byproduct of the expansion theory is sometimes (misleadingly!) viewed as the only thing which is absolutely indispensable in the Pirogov-Sinai theory.

a *mixed model* below. (This idea is, of course, not new; however in our approach it is really the cornerstone of the theory.)

2) A procedure describing the transition from a given partially expanded model to another, "more expanded" model.

"More expanded" means here that some contours will be "removed" from the new, more expanded model (i.e. they will be "recolored" – in the language we are using below). Of course, such a simplification of the configuration space (configuration space of the given partially expanded model) must be accompanied by a suitable *definition* of some *new* cluster expansion terms appearing in the new, more expanded model.

It should be not surprising that these new cluster expansion terms are employing the quantity already well known to us – namely the contour functional F (of the contours which are just recolored).

3.1 – The notion of a partially expanded model ("mixed" model)

The concept of a mixed model will be now defined as a natural generalization of the abstract Pirogov-Sinai model (1.5). We just will generalize the notion of an external field suitably: namely, in addition to the quantities $E(\mathbf{\Gamma})$ and e_t we will assume that another "cluster field" $\{k_T\}$ (sitting on connected clusters T and depending also on the underlying "colour" induced on T by the configuration $\{\mathbf{\Gamma}_i\}$) is given.

In the following, it will be useful to simplify the notion of a cluster variable by putting, whenever we have an expansion of the type (3.8),

$$k_T^q = \sum_{\mathcal{T}} \alpha_{\mathcal{T}} w_{\mathcal{T}}^q \tag{3.13}$$

where the summation is over all clusters $\mathcal{T} = \{\mathbf{\Gamma}_i^q\}$ with the same collection of supports $T = \operatorname{Supp}\mathcal{T} = \{\operatorname{supp}\mathbf{\Gamma}_i\}$. In such situations the notion of a cluster will have the meaning of a collection of *sets* whose "connectedness" will be meant in the sense that

$$|k_T^q| \le \varepsilon^{\operatorname{con} T} \tag{3.14}$$

where we denote by

$$\operatorname{con} T = \sum_{\operatorname{supp}\mathbf{\Gamma}_i \in \operatorname{Supp}\mathcal{T} = T} \operatorname{con}\mathbf{\Gamma}_i.$$

We *define the new Hamiltonian* $H_{\mathcal{M}}$ of the mixed model as follows.

$$H_{\mathcal{M}}(x_\Lambda|x^q_{\Lambda^c}) = \sum_q \sum_{t\in\Lambda_q} e_q + \sum_{\mathbf{\Gamma}} (E(\mathbf{\Gamma}) + e|\operatorname{supp}\mathbf{\Gamma}|) + \\ - \sum_{T\subset\subset\Lambda\setminus\cup\operatorname{supp}\mathbf{\Gamma}} k_T^{q(x)} \tag{3.15}$$

where $q(x)$ denotes the "colour"(from Q) induced by x on T and the quantities $k^q_T = k^q_T(\mathcal{M})$ are the "external fields" of the given mixed model $\mathcal{M}$. In the following we will use the notations, analogous to what was mentioned above (3.5),

$$h_q = h^{\mathcal{M}}_q = e_q - s_q \quad \text{where} \quad s_q = \sum_{T:0\in T} \frac{k^q_T}{|T|} \tag{3.15h}$$

and call these quantities as "densities of energy" of the given mixed model[(41)].

NOTE. In the following, the terms k_T in any mixed model considered by us will be given just by an *expansion of some metastable model constructed so far*. More precisely we will have from (3.8) (whose validity we are assuming here) and (3.13) the relation

$$\log Z^q_{\mathcal{M}}(\Lambda) = -e_q|\Lambda| + \sum_{T\subset\subset\Lambda} k^q_T \ ; \ k^q_T = k^q_T(\mathcal{M}) \tag{3.16}$$

where $\mathcal{M}$ denotes the metastable model.

Then, if (3.15) is already an expression of such an expansion of the metastable model "living outside of all $\mathbf{\Gamma}$" (notice that different metastable models live in the regions with different $q(x)$ and the quantities k^q_T are functions of q!) in a volume $\Lambda = \operatorname{int}\mathcal{D}$ where $\mathcal{D}$ is some (admissible system of) contour(s) we can rewrite our quantity (3.2) as follows, after substituting 3.16) into it :

$$F_{\mathcal{M}}(\mathcal{D}) = E_q(\mathcal{D}) + \sum_{q'\in Q} \sum_{T\subset\subset \operatorname{int}_{q'}\mathcal{D}} (k^q_T - k^{q'}_T) - \sum_{q'\in Q} (e_q - e_{q'})|\operatorname{int}_{q'}\mathcal{D}|. \tag{3.17}$$

[(41)]Notice that we ignore all contours of the mixed model. The quantities $h^{\mathcal{M}}_q$ are in fact the *free energy densities* of the metastable models whose expansion the given mixed model is!

In a more transparent expression we have

$$F_{\mathcal{M}}(\mathcal{D}) \approx E_q(\mathcal{D}) + \sum_{q' \in Q} (h_q - h_{q'}) |\operatorname{int}_{q'} \mathcal{D}|.$$

Extend now the notion of a recolorability, residuality, metastability also to a situation of a *general mixed model*[(42)]: Just write the mixed Hamiltonian $H_{\mathcal{M}}$ instead of H everywhere in the definitions like (3.4) above. In particular, for the interior systems $\mathcal{D}$ we have the expression (3.17), and therefore (3.4) says that

$$\begin{aligned} E_q(\mathcal{D}) + \sum_{q' \in Q} \sum_{T \subset \subset \operatorname{int}_{q'} \mathcal{D}} (k_T^q - k_T^{q'}) - \sum_{q' \in Q} (e_q - e_{q'}) |\operatorname{int}_{q'} \mathcal{D}| \geq \\ \geq \frac{\tau}{12\nu} \operatorname{con} \mathcal{D} \end{aligned} \tag{3.18}$$

if the considered mixed model is given as the expansion of the metastable model constructed so far in the volume $\operatorname{int} \mathcal{D}$. Later, in our investigation of the validity of (3.4) we will rather try to check another inequality

$$E_q(\mathcal{D}) + \sum_{q' \in Q} (h_q - h_{q'}) |\operatorname{int}_{q'} \mathcal{D}| \geq \frac{\tau'}{12\nu} \operatorname{con} \mathcal{D} \tag{3.18$'$}$$

which is a stronger (for slightly bigger $\tau' > \tau$) and more flexible alternative to the above inequality (3.18).

In fact, below we will be using the notion of a recolorable subsystem of a mixed model only for a *special class of admissible systems*, which are the "smallest possible" in the following sense:

Removable systems of the model. Say that an admissible system $\mathcal{D}$ of a given abstract Pirogov-Sinai mixed model is *removable* if it is recolorable and moreover it satisfies the condition that *no* admissible recolorable system $\mathcal{D}'$ with a smaller volume $V(\mathcal{D}')$ (smaller is meant in the sense that $V(\mathcal{D}') \subsetneq V(\mathcal{D})$) already exists in this mixed model.

[(42)] We are using here the same symbol $\mathcal{M}$ for the mixed as well as for the (original abstract Pirogov-Sinai) metastable model. There will be no confusion in the notation of $F_{\mathcal{M}}(\mathcal{D})$ (which will be later denoted also as $F_{\text{meta}}(\mathcal{D})$) because our mixed model will be always defined as an expansion (in a given volume Λ) of the metastable model constructed so far.

NOTE. Thus, like recolorability (see the text below (3.4)), removability is a property of the *set* $\operatorname{supp}\mathcal{D}$ and of the *external colour* of $\mathcal{D}$, not the property of a particular admissible system $\mathcal{D}$. (In particular, internal colours of $\mathcal{D}$ are not considered in this definition.) By a removal of $\mathcal{D}$ we will in fact mean the removal of all $\tilde{\mathcal{D}}$ with the same support and with the same external colour *at once*; see below.

The following technical lemma is absolutely crucial for our approach.

EQUIVALENCE OF MIXED MODELS. Say that two mixed models are equivalent if all diluted partition functions are the *same* for both models.

RECOLORING LEMMA. *Consider a mixed model $\mathcal{M}$ from* (3.15). *If $\mathcal{D}$ is removable then it is possible to define a new, equivalent mixed model $\mathcal{M}_{\mathrm{new}}$ living on a smaller configuration space given by the exclusion of all configurations containing some $\tilde{\mathcal{D}}$ such that* $\operatorname{supp}\tilde{\mathcal{D}}$ *is a shift of* $\operatorname{supp}\mathcal{D}$ *and such that the external colours of $\tilde{\mathcal{D}}$ and $\mathcal{D}$ are the same, and containing some* new *cluster quantities $k_{\mathcal{T}}$ satisfying, together with the "old" quantities $k_T(\mathcal{M})$, a bound*

$$\sum_{\mathcal{T}:\ \operatorname{Supp}\mathcal{T}=T} k_{\mathcal{T}} \le \varepsilon^{\operatorname{con} T} \tag{3.19}$$

where we put (*see also* (3.14)) $\operatorname{con} T = \sum_i \operatorname{con} T_i$ *for any collection T of sets $\{T_i\}$.* (*Recall that contours resp. admissible systems $\mathbf{\Gamma}$ are counted here with their multiplicity!*). *These new cluster quantities are defined only for clusters containing some shift of* $\operatorname{supp}\mathcal{D}$, *and they depend only on the values $F(\mathcal{D})$ as well as on the "old" values $k_{\mathcal{T}'}$,* $\operatorname{supp}\mathcal{T}' \subset V(\mathcal{T})$.

NOTATION. The passage from the original mixed model to the new one (as described in the lemma above) will be called the *recoloring* of $\mathcal{D}$.

PROOF OF RECOLORING LEMMA. Given a configuration x of a given mixed model write it as

$$x = (\mathcal{D}^1 \cup \mathcal{D}^2 \cup \dots) \cup \tilde{\mathcal{D}} \tag{3.20}$$

where $\mathcal{D}^1, \mathcal{D}^2, \dots, \mathcal{D}^k$ are some shifts of $\mathcal{D}$ (which mutually do not touch and which are *interior subsystems* of x) and $\tilde{\mathcal{D}}$ is the admissible system of the "remaining contours of x" (which appears if all the above shifts

of $\mathcal{D}$ are removed). The fundamental observation – steming from (3.2) (which is rewritten by (3.17)) is the following one. We write it here only for $k = 1$; $\mathcal{M} = \mathcal{M}_{\text{old}}$ is the mixed model given in Recoloring Lemma; $F_{\mathcal{M}}$ is the value of the contour functional (3.17) in the model $\mathcal{M}_{\text{old}}$; we are, of course, working in some finite volume Λ but we omit the symbols Λ, x_Λ here for the simplicity of notations:

$$\exp(-H_{\mathcal{M}}(\mathcal{D} \cup \tilde{\mathcal{D}}) = \exp(-F_{\mathcal{M}}(\mathcal{D})) \exp(-H_{\mathcal{M}}(\tilde{\mathcal{D}})) \exp(-\sum_T k_T^q) \tag{3.21}$$

where the sum in the exponent is over all $T \subset\subset \Lambda$ having the colour q and such that

$$\operatorname{dist}(T, (\operatorname{int} \mathcal{D})^c) \leq 1 \quad \& \quad \operatorname{dist}(T, \operatorname{supp} \mathcal{D}) \leq 1$$

and where q denotes the external colour of $\mathcal{D}$.

Then the new Gibbs factor $\exp(-H_{\mathcal{M}_{\text{new}}}(\tilde{\mathcal{D}}))$ of the remaining configuration $\tilde{\mathcal{D}}$ should be equal to the sum of all corresponding ("old") factor $\exp(-H_{\mathcal{M}}(\tilde{\mathcal{D}}))$ as well as of the factors $\exp(-H_{\mathcal{M}_{\text{old}}}(\tilde{\mathcal{D}} \cup \mathcal{D}^1 \cup \cdots \cup \mathcal{D}^k))$ expressed by (3.21) above (for a general $k \geq 1$). In other words, we require that (we are writing below the result again for a general $k \in \mathbb{N}$)

$$\begin{aligned}
\exp(-H_{\mathcal{M}_{\text{new}}}(\tilde{\mathcal{D}})) &= \exp(-H_{\mathcal{M}}(\tilde{\mathcal{D}})) + \\
&+ \sum_{\{\mathcal{D}^1, \ldots, \mathcal{D}^k\}} \exp(-H_{\mathcal{M}}(\tilde{\mathcal{D}} \cup \mathcal{D}^1 \cup \cdots \cup \mathcal{D}^k)) = \\
&= \exp(-H_{\mathcal{M}}(\tilde{\mathcal{D}})) + \\
&+ \sum_{\{\mathcal{D}^1, \ldots, \mathcal{D}^k\}} \exp\left(- \sum_{\substack{T: \operatorname{dist}(T, (\cup_i \operatorname{int} \mathcal{D}^i)^c) \leq 1 \\ \& \ \operatorname{dist}(T, \cup_i \operatorname{supp} \mathcal{D}^i) \leq 1}} k_T^q \right) \times \\
&\times \exp(-H_{\mathcal{M}}(\tilde{\mathcal{D}})) \prod_{i=1}^k \exp(-F_{\mathcal{M}}(\mathcal{D}^i))
\end{aligned} \tag{3.22}$$

where $F_{\mathcal{M}} = F_{\mathcal{M}_{\text{old}}}$ denotes the contour functional (3.17). Write

$$w_D^q = \sum_{\mathcal{D}': \operatorname{supp} \mathcal{D}' = D} \exp(-F_{\mathcal{M}}(\mathcal{D})) \tag{3.23}$$

where q denotes the external colour of $\mathcal{D}$ and the sum is over all removable $\mathcal{D}'$ which have the same support $\operatorname{supp}\mathcal{D}' = D$ and the same external colour as $\mathcal{D}$[43].

This expression can be written as follows. Let us first *ignore*, just for the clarity of the exposition below, the appearance of the cluster terms k_T^q above i.e. write (3.22) in a simplified form

$$\begin{aligned} \exp(-H_{\mathcal{M}_{\text{new}}}(\tilde{\mathcal{D}})) &= \exp(-H_{\mathcal{M}_{\text{old}}}(\tilde{\mathcal{D}}))+ \\ &+ \sum_{\{\mathcal{D}^1,\dots,\mathcal{D}^k\}} \exp(-H_{\mathcal{M}_{\text{old}}}(\tilde{\mathcal{D}})) \prod_{i=1}^{k} \exp(-F_{\mathcal{M}}(\mathcal{D}^i)). \end{aligned} \tag{3.24}$$

See (4.22) (and below it) for the errata concerning this simplification of the formula (3.22). Using (3.23), the relation (3.24) can be written as

$$\exp(-H_{\mathcal{M}_{\text{new}}}(\tilde{\mathcal{D}})) = \exp(-H_{\mathcal{M}_{\text{old}}}(\tilde{\mathcal{D}}))Z_{\operatorname{out}\tilde{\mathcal{D}}} \tag{3.25}$$

where (the partition function Z_M below will be expanded below; see (4.19))

$$Z_M = \sum_{\{D_i\}} \prod_i w_{D_i}^q \tag{3.26}$$

is the polymer partition function and the weights w_D^q are given by (3.23). The set $\operatorname{out}\tilde{\mathcal{D}}$ here is just the collection of all points of the given volume Λ which are outside $\operatorname{supp}\tilde{\mathcal{D}}$ and whose colour in $(\operatorname{supp}\tilde{\mathcal{D}})^c$ is the same as the external colour of the subsystems $\mathcal{D}_i$ which were just recolored.

4 – Elements of the Cluster Expansion Method. (Conclusion of the proof of Main Theorem)

Now, to finish the proof of Recoloring Lemma we have to expand the *polymer* partition function $Z_{\operatorname{out}\tilde{\mathcal{D}}}$: This is essentially the polymer partition function for polymers which are just shifts of *points* of $\mathbb{Z}^\nu$ with

[43] The terms $\exp(-F_{\mathcal{M}}(\mathcal{D}))$ will be thus "glued together", into one term k_D^q. We are "sweeping out" all the $\mathcal{D}$ with the same support and the same external colour at once.

some translation invariant compatibility relation $\approx$ between them. Notice, however, that *different* polymer models live in regions of $\tilde{\mathcal{D}}$ marked by different colours $q \in Q$.

Maybe it is useful to expose in some detail, in this section, the standard way how polymer partition functions of such a type are expanded: Write the weights w^q_{D+t} (where $D+t$ are various possible, mutually compatible shifts of $D = \operatorname{supp} \mathcal{D}$) simply as w_t and consider them as complex variables indexed by points of $\mathbb{Z}^\nu$. The exposition below was motivated by the paper [25], however we use a slightly different approach here, not using Cauchy formulas and giving instead a more emphasis on the Kirkwood Salsburg equations (which *were* traditionally used in these studies). We develop first the simplified (according to the simplifications made in (3.22) $\to$ (3.26)) case when the polymers "are" just points of the lattice. This is also for the "pedagogical reasons" (to obtain a maximal possible clarity of the formulation of the assumptions imposed on the contour weights). The general case is briefly commented at the end of the section, together with the necessary comments needed to clarify the omissions made in (3.24).

Main Lemma for (point) polymer models. *Let $\approx$ be some compatibility pair relation between points of $\mathbb{Z}^\nu$, such that $t \approx s$ whenever t and s are sufficiently distant: $|t-s| > r$. (The actual choice of r depends on the size of the contours above; recall that the points $t \in \mathbb{Z}^\nu$ actually represent various shifts of possible recolorable systems with a given support* $\operatorname{supp} \mathcal{D}$*). Assume the validity of the Peierls condition*

$$|w_t| \le \varepsilon \tag{4.0}$$

for each $t \in \mathbb{Z}^\nu$, where $\varepsilon \le \varepsilon(\approx)$ for some $\varepsilon(\approx)$ which is sufficiently small (see Note below). Then the partition function over all compatible families $\{t_i\}$ (the analogy of (2.8))

$$Z_\Lambda = 1 + \sum_{\{t_1,\dots,t_k\}: t_i \subset\subset \Lambda}^{t_i \approx t_j} \prod_i w_{t_i} \tag{4.1}$$

is nonzero and it can be expanded into a convergent sum

$$\log Z_\Lambda = \sum_{n_T} \frac{(-1)^n}{n_T!} \alpha_{n_T} w_{n_T} \tag{4.2}$$

where the summation is over special multiindices $n_T = \{n_t, t \in T\}$ *namely those indexed by "indecomposable" collections (" clusters")* $T \subset\subset \Lambda$ *of points. Here we denote by*

$$w_{n_T} = \prod_{t\in T} (w_t)^{n_t},\ n = \sum_t n_t,\ n_T! = \prod_t n_t! \tag{4.3}$$

for any multiindex n_T *whose support* $\{t : n_t \neq 0\}$ *is a cluster* $T = \{t_1, \dots, t_k\}$. *The indecomposability of* T *is meant in the following sense:* T *cannot be splitted into two mutually compatible parts* T_1, T_2 *such that* $t \approx s$ *whenever* $t \in T_1$ *and* $s \in T_2$.

The coefficients α_{n_T} *satisfy the bound, with a constant* $C = C(\approx)$ *giving an upper bound for the number of incompatible neighbours to a given point of* $\mathbb{Z}^\nu$,

$$0 \leq \alpha_{n_T} \leq C^n \tag{4.4}$$

for any n_T. *The series* (4.2) *quickly converge, e.g. we have*

$$\sum_{n_T : n_t > 0 \text{ for all } t\in S} |\alpha_{n_T} w_{n_T}| \leq (C'\varepsilon)^{|S|} \tag{4.4'}$$

with another constant $C' = C'(\approx)$.

NOTE. The statement "$\varepsilon(\approx)$ is sufficiently small" depends therefore on the dimension ν and also on the value r (more precisely it depends on the properties of the relation $\approx$). For example, for $\nu = 2$ and $r = 1$ one can take $\varepsilon(\approx)$ something like $1/8$.

The proof of the lemma is given by taking the Taylor expansion of $\log Z_\Lambda$ with respect to the variables w_t. One has to show first that the terms α_{n_T} with *decomposable* $T = T_1 \cup T_2$ *disappear* i.e. that the derivative of $\log Z_\Lambda$ at the point $\{w_t = 0\}$, with respect to the quantities $\{w_t, t \in T_1\}$ and $\{w_s, s \in T_2\}$ such that $s \approx t$ for each $t \in T_1$ and $s \in T_2$ is zero. To understand this notice that the part of the infinite Taylor sum for $\log Z_\Lambda$ which does not disappear by taking the derivative above can be written as the logarithm of the *product* of the corresponding two quantities Z_{T_1} and Z_{T_2} ! Noticing that one of these two variables depends only on $\{s \in T_1\}$ and analogously, the other only on $\{t \in T_2\}$ we conclude

that the derivative of the logarithm of the product $Z_{T_1} Z_{T_2}$, being the sum of two terms each depending on the corresponding group of variables, disappears.

It remains to prove the bound (4.4) for multiindices n_T with indecomposable T. There are several ways how to estimate the multindices α_{n_T}. The most standard one (see [24]) is perhaps the method based on the following expression of the integer α_{n_T}, which we write for the simplicity of notations for the special case $\{n_t = 1, t \in \operatorname{supp} n_T\}$ only :

$$\alpha_{n_T} = \sum_G (-1)^l \tag{4.5}$$

where the sum is over all graphs G having the support T which are *connected subgraphs* of the graph $G_{\not\sim}$, defined as the collection of all incompatible pairs of points in T, and where $l = |G|$ denotes the number of bonds in G.

Namely, one can compute the term n_T in the expansion of $\log Z_\Lambda$ from the following expression.[(44)] Notice that (by the same argument as used above) it is sufficient to investigate the expansion of Z_T only: the sum below is over *all* subgraphs of $G_{\not\sim}$

$$Z_T = 1 + \sum_G (-1)^l \prod_{t \in T \setminus \operatorname{supp} G} (1 + w_t) \prod_{t \in \operatorname{supp} G} w_t \qquad \text{i.e.} \tag{4.6}$$

$$Z_T = \prod_{t \in T} (1 + w_t) \Big(1 + \sum_G (-1)^l \prod_{t \in \operatorname{supp} G} w_t (1 + w_t)^{-1} \Big).$$

Now take the logarithm of the product and expand $\log(1 + z) = z + \dots$ We get the expression

$$\begin{aligned} \log Z_T &= \sum_{n, \{G_i\}}^{i=1,2,\dots,n} \frac{(-1)^{\sum_i l_i}}{n} \prod_{t \in \cup_i \operatorname{supp} G_i} \Big(\frac{w_t}{1 + w_t} \Big) + \dots = \\ &= \sum_G (-1)^l \prod_{t \in \operatorname{supp} G} w_t + \dots \end{aligned} \tag{4.6'}$$

[(44)] The validity of such an expression is checked by usual inclusion – exclusion formulas. Namely, by the binomial theorem, the terms from $\prod t \in T(1 + w_t) - Z_T$ disappear from the right hand side of the equation (4.6). Any term contributing to Z_T is counted exactly once on the right hand side of (4.6) while the terms from $\prod_{t \in \operatorname{supp} G} w_t Z_{T \setminus \operatorname{supp} G}$ are counted $(1 - 1)^{|G|} = 0$ times !

where the first resp. second sum is taken over all collections of graphs $\{G_i\}$ resp. all graphs G on T. (In the first case the sum is even over graphs with multiple bonds resp. multiple vertices; however these terms obviously do not affect our value of α_{n_T} and so we do not write them yet in the second expression.) The dots denote remaining terms irrelevant for the determination of the coefficient (4.5). Notice that the graphs G in (4.6′) which can be splitted into mutually disconnected pairs $G = G_1 \cup G_2$ also do not contribute to the value of α_{n_T} because (and this is a similar argument as above) these contributions can be computed from the logarithm of the product of suitable functions of $\{w_t : t \in \operatorname{supp} G_i\}$ for $i = 1$ resp. for $i = 2$.

Thus, the sum in (4.6′) can be considered only over all the connected subgraphs G of $G_{\not\approx}$ on T (in principle even over those with a *smaller* support than T; however, the latter terms are again obviously irrelevant for the determination of the value (4.5) of α_{n_T}).

We will not investigate here appropriate estimates of n_T implied by (4.5) i.e. best possible bounds for the sum of $(-1)^{|G|}$ over all connected graphs G with $\operatorname{supp} G = T$ in (4.5). It seems to us that the estimate of α_{n_T} thus obtained is not better (apparently, it is of the same order) than the estimate of α_T given by another, more "traditional" method (see [22], [8]) which is based on the study of the "correlation functions" of the polymer model and which we present in detail below, just for comparison with (4.5).

Let us start with the study of the *first order* derivatives of $\log Z_\Lambda$: Given $t \in \Lambda$ the first derivative of $\log Z_\Lambda$ with respect to w_t is equal to

$$\frac{\partial \log Z_\Lambda}{\partial w_t} = \frac{Z_{\Lambda\setminus t}}{Z_\Lambda} \quad := \rho_t^\Lambda .$$

This is the "correlation function" of t[(45)]. For the estimate of the right hand side of this equation we will use the Kirkwood Salsburg equations (see [22] for example): Assume that we have some selection rule $\{\ A \mapsto t_A\ \}$ for any finite set $A \subset \mathbb{Z}^\nu$. Such a selection rule induces also some partial ordering $\prec$ on the family of all finite sets A from $\mathbb{Z}^\nu$ extending the

[(45)] To see this notice that nonzero terms of the derivative of Z_Λ appear only from the products of the type $w_t \prod w_s$ where all s are compatible with t.

relation $A = A$ and

$$A \setminus t_A \prec A \quad .$$

Write now the partition functions $Z_{\Lambda\setminus A}$, using the notation $\hat{A} = (A \cup \{\hat{t}_A\}) \setminus \{t_A\}$ as

$$Z_{\Lambda\setminus A} = Z_{(\Lambda\setminus A)\cup t_A} - \sum_{\hat{t}_A} w_{\hat{t}_A} Z_{\Lambda\setminus\hat{A}} \tag{4.7}$$

where the sum is over all $\hat{t}_A$ incompatible with t_A. Iterate this equation sufficiently many times for all the terms of the type $Z_{\Lambda\setminus B}$ appearing on the right hand side such that $B \neq \emptyset$. We get the following relation. First write down the result of the iterative ($|A|$ times) substitution of (4.7) into the *first* term on its right hand side:

$$Z_{\Lambda\setminus A} = Z_\Lambda - \sum_{B\prec A; B\neq\emptyset} \sum_{\hat{t}_B} w_{\hat{t}_B} Z_{\Lambda\setminus\hat{B}}. \tag{4.7B}$$

Now substitute (4.7B) iteratively $n-1$ times into the each term on its right hand side which does *not* already contain the term Z_Λ :

$$Z_{\Lambda\setminus A} = Z_\Lambda \, (1 + \sum_{k=1}^{n} (-1)^k \sum_{(B_1,\dots,B_k): B_i\prec\hat{B}_{i-1}}^{B_1\prec A} \prod_{i=1}^{k} w_{\hat{t}_{B_i}}) + R_{n+1} \tag{4.8}$$

where the summation is over collections of nonempty sets B_i and the remainder R_{n+1} is

$$R_{n+1} = \, (-1)^{n+1} \sum_{(B_1,\dots,B_{n+1}): B_i\prec\hat{B}_{i-1}}^{B_1\prec A} \prod_{i=1}^{n+1} w_{\hat{t}_{B_i}} Z_{\Lambda\setminus\hat{B}_{n+1}}. \tag{4.8R}$$

Below we will see that $R_n \to 0$ for $n \to \infty$. This will yield the expression of the quantity $\rho_A^\Lambda = Z_{\Lambda\setminus A}/Z_\Lambda$ by the infinite sum

$$\rho_A^\Lambda = 1 + \sum_{k=1}^{\infty} (-1)^k \sum_{(B_1,\dots,B_k): B_i\prec\hat{B}_{i-1}}^{B_1\prec A} \prod_{i} w_{\hat{t}_{B_i}}. \tag{4.9}$$

Investigate the convergence of this sum. Let us first denote by $N(k)$ the number of possible chains of integers $(|B_1|, \dots, |B_k|)$, $k \geq 1$ (i.e. the

number of possible sequences of substitutions of the first resp. the second term on the right hand side of (4.7) again into (4.7)). We have to make exactly $|A|$ consequent substitutions of the left hand side of the equation (4.7) interrupted somewhere by k substitutions of the right hand side of (4.7)[46].

Let us concentrate on the case $A = \{t\}$. Then obviously $N(k) = 1$ and this must be multiplied by the factor C^k estimating the number of various choices of the sequence $(\hat{t}_{A_1}, \dots, \hat{t}_{A_k})$ where $C = C(\approx)$ denotes[47] the maximal possible number of points which are incompatible with a given point $t \in \Lambda$. Therefore, $(C)^k \varepsilon^k$ is an upper bound for the contribution of all these chains to the sum (4.9). This proves the convergence of (4.9) and therefore also the relation $R_k \to 0$ for ε sufficiently small. The relation $Z_\Lambda \neq 0$ follows from (4.9) by induction over the volume, if we use the obvious relation

$$Z_\Lambda^{-1} = \prod_{M \prec \Lambda} \rho_{t_M}^M \tag{4.10}$$

and the fact (established by induction over volume from (4.9)) that ρ_t^Λ is nonzero (and in fact close to 1). The conclusion of all these estimates is the following bound (which is useful for the sets A of a small cardinality; the term $\sum_{t \in A} \sum_{\hat{t}} w_{\hat{t}}$ below corresponds to all chains of the length $k = 1$, and the remainder is the estimate of the contribution of all the longer chains):

$$|\rho_A^\Lambda - 1 + \sum_{t \in A} \sum_{\hat{t}} w_{\hat{t}}| \leq C'(\varepsilon)^2 \tag{4.11}$$

where C' is a suitable new constant. This proves, considering the special case $A = \{t\}$ a stronger form of the desired relation (4.4) for all the multiindices of the special type (with lowest possible cardinality) $\{n_t = 1$ if $t = t_0$; $n_s = 0$ otherwise $\}$.

NOTE. For $|A| \gg 1$ it is advisable to substitute (4.11) with $|A| = 1$ into an equation (generalization of (4.10))

$$\rho_A^\Lambda = \prod_{B \prec A} \rho_{t_B}^{\Lambda \setminus B}$$

[46] The argument above leading to (4.9) can be formulated by using only (4.7) instead of (4.7B).

[47] There are at most $C(\approx)$ terms in the sum at the right hand side of (4.7).

to obtain a bound

$$|\log \rho_A^\Lambda + \sum_{t\in A}\sum_{\hat t} w_{\hat t}| \le \ C'(\varepsilon)^2|A|. \tag{4.12}$$

For higher multiindices, a most convenient way to establish suitable bound for the (integer!) coefficients α_{n_T} is apparently to derive (4.9) with respect to the remaining variables w_t: Given a multiindex n_T, select a point t_0 with $n_{t_0} > 0$ and denote by m_T the new multiindex $\{m_t = n_t \text{ if } t \neq t_0;\ m_{t_0} = n_{t_0} - 1\}$. Put $m = \sum_t m_t$, $n = \sum_t n_t$.

Write

$$\frac{\partial^n \log Z_\Lambda}{\prod_t \partial(w_t)^{n_t}}(0) = \frac{\partial^m \rho_{t_0}^\Lambda}{\prod_t \partial(w_t)^{m_t}}(0).$$

We get, for $m \ge 1$, the following consequence of (4.9):

$$\alpha_{n_T} = \frac{\partial^m \rho_{t_0}}{\prod_t \partial(w_t)^{m_t}}(0) = (-1)^m \sum_{(B_1,\dots,B_m):B_i=\hat B_{i-1}}^{B_1=t_0} 1 \tag{4.13}$$

where the summation runs over those chains $(B_1, \dots, B_m)$ only which contain any point $t \in T$ *exactly* m_t times in the list of the successive choices of $\hat t_{B_1}, \dots, \hat t_{B_k}$ and where we use again the notation $\hat B = (B \cup \{\hat t_B\}) \setminus \{t_B\}$. Therefore we have the inequality

$$|\alpha_{n_T}| \le M(m) \tag{4.14}$$

where $M(m)$ denotes the number of such chains $(B_1, \dots, B_m)$) and this is (4.4), if we take into account the already established bound $M(m) \le (C)^m$, $C = C(\nu, \approx)$.

NOTE. The coefficients α_{n_T} can be computed also from the Cauchy formulas (like in [25]) but the estimates of these contour integrals do not seem to give better bounds for α_{n_T} than what is obtained above by combinatorial methods.

CLUSTER EXPANSION OF GENERAL POLYMER MODELS. Now we shortly mention the modifications needed to extend the results above

to the case of general polymer models. Consider now a general polymer partition function

$$Z_\Lambda = \sum_{\{S_i\}} \prod_i w_{S_i} \tag{4.15}$$

where S_i are some connected "polymers" with some compatibility relation $\approx$ between them. Below, we will have $S \approx S'$ if and only if S and S' do not "touch'; this phrase will have the meaning $\mathrm{dist}(S, S') > 1$. There is almost nothing requiring a nontrivial change in the formulation and the proof of a corresponding analogy of Main polymer lemma: Assume that the polymers S are connected sets and their weights w_S satisfy a bound

$$|w_S| \leq \varepsilon^{|S|}. \tag{4.16}$$

We have the following equation which is an appropriate generalization of (4.9) :

$$\rho_A^\Lambda = 1 + \sum_{k=1}^{\infty} (-1)^k \sum_{\substack{(B_1, S_1, \ldots, B_k, S_k): \\ B_i \prec B_{i-1} \cup S_{i-1} \,\&\, \mathrm{dist}(t_{B_i}, S_i) \leq 1}}^{B_1 \prec A} \prod_i w_{S_i}. \tag{4.17}$$

The contribution of all such chains of the total length k and with $\sum |S_i| = K$ is estimated (roughly) for $|A| = 1$ as

$$\binom{K}{k} (C\varepsilon)^K \tag{4.18}$$

where $C = C(\nu)$ is suitable geometrical constant. This proves the quick convergence, for small ε, of the series (4.17). (We do not try here to obtain a best possible generalization of the bounds outlined below (4.9).)

Corollary. *Under* (4.16)*, the polymer partition sum* (4.15) *can be expanded as*

$$\log Z_\Lambda = \sum_{\mathcal{N} \subset\subset \Lambda} \alpha_{\mathcal{N}} w_{\mathcal{N}} \tag{4.19}$$

where for any "cluster" of polymers $\mathcal{N} = \{S_i; n_i\}$, $n_i > 0$ *is the multiplicity of the polymer* S_i *and the statement "* $\mathcal{N}$ *is a cluster* $\subset\subset \Lambda$ *" has the meaning that* $\cup S_i$ *is connected and* $\operatorname{supp}\mathcal{N} \subset\subset \Lambda$*". We write* $w_{\mathcal{N}} = \prod_i (w_{S_i})^{n_i}$. *The coefficients* $\alpha_{\mathcal{N}}$ *satisfy the bound*

$$|\alpha_{\mathcal{N}}| \leq C^{\sum_i n_i |S_i|} \tag{4.20}$$

with suitable $C = C(\nu)$ *and the series* (2.39) *quickly converges. In particular we have, with another small* ε', *the following bounds which are valid for any finite set* S*:*

$$\sum_{\mathcal{N} \in \mathcal{S}'} |\alpha_{\mathcal{N}} w_{\mathcal{N}}| \leq (\varepsilon')^{|S|} \tag{4.20'}$$

resp.

$$\sum_{\mathcal{N} \in \mathcal{S}''} |\alpha_{\mathcal{N}} w_{\mathcal{N}}| \leq (\varepsilon')^{|S|} \tag{4.20''}$$

where $\mathcal{S}'$ *resp.* $\mathcal{S}''$ *denote the family of clusters whose support contains resp. intersects* S.

4.1 – Recoloring. Conclusion

Return now to (3.24). Using Corollary of the preceding section we can finally write (3.25) as (the set $\operatorname{out}\tilde{\mathcal{D}}$ was defined in (3.26))

$$\begin{aligned} \exp(-H_{\mathcal{M}_{\text{new}}}(\tilde{\mathcal{D}})) &= \exp(-H_{\mathcal{M}_{\text{old}}}(\tilde{\mathcal{D}})) Z_{\operatorname{out}\tilde{\mathcal{D}}} = \\ &= \exp(-H_{\mathcal{M}_{\text{old}}}(\tilde{\mathcal{D}})) \exp\left(\sum_{\mathcal{T} \subset\subset \operatorname{out}\tilde{\mathcal{D}}} k_{\mathcal{T}} \right) \end{aligned} \tag{4.21}$$

where $\mathcal{T}$ are the new clusters formed by indecomposable collections of shifts of D. This essentially completes our recoloring procedure; however the simplification made at (3.24) must be supplemented now by a more complete discussion: Actually, instead of (3.24) one has a more precise

relation (3.22):

$$\exp(-H_{\mathcal{M}_{\rm new}}(\tilde{\mathcal{D}})) = \exp(-H_{\mathcal{M}_{\rm old}}(\tilde{\mathcal{D}}))+ \\ + \sum_{\{\mathcal{D}^1,\dots,\mathcal{D}^k\}} \exp\left(- \sum_{\substack{T:\ \mathrm{dist}(T,(\cup_i \mathrm{int}\,\mathcal{D}^i)^c\le 1 \\ \&\ \ \mathrm{dist}(T,\cup_i \mathrm{supp}\,\mathcal{D}^i)\le 1}} k_T^q \right) \times \\ \times \exp(-H_{\mathcal{M}_{\rm old}}(\tilde{\mathcal{D}})) \prod_{i=1}^{k} \exp(-F_{\mathcal{M}}(\mathcal{D}^i)) \tag{4.22}$$

where the last sum is over all "old" clusters T whose supports are *not* contained ($\subset\subset$) in a suitable shift $\mathrm{int}(D_i)$ of $\mathrm{int}(D) = \mathrm{int}(\mathcal{D})$ but which "touch" the union of these shifts (which are all recolored at once): $\cup_i \mathrm{int}\, D_i \cap T \neq \emptyset$. The modification which must be made here is the following one: writing

$$\exp(k_T^q) = 1 + \tilde{k}_T^q \tag{4.23}$$

one obtains in fact the following more precise relation :

$$\exp(-H_{\mathcal{M}_{\rm new}}(\tilde{\mathcal{D}})) = \exp(-H_{\mathcal{M}_{\rm old}}(\tilde{\mathcal{D}}))\tilde{Z}_{\mathrm{out}\,\tilde{\mathcal{D}}} \tag{4.24}$$

where the new polymers acting in the definition of the partition functions $\tilde{Z}_{\mathrm{out}\,\tilde{\mathcal{D}}}$ are some connected "conglomerates" $\mathcal{C}$ of the shifts of $\mathrm{supp}\,\mathcal{D}$ *and* sets T from (4.22). The weights $w_{\mathcal{C}}$ are

$$w_{\mathcal{C}} = \prod_i w_{D_i} \prod_j \tilde{k}_{T_j}^q \tag{4.25}$$

for any such conglomerate $\mathcal{C} = \{D_i \& T_j\}$ of shifts D_i (of $D = \mathrm{supp}\,\mathcal{D}$) and of clusters T_j. Let us finally note that the convergence of the cluster series with terms decaying like (3.14) is studied in more detail in [9] (but this is essentially the type of convergence obtained already in classical Mayer expansions).

Conclusion of the proof of Main Theorem. Investigation of the notion of a smallness. The skeleton of a residual system $\mathcal{D}$. Let us return to the notion of a smallness already introduced

in (3.6). In fact, this notion will be discussed in more detail only now where we have at our disposal the notion of a mixed model – which is assumed to be the result of a partial expansion of the original Pirogov-Sinai model. More precisely the contours which are left in such a mixed model are either nonrecolorable or bigger than a certain size. If we are inquiring now the removability of a given $\mathcal{D}$ (of the latter type; nonrecolorable contours resp. admissible systems will be left "for ever" – except of the cases when they are finally swallowed by some bigger, external system making the whole collection recolorable) we have already defined the metastable model *whose all contours have a size smaller than that of* $\mathcal{D}$. Expanding this we obtain the appropriate mixed model suitable for the discussion of the recolorability of $\mathcal{D}$. We define the quantities h_q and a_q in (3.5) as in (3.15h) and thus we have the precise meaning of (3.4), expressed by (3.18) or, more suitably by (3.18′).

Thus, we rather take now, in (3.6), some "temporary" value of h_q, not the final value of h_q which is used in the Main Theorem. A question arises how much is changed if we do not take in (3.6) the final value of a_q (which is not yet constructed !). (More generally one could ask what happens if we take in (3.6) some other value of h_q, obtained for some other, "not so carefully chosen" variant of the mixed model.) The answer is that the discussion of the notion of smallness is not seriously affected by such subtlety of the choice of h_q. The difference between the two above candidates for h_q is of the order $\varepsilon^{\operatorname{diam}\Box}$, really a negligible quantity in formulas (3.6) compared to our freedom in the choice of C. Thus, take suitably smaller (than in (3.6)) constant C namely take $C = 1$ *now* (and remember that one should take slightly bigger $C > 1$ in the original formulation of (3.6)) – to assure that the smallness in the original sense (3.6) would guarantee also the smallness in the modified sense now discussed.

So, let $\mathcal{D}^q$ be a residual admissible system in a cube $\Box$ which is q – small in the temporary mixed model constructed up to now. We will show that such a system *is* recolorable. In particular, such a situation therefore cannot happen in the context of Main Theorem where *total expansion* – with no recolorable systems left in the expanded model – can be considered. The sense of the recoloring procedure formulated above was that for any mixed model with some remaining recolorable systems, we can define an equivalent model on a *smaller configuration*

space. Thus, applying such a procedure sufficiently many times, we may assume that an expansion of the model considered in Main Theorem was already found yielding *no removable* (no recolorable) contours in the corresponding mixed model. The core of the proof of Main Theorem lies then in establishing of the fact that there are also *no small* contours resp. admissible systems in such an expanded model. Namely this is proven in this last section.

So, consider a residual system $\mathcal{D}$ with an external colour q and assume that it is small i.e. there is some q –small cube $\tilde{\square}$ packing $\mathcal{D}$: $V(\mathcal{D}) \subset \tilde{\square}$. Let us show that $\mathcal{D}$ is recolorable.

Take successively some *nonsmall* (in their colour induced by $\mathcal{D}$; of course this interior colour is already different from q !) cubes $\square_i$ which are disjoint from $\operatorname{supp}\mathcal{D} \cup (\tilde{\square})^c$ and which are also *mutually* disjoint. The *skeleton* of $\mathcal{D}$ will be defined as a maximal possible collection $\{\square_i\}$ of such cubes, giving no room for *additional* nonsmall cubes *not* intersecting $\operatorname{supp}\mathcal{D} \cup (\tilde{\square})^c$ and the cubes $\square_i$ already constructed. Writing

$$E(\square_i) = \tau \operatorname{diam} \square_i \tag{4.26}$$

one can treat the cubes of the skeleton as some "additional contours".[(48)] If a new, best possible colour q'' (minimizing $h_{q''}$) is assigned to a cube $\square_i = \square$ living, say, in a q' regime i.e. having the colour q' at the boundary of the complement of $\square$ then we have, using the nonsmallness of $\square$, the inequality[(49)]

$$\log Z^{q''}(\square) - \log Z^{q'}(\square) \geq a_{q'}|\square| - \varepsilon|\partial\square| \geq \tau \operatorname{diam} \square. \tag{4.27}$$

Imagine the new "admissible system" $\mathcal{D}^*$ which is defined as the original system $\mathcal{D}$ *enriched* by all the cubes $\square_i$ of the skeleton. Imagine that the configuration determined by $\mathcal{D}^*$ "jumps into the best possible q'' inside any cube of the skeleton" and the energy of any $\square_i$ from the skeleton is

[(48)] This is a suitable convention to be used below; imagine that we define the new, artificial "contours" $\square_i$ and assign the energy $E(\square_i)$ to them.

[(49)] This is the consequence of the fact that the cubes of the skeleton are not small i.e. the condition (3.6) does *not* hold for them. The constant C from (3.6) should be taken such that (4.27) is fulfilled: take C (slightly) bigger than 1 such that $C(\tau - \varepsilon) > \tau$.

given by (4.26).[(50)] Then we have from (4.27) the inequality

$$F(\mathcal{D}^*) \le F(\mathcal{D}). \tag{4.28}$$

Really, from (2.4) we have, inserting into it the relation (4.27), the inequality

$$F(\mathcal{D}^*) - F(\mathcal{D}) + \sum_i (\tau \operatorname{diam} \square_i - a_{q_i}|\square_i| + \varepsilon|\partial\square_i|) \le F(\mathcal{D})$$

where q_i denotes the colour of the square $\square_i$. This is, by (3.6), the desired bound (4.28).

On the other hand, the smallness of $\mathcal{D}$ (which we assume) implies even the smallness of the enriched system $\mathcal{D}^*$! Namely, $\mathcal{D}^*$ still belongs to $\tilde{\square}$. (This "minor" observation shows the advantage of our very definition of smallness, through the smallness of the covering cubes $\square$.[(51)] Noticing that we have the bound, say

$$|\operatorname{supp} \mathbf{\Gamma}| - \operatorname{diam} \mathbf{\Gamma} > 1/2|\operatorname{supp} \mathbf{\Gamma}|$$

we have the relation, from the *smallness* of $\mathcal{D}^*$

$$F(\mathcal{D}^*) \ge \frac{\tau}{2}|\operatorname{supp} \mathcal{D}^*|. \tag{4.29}$$

Concerning the relation between $|\mathcal{D}^*|$ and $\operatorname{con} \mathcal{D}^*$ we will now have the following important bound (see the topological section below, concluding the Section 4):

$$\operatorname{con} \mathcal{D}^* \le 6\nu|\operatorname{supp} \mathcal{D}^*|. \tag{4.30}$$

[(50)] This is just a play with values $\tau|\operatorname{diam} \square|$ versus $a_{q'}|\square|$; it is not at all necessary to interpret the cubes $\square_i$ as some "real" contours. One could not replace $\operatorname{diam} \square$ by the more "physical" quantity $|\partial\square|$ in this play without substantial changes in the notion of smallness.

[(51)] In connection with rather subtle considerations used here, when discussing the relation between the notions of smallness and recolorability, one should make one general "philosophical" remark: Sometimes, one is fighting severe technical problems in the Pirogov-Sinai theory which however start to be relevant only in volumes which are really astronomically large. For example the problem discussed above i.e. establishing of the fact that residual implies recolorable seems to be much simpler for volumes of a "moderate" size (like 10^{27})!

It is clear that (4.29) and (4.30) would conclude the proof of the fact that a residual small $\mathcal{D}$ *is* recolorable. More precisely, noticing that the enrichened system of contours $\mathcal{D}^*$ is "tight" (see the next section), we obtain the bound

$$F(\mathcal{D}) \geq F(\mathcal{D}^*) \geq \frac{\tau}{2}|\operatorname{supp}\mathcal{D}^*| \geq \frac{\tau}{12\nu}\operatorname{con}\mathcal{D}^* \geq \frac{\tau}{12\nu}\operatorname{con}\mathcal{D} \tag{4.31}$$

i.e. $\mathcal{D}$ really is recolorable. This would, however, contradict the notion of residuality. Thus, there are *no* small residual systems and our Main Theorem is proven.

It remains to say something about the differentiability of the mapping (3.11). Look on the formulas (3.8), (3.10) and notice that the only source of nondifferentiability (even noncontinuity) in the formulas like (3.10) is the fact that the property "to be recolorable" defined by (3.4) is of course not continuous as the function of the Hamiltonian. Thus h_q defined by (3.4) are even *not* continuous for q *non*stable!

However, this obstacle does not appear for stable q. To prove the best possible smoothness properties of the quantities h_q (and hence of the mapping (3.11)) it is advisable to *modify* somehow the notion of recolorability; namely to define some (infinitely) smooth (as a function of the Hamiltonian) alternative to the "yes or no" condition (3.4). One possibility is to multiply $\exp(-F(\mathbf{\Gamma})$ in formulas like (3.10) by some smooth "weight" – which is equal exactly to one if (3.4) is satisfied with a large τ and which goes quickly to zero otherwise. This gives an infinitely smooth alternative to the constructions given above. See e.g. [11] for more details.

NOTE. The local *analyticity* (of the manifolds where the collection of stable q is fixed, and of the value h) is a much more subtle problem. It can be solved by a more careful study of the equations determining $F(\mathbf{\Gamma})$; variants of the quantities h_q which are analytic can be found. See [11]; the method is (one can *not* of course find a decision alternative to (3.4) which would be constantly "yes" for sufficiently large τ and analytic at the same time) to *fix* (independently of the Hamiltonian) the notion of recolorability (and metastability) for nonstable q, locally in λ, μ, and then to use an approach which combines the original, Pirogov-Sinai , "fixed point" method of Section 2 – applied for stable q only– with our

"physical" treatment of $F(\mathbf{\Gamma}_q)$ applies to nonstable q[52]. See [11] for more details.

4.2 – A topological appendix: tight sets

For any set $T \subset \mathbb{Z}^\nu$ denote by $\square(T)$ the smallest cube containing T. Say that a set $S \subset T$ is isolated in T if

$$\operatorname{dist}(\square(S), T \setminus S) \geq \operatorname{diam} \square(S). \tag{4.32}$$

Say that a set $T \subset \Lambda$ is *tight* if it has *no* isolated subsets.

LEMMA. *If T is tight then*

$$\operatorname{con} T \leq 6\nu|T|. \tag{4.33}$$

The proof of this topological statement can be found in [8][53].

When applying this result to the enrichened collections $\mathcal{D}^*$ one has to identify the "supports" of cubes of $\mathcal{D}^*$ as suitable subsets of $\mathbb{Z}^\nu$ of cardinality $\operatorname{diam} \square$. (Contours of $\mathcal{D}$ have supports already defined as connected subsets of $\mathbb{Z}^\nu$.) Let us assume e.g. that any square is identified with its "diagonal". Then clearly $\operatorname{supp} \mathcal{D}^*$ is tight in the sense of the definition above.

4.3 – Bounds for diluted partition functions

The bounds stated below summarize some of the main consequences of our Main Theorem. Such bounds were always important in the

[52] There is, of course, a big freedom in the choice of the constant "τ" in the definition of recolorability. Thus, if we fix this notion with respect to one particular choice of $H(\lambda_0, \mu_0)$ we know that the inequality (3.4) will survive also for Hamiltonians which are sufficiently close to $H(\lambda_0, \mu_0)$, for all contours which have a uniformly bounded size. Studying any fixed phase picture i.e. family $\tilde{Q}$ of stable q containing some $H(\lambda, \mu)$ we can take a sufficiently small region of parameters where the notion of recolorability can be kept fixed for all nonstable values $q \notin \tilde{Q}$. Then we consider the relations (4.38), below (see the next section) as the integral relations for the "unknowns" $\{F(\mathbf{\Gamma}^q), q \in \tilde{Q}\}$ while for the analytic functions $\{F(\mathbf{\Gamma}^q), q \in Q \setminus \tilde{Q}\}$ we use the "physical" prescription (3.2). The rest of the argument is analogous to that of Section 2.

[53] The constant 6ν can be probably considerably improved.

Pirogov-Sinai theory (see [2],[17], [7]). They can be, however, formulated now in a sharper form than before:

COROLLARY. *The partition functions* $Z^q(\Lambda)$ *and the metastable partition functions* $Z^q_{\mathrm{meta}}(\Lambda)$ *can be estimated as follows* (*compare* (3.8)-(3.10))*:*
1) *The metastable partition functions are expressed as*

$$Z^q_{\mathrm{meta}}(\Lambda) = \exp(-h_q|\Lambda| + \Delta^q(\Lambda)) \tag{4.35}$$

where the boundary terms defined as (*we define* $\operatorname{inn}\Lambda = \{t : \operatorname{dist}(t, \Lambda^c) \geq 2\}$)

$$\Delta^q(\Lambda) = -\sum_{T:\ \operatorname{dist}(T,\Lambda^c)\leq 1} \ \sum_{\mathcal{T}:\ \operatorname{supp}\mathcal{T}=T} \alpha_{\mathcal{T}} w_{\mathcal{T}} \frac{|T \setminus \operatorname{inn}\Lambda|}{|T|} \tag{4.36}$$

satisfy the obvious bound, with some small ε' *:*

$$|\Delta^q(\Lambda)| \leq \varepsilon' |\partial\Lambda^c|. \tag{4.37}$$

2) *The general diluted partition functions differ from the metastable ones by factors*

$$Z^q(\Lambda) - Z^q_{\mathrm{meta}}(\Lambda) = \ Z^q_{\mathrm{residual}}(\Lambda) \tag{4.38}$$

where the difference $Z^q_{\mathrm{residual}}(\Lambda)$ *is expressed as the sum over all possible residual* $\mathcal{D}$

$$Z^q_{\mathrm{residual}}(\Lambda) = \sum_{\mathcal{D}} Z^q_{\mathrm{meta}}(\operatorname{ext}\mathcal{D}) \exp(-E(\mathcal{D}) - e|\operatorname{supp}\mathcal{D}|) \prod_{q'\in Q} Z^{q'}_{\mathrm{meta}}(\operatorname{int}_{q'}\mathcal{D}).$$

This is equal, when using (4.35) *for the expression of both the exterior and the interior partition*[(54)] *partition functions* $Z^q_{\mathrm{meta}}(\mathrm{ext})$ *and* $Z^{q'}_{\mathrm{meta}}(\operatorname{int}_{q'}\mathcal{D})$*, to the following expression :*

$$Z^q_{\mathrm{residual}}(\Lambda) = \exp(-h|\Lambda|) \sum_{\mathcal{D}} \exp(-a_q|\operatorname{ext}| - E_q(\mathcal{D})) \exp(\Delta'(\Lambda, \mathcal{D})) \tag{4.39}$$

[(54)] *metastable* functions, by the very definition of residuality

using the notation $\Delta'(\Lambda,\mathcal{D}) = \Delta^q(\operatorname{ext}\mathcal{D}) + \sum_{q'}(\Delta^{q'}(\operatorname{int}_{q'}\mathcal{D}) - a_{q'}|\operatorname{int}_{q'}\mathcal{D}|)$. *The quantity* $Z^q_{\text{residual}}(\Lambda)$ (*it is nonzero only if the above sum over* $\mathcal{D}$ *is nonvoid*) *satisfies the bound*

$$Z^q_{\text{residual}}(\Lambda) \le \exp(-h|\Lambda| + \Delta^q(\Lambda))\exp(-A^q_\tau(\Lambda)) \tag{4.40}$$

where $A^q_\tau(\Lambda)$ *is defined as follows:*

$$A^q_\tau(\Lambda) = \min_T\{A^q_T(\Lambda)\} \tag{4.41}$$

with $A^q_T(\Lambda)$ *being defined (not using the notion of contour anymore) as*

$$A^q_T(\Lambda) = a_q|\operatorname{ext}T \cap \Lambda^c| + (\tau - \varepsilon)|T|$$

the minimum in (4.41) *being taken over all connected* $T \subset \Lambda$ *whose diameter is greater or equal to* τ/a_q. *By* $\operatorname{ext}T$ *we denote the external component of* T^c. *If there is no such* T *then we put* $A^q_\tau = \infty$. *In particular, for volumes of a size* $\approx \tau/a_q$ *we get*

$$Z^q_{\text{residual}}(\Lambda) \le \exp(-h|\Lambda|)\exp(-C\frac{\tau^2}{a_q}) \tag{4.42}$$

with a suitable constant $C = C(\nu)$, *whereas for volumes of the size* $\gg \tau/a_q$ *the bound* (4.40) *can be written as*

$$Z^q_{\text{residual}}(\Lambda) \le \exp(-h|\Lambda|)\exp(-(1-\varepsilon')\tau|\mathcal{D}_\Lambda|)$$

where $\mathcal{D}_\Lambda$ *is the smallest set "swallowing (in its interior) most of* Λ*".*

The *proofs* of this statements follow from (3.8) and (3.10) (from which (4.36) follows just by resummation) and also from the fact – proven in the last section – namely that any residual (therefore nonsmall!) system $\mathcal{D}$ has a size[(55)] *at least* τ/a_q. Therefore we have the inequality, for any

[(55)] More precisely one could argue that even the ratio between $V(\mathcal{D})$ and $|\operatorname{supp}\mathcal{D}|$ for really "dangerous" (nonrecolorable) systems should be at least so big. However, such a stronger statement does not follow immediately from our definition of smallness. Apparently, to improve the bound (4.42) (and, therefore, to achieve the quantity of the order $\tau^\nu/(a_q)^{\nu-1}$ or so in the exponent a suitable modification of the notion of smallness is desirable.

nonsmall $\mathcal{D}$,

$$E(\mathcal{D}) \geq \tau |\operatorname{supp} \mathcal{D}| \geq \frac{\tau^2}{a_q}. \tag{4.43}$$

It is easy to see that (for τ sufficiently large) with the help of this bound, the summation over all possible residual $\mathcal{D}$ gives the bound (4.42).

These bounds say, roughly speaking, that in the volumes of the size $\approx \tau/a_q$ (or smaller) there is no noticeable difference between the behaviour of the stable phases and the q – th "phase". Therefore, the presence of the large "bubbles" of the stable phases (inside of the residual contours) starts to be decisive (concerning the relative contribution, through these configurations, of $Z^q_{\text{residual}}(\Lambda)$ to the partition function $Z^q(\Lambda)$) only when the quantity $A^q_\tau(\Lambda)$ starts to be "considerably smaller than $a_q|\Lambda|$".

NOTE. This opens a way to the estimates of "finite size effects" for realistic boundary conditions (not only the "weak" ones like in [30]). To do this, one of course needs a more precise evaluation of the quantities $Z^q_{\text{residual}}(\Lambda)$ – through the quantities of the type $A^q_\tau(\Lambda)$.

Some bounds of this type were already obtained in [11]. They can be apparently made more sharp now, using the expressions (4.38),(4.39) above.

SURFACE TENSION. It is the characteristic feature of the Pirogov-Sinai theory that quantities like the free energy but also the *surface tension* around a *rigid* interface (and other, less dimensional "tensions", see below) can be computed by quickly converging cluster expansion formulas (and not only by limit procedures using logarithms of partition functions – which is far less flexible and often suspect if more delicate quantities than the "bulk" ones are considered). It can be even said that the very control over all these delicate quantities (like various surface tensions) is one of the most characteristic features of the Pirogov-Sinai theory, thus distinguishing it from the other, less detailed theories.

Simply speaking, all these quantities are defined by suitable *resummation* of the cluster expansion formulas. This resummation uses formulas of the type

$$\sum_T k_T = \sum_t k_t \quad \text{where} \quad k_t = \sum_{T:\ t\in T} \frac{k_T}{|T|} \tag{4.44}$$

which were already used in many parts of these lectures, starting from the definition (1.1). We will not go into details of the computation of quantities like the surface tension around a rigid interface (see [9]) but mention here only the most basic question: what are possible more precise variants of the formula (4.35), taking in account the parts of $\partial\Lambda$ which are *flat*? In a more detailed expression we could consider also the "$\nu-2$ dimensional parts of $\partial\Lambda$" (the collection of all "edges of quadrants sticked to $\partial\Lambda$") etc.

Obviously, the formulas (4.36), (4.37) can be apparently made more precise for volumes whose boundary is sufficiently regular, containing flat pieces having a significant area: Consider the dimension $\nu = 3$ for the brevity, write (3.8) as

$$\log Z^q_{\mathrm{meta}}(\tilde\Lambda) = -e_q|\tilde\Lambda| + \sum_{T\subset\Lambda} k^q_T \tag{4.45}$$

where $\Lambda = \tilde\Lambda \setminus \partial\tilde\Lambda$ and introduce the following six resp. fifteen quantities (which can be called the "plane" surface tensions, the "edge" surface tensions): Put, for example[(56)]

$$\begin{aligned} \sigma_{3+} &= -\sum_{T:0\in T} \frac{n^3_+}{n^3_0}\,\frac{k^q_T}{|T|} \\ \sigma_{\{2+,3+\}} &= -\sum_{T:0\in T} \frac{1}{n^{2,3}_{0,0}}\Big(n^{2,3}_{+,+} - \frac{n^3_+ n^{2,3}_{+,0}}{n^3_0} - \frac{n^2_+ n^{2,3}_{0,+}}{n^2_0}\Big)\,\frac{k^q_T}{|T|} \end{aligned} \tag{4.46}$$

where the quantities $n^3_+, n^3_0, n^{2,3}_{+,+}, n^{2,3}_{+,0}, n^{2,3}_{0,0}$ etc. are defined as follows:

$$\begin{aligned} n^3_+ = |\mathbb{Z}^3_+ \cap T| \;\;;\;\; n^3_0 = |\mathbb{Z}^3_0 \cap T| \;\;; \\ n^{2,3}_{+,+} = |\mathbb{Z}^{2,3}_{+,+} \cap T| \;\;;\;\; n^{2,3}_{0,0} = |\mathbb{Z}^{2,3}_{0,0} \cap T| \\ n^{2,3}_{+,0} = |\mathbb{Z}^{2,3}_{+,0} \cap T| \;\;\text{etc.} \end{aligned} \tag{4.47}$$

[(56)] Formally, the sum $\sum k_T$ (over $T \subset \mathbb{Z}^3_+$) can be written as $(e_q - h_q)|\mathbb{Z}^3_+ + \sigma_{3+}|\mathbb{Z}^3_0|$. Analogously $\sigma_{\{2+,3+\}}|\mathbb{Z}^{2,3}_{0,0}|$ compensates the "edge error" $\sum k_T$ (over $T \subset\subset \mathbb{Z}^{2,3}_{+,+}$) $-\sigma_{3+}|\mathbb{Z}^{2,3}_{+,0}| - \sigma_{2+}|\mathbb{Z}^{2,3}_{0,+}|$.

where

$$\mathbb{Z}^3_+ = \{t \in \mathbb{Z}^3 : t_3 \geq 0\} \ ; \mathbb{Z}^3_0 = \{t \in \mathbb{Z}^3 : t_3 = 0\}$$
$$\mathbb{Z}^{2,3}_{+,+} = \{t \in \mathbb{Z}^3 : t_3 \geq 0 \ \ \& \ \ t_2 \geq 0\} \ ; \ \ \mathbb{Z}^{2,3}_{0,0} = \{t \in \mathbb{Z}^3 : t_3 = 0 \ \ \& \ \ t_2 = 0\}$$
$$\mathbb{Z}^{2,3}_{+,0} = \{t \in \mathbb{Z}^3 : t_3 \geq 0 \ \ \& \ \ t_2 = 0\} \quad \text{etc.}$$

Of course, if the Hamiltonian of the given model is symmetric with respect to permutations and reflections of coordinates then we have the relations like $\sigma_{3\pm} = \sigma_{1\pm}$, $\sigma_{\{2\pm,3\pm\}} = \sigma_{\{1\pm,2\pm\}}$ etc. Then (4.45) can be written as

$$\log Z^q_{\text{meta}}(\tilde{\Lambda}) = -e_q|\partial\tilde{\Lambda}| - h_q|\Lambda| + \sigma_{3+} \ |\partial^3_+\Lambda| + \cdots + \\ + \ \sigma_{\{2+,3+\}}|\partial^{2,3}_{+,+}\Lambda| + \cdots + \Delta(|\partial_{\text{extr}}\Lambda|) \tag{4.48}$$

where e.g. $|\partial^3_+\Lambda|$ denotes the overall cardinality of the flat part of $\partial\Lambda$ which is "of the type $\mathbb{Z}^3_+$" (including its boundary) i.e. the cardinality of that area of $\partial\Lambda$ where Λ "sticks to a halfspace parallel to Z^3_+" and, analogously e.g. $|\partial^{2,3}_{+,+}\Lambda|$ denotes the overall length of all the segments of $\partial\Lambda$ which correspond to the "edges of quadrants parallel to $Z^3_{\{2+,3+\}}$ sticked (in the given segment) to Λ". The second resp. the third term on the right hand side of (4.46) thus represent, together with the other terms of this type, the "flat" resp. the "edge" part of the surface $\partial\Lambda$. (Of course, in higher dimensions there are more quantities of this type.) The last term in (4.48) corresponds to the sum, over T touching $\partial_{\text{extr}}\Lambda$, of the "rests of k_T" and it can be estimated, at least in the case when the mutual distances of the various "flat" and "edge" parts of Λ are bigger than the logarithms of their diameters, as

$$|\Delta(\partial_{\text{extr}}\Lambda)| \leq \varepsilon'|\partial_{\text{extr}}\Lambda| \tag{4.49}$$

where $\partial_{\text{extr}}\Lambda$ denotes the collection of all the extremal points of Λ i.e.the collection of points which do not belong to any segment of $\partial\Lambda$ of notrivial length.

I plan, in future, to add some additional chapters to this text dealing briefly with other aspects of the theory and with other models not covered by the basic setting of Sections 3 and 4: 1) Description of some details of the more general method of [9] which gives in particular the rigorous description of the phenomenon of a "rigid interface", appearing in

a wide class of "stratified" (depending on one coordinate only) phases of the Dobrushin type (with one or *several* rigid interfaces). Such stratified phases emerge in many translation invariant Ising type models (and, of course, also in more general, "stratified" versions of these models, where also the Hamiltonian depends on the last coordinate). It seems that this method has a wider application for situations where "noncrusted" contours (with not so clear a meaning of what should be $\operatorname{int}\boldsymbol{\Gamma}$) appear. 2) Short description of some other aspects of the theory like the smoothness and local analyticity properties of the phase diagram, and the completeness of the phase picture described by our Main Theorem. 3) Finally, I plan to mention some other applications (either already made or with the investigation in progress) to more complicated models like the models with continuous spin [50], models with random impurities, long range models of Kac type etc.

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