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Weighted norm inequalities for the Hardy-Littlewood maximal operator on radial and nonincreasing functions

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RIASSUNTO: *Si assegna una condizione necessaria e sufficiente sulle funzioni peso* $u(.)$ *e* $v(.)$ *affinché risulti limitato l'operatore massimale di Hardy-Littlewood* M : $L^p(\mathcal{R}.D., v(x)dx) \longrightarrow L^q(u(x)dx)$. *Si intende che* $L^p(\mathcal{R}.D., v(x)dx)$ *sia l'insieme di tutte le funzioni radiali non-decrescenti che appartengono allo spazio pesato di Lebesgue* $L^p(v(x)dx)$.

ABSTRACT: *We give a necessary and sufficient condition on the weight functions* $u(.)$ *and* $v(.)$ *for which the Hardy-Littlewood maximal operator* M *is bounded from* $L^p(\mathcal{R}.D., v(x)dx)$ *into* $L^q(u(x)dx)$. *Here* $L^p(\mathcal{R}.D., v(x)dx)$ *is the set of all radial and nonincreasing functions which belong to the weighted Lebesgue space* $L^p(v(x)dx)$.

1 – Introduction and the Result

The Hardy-Littlewood maximal operator on $\mathbb{R}^n$, $n \in \mathbb{N}^*$, is defined as

$$(Mf)(x) = \sup\Big\{|Q|^{-1}\int_Q |f(y)|dy; \quad Q \quad \text{a cube with} \quad Q \ni x\Big\}.$$

Here Q is a cube whose sides are parallel to the coordinates-axes.

KEY WORDS AND PHRASES: *Weighted inequalities – Maximal operator*
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Many authors (see [1], [6], [9]) investigated conditions on weight functions $u(.)$ and $v(.)$ for which M is bounded from $L^p_v = L^p(\mathbb{R}^n, v(x)dx)$ into $L^q_u = L^q(\mathbb{R}^n, u(x)dx)$ i.e.

$$(1.1)\quad \Big(\int_{\mathbb{R}^n}(Mf)^q(x)u(x)dx\Big)^{\frac{1}{q}} \leq C\Big(\int_{\mathbb{R}^n} f^p(x)v(x)dx\Big)^{\frac{1}{p}} \quad \text{for all} \quad f(.)\geq 0,$$

with $1 < p, q < \infty$. Here $C > 0$ is a fixed constant. Let us remind some facts about (1.1).

i) A well known necessary condition for (1.1) with $1 < p < \infty$, is that $v^{-\frac{1}{p-1}}(.) \in L^1_{\mathrm{loc}}(\mathbb{R}^n, dx)$, (see [1] p. 390).

ii) By the Lebesgue differentiation theorem, the boundedness (1.1) has only a nontrivial sense for $q \leq p$.

iii) A characterization of weight functions $u(.)$ and $v(.)$ for which (1.1) holds with $p = q$ is due to SAWYER [6]. And the case $q < p$ is solved by VERBITSKY [9].

iv) In general the characterizing condition found in [6] is not easy to check since it is expressed in term of the operator M and arbitrary cubes. The Verbitsky condition for (1.1) (with $q < p$) is more difficult to handle than the Sawyer's one.

v) The boundedness (1.1) does not hold (in general) for $p = 1$ (see [1], p. 146) so usually a weak version of (1.1) is considered and the case $0 < p < 1$ remains open.

If in (1.1) only functions $f(x) = \varphi(|x|) \geq 0$, with $\varphi(.)$ a nonincreasing function, are considered then the corresponding inequality will be denoted by $M : L^p_v(\mathcal{R}.D.) \to L^q_u$. Our purpose is to characterize the weight functions $u(.)$ and $v(.)$ for which this boundedness holds.

The present work is first motivated by the fact that during these last years the problem $T : L^p_v(\mathcal{R}.D.) \to L^q_u$, for linear operators, has been considered and studied by many authors [7], [2], [3], [4]. So it is also natural to investigate on the same question for the maximal function M which is only a sublinear operator. The second motivation in writing this note is to collect the observations that for the boundedness $M : L^p_v(\mathcal{R}.D.) \to L^q_u$, the points i), ii), iv) and v) are in some way violated. In other words the following remarks can be found:

– it is not required that $v^{-\frac{1}{p-1}}(.) \in L^1_{\mathrm{loc}}(\mathbb{R}^n, dx)$;

– there are nontrivial weight functions $u(.)$ and $v(.)$ for which $M : L^p_v(\mathcal{R}.D.) \to L^q_u$ with $p < q$;
– a necessary and sufficient condition for the boundedness can be derived when $0 < p \leq 1$ and $p \leq q$;
– all of the characterizing conditions for $M : L^p_v(\mathcal{R}.D.) \to L^q_u$ we will introduce are not expressed in term of neither the maximal operator M nor arbitrary cubes, so for many cases (as for instance for radial weights), they are easy to check.

The main result reads as follows:

Theorem. *Let $0 < p, q < \infty$.*

For $p \leq 1$ and $p \leq q$, then $M : L^p_v(\mathcal{R}.D.) \to L^q_u$ if and only if for some constant $C > 0$ these two inequalities hold for all $R > 0$:

$$\left(\int_{|x|<R} u(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_{|x|<R} v(x)dx\right)^{\frac{1}{p}}, \tag{1.2}$$

$$R^n\left(\int_{R<|x|} |x|^{-nq}u(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_{|x|<R} v(x)dx\right)^{\frac{1}{p}}. \tag{1.3}$$

For $1 < p \leq q$ and $p' = \frac{p}{p-1}$, then $M : L^p_v(\mathcal{R}.D.) \to L^q_u$ if and only if both (1.2) and

$$\left(\int_{R<|x|} |x|^{-nq}u(x)dx\right)^{\frac{1}{q}}\left(\int_{|x|<R}\left[\int_{|y|<|x|} v(y)dy\right]^{-p'}|x|^{np'}v(x)dx\right)^{\frac{1}{p'}} \leq C \tag{1.4}$$

are satisfied for all $R > 0$.

Let $q < p$ with $q \neq 1$, $\frac{1}{\theta} = \frac{1}{q} - \frac{1}{p}$, $q' = \frac{q}{q-1}$, (so $q' < 0$ if $q < 1$). Then $M : L^p_v(\mathcal{R}.D.) \to L^q_u$ if and only if the following inequalities hold:

$$\int_{\mathbb{R}^n}\left[\left(\int_{|y|<|x|} u(y)dy\right)^{\frac{1}{p}}\left(\int_{|z|<|x|} v(z)dz\right)^{-\frac{1}{p}}\right]^{\theta} u(x)dx < \infty \tag{1.5}$$

$$\begin{aligned}&\int_{\mathbb{R}^n}\left[\left(\int_{|x|<|y|} |y|^{-nq}u(y)dy\right)^{\frac{1}{q}} \times\right.\\ &\quad\times\left.\left(\int_{|z|<|x|}\left[\int_{|y|<|z|} v(y)dy\right]^{-p'}|z|^{np'}v(z)dz\right)^{\frac{1}{q'}}\right]^{\theta}\times\\ &\quad\times\left[\int_{|y|<|x|} v(y)dy\right]^{-p'}|x|^{np'}v(x)dx < \infty.\end{aligned} \tag{1.6}$$

To justify the above claims in the introduction, examples would be needed.

Corollary. *Let* $0 < p, q < \infty$, $u(x) = |x|^{\beta-n}$ *and* $v(x) = |x|^{\gamma-n}$.

A) *For* $p \le q$, $0 < \beta < nq$ *and* $0 < \gamma < np$ *then* $M : L^p_v(\mathcal{R}.D.) \to L^q_u$ *if and only if* $\frac{\beta}{q} = \frac{\gamma}{p}$.

B) *If* $\frac{\beta}{q} = \frac{\gamma}{p}$ *and* $q < p$, *then the boundedness* $M : L^p_v(\mathcal{R}.D.) \to L^q_u$ *does not hold.*

C) *Let* $w(x) = |x|^{\beta-n} 1\!\!\mathrm{I}_{\{|x|\le 1\}}(x) + |x|^{\delta-n} 1\!\!\mathrm{I}_{\{|x|>1\}}(x)$. *For* $q < p$ *with* $q \ne 1$ *then* $M : L^p_v(\mathcal{R}.D.) \to L^q_w$ *whenever* $0 < \frac{\delta}{q} < \frac{\gamma}{p} < n < \frac{\beta}{q}$.

Remarks. For the following observations 1) to 3), the integer n is taken equal to 1.

1) For $u(x) = |x|^{-\frac{1}{2}}$ (i.e. $\beta = \frac{1}{2}$) then $M : L^p_u(\mathcal{R}.D.) \to L^p_u$ for all $1 < p < \infty$. Better, it is well-known ([1], Corollary 1.13 p393) that (1.1) is valid since $u(.)$ satisfies the A_1-Muckenhoupt condition. Part A leads to state that for all p with $\frac{1}{2} < p \le 1$ the above boundedness also holds.

2) For $u(x) = |x|^{\frac{1}{2}}$, $v(x) = 1$ (i.e. $\beta = \frac{3}{2}$, $\gamma = 1$), then by Part A, $M : L^2_v(\mathcal{R}.D.) \to L^4_u$ (i.e. $p = 2 < q = 4$).

3) Part B says that for power weight functions the boundedness $M : L^p_v(\mathcal{R}.D.) \to L^q_u$ is false when $q < p$. However nontrivial weights for which this boundedness holds can be obtained by modifying power weight functions as it is stated in Part C. Take for instance $v_1(x) = 1$, $u_1(x) = |x|^2 1\!\!\mathrm{I}_{|x|\le 1}(x) + |x|^{-\frac{3}{4}} 1\!\!\mathrm{I}_{|x|>1}(x)$, (i.e. $\beta = 3$, $\gamma = 1$, $\delta = \frac{1}{4}$); then $M : L^2_{v_1}(\mathcal{R}.D.) \to L^{\frac{3}{2}}_{u_1}$. Also for $v_2(x) = 1$, $u_2(x) = 1\!\!\mathrm{I}_{|x|\le 1}(x) + |x|^{-\frac{7}{8}} 1\!\!\mathrm{I}_{|x|>1}(x)$, (i.e. $\beta = 1$, $\gamma = 1$, $\delta = \frac{1}{8}$); then $M : L^2_{v_2}(\mathcal{R}.D.) \to L^{\frac{1}{2}}_{u_2}$.

4) While this paper has been submitted, the author [5] has got a characterization of inequality (1.1) for general functions $f(.)$ and $0 < p = q \le 1$. It appears from this last work [5] that many usual weights must be excluded contrary to the situation for $M : L^p_v(\mathcal{R}.D.) \to L^q_u$ studied above.

2 – Proof of results

Proof of The Theorem. The first key for the proof is

Proposition 1. *Let $0 < p, q < \infty$. The following statements are equivalent:*

1) *the boundedness $M : L^p_v(\mathcal{R}.D.) \to L^q_u$ holds;*

2) *for some constant $C > 0$ and for all $\psi(.) \geq 0$ with $\psi(.) \searrow$*

$$\Big(\int_0^\infty (H\psi)^q(t)\overline{u}(t)dt\Big)^{\frac{1}{q}} \leq C\Big(\int_0^\infty \psi^p(t)\overline{v}(t)dt\Big)^{\frac{1}{p}}. \tag{2.1}$$

Here $(H\psi)(t) = \dfrac{1}{t}\displaystyle\int_0^t \psi(s)ds$, $\overline{v}(t)$ and $\overline{u}(t)$ are weight functions on $]0,\infty[$ defined by

$$\overline{v}(t) = t^{\frac{1}{n}[1-n]}\widetilde{v}(t^{\frac{1}{n}}), \quad \widetilde{v}(r) = r^{n-1}\int_{S_{n-1}} v(r\omega)d\omega$$

$$\overline{u}(t) = t^{\frac{1}{n}[1-n]}\widetilde{u}(t^{\frac{1}{n}}), \quad \widetilde{u}(r) = r^{n-1}\int_{S_{n-1}} u(r\omega)d\omega\,;$$

and $d\omega$ is the area-measure on the unit sphere S_{n-1} of $\mathbb{R}^{n-1}$.

The second key for the proof is a result about inequality (2.1).

Proposition 2. *Let $0 < p, q < \infty$.*

A) *For $p \leq 1$ and $p \leq q$, the Hardy inequality (2.1) holds if and only if for some constant $C > 0$ the following two inequalities hold for all $R > 0$:*

$$\Big(\int_0^R \overline{u}(t)dt\Big)^{\frac{1}{q}} \leq C\Big(\int_0^R \overline{v}(t)dt\Big)^{\frac{1}{p}}, \tag{2.2}$$

$$R\Big(\int_R^\infty t^{-q}\overline{u}(t)dt\Big)^{\frac{1}{q}} \leq C\Big(\int_0^R \overline{v}(t)dt\Big)^{\frac{1}{p}}. \tag{2.3}$$

B) *For $1 < p \leq q$ and $p' = \frac{p}{p-1}$, inequality (2.1) holds if and only if both (2.2) and*

$$\Big(\int_R^\infty t^{-q}\overline{u}(t)dt\Big)^{\frac{1}{q}}\Big(\int_0^R\Big[\int_0^t \overline{v}(s)ds\Big]^{-p'} t^{p'}\overline{v}(t)dt\Big)^{\frac{1}{p'}} \leq C \tag{2.4}$$

are satisfied for all $R > 0$.

C) *For $q < p$ with $q \neq 1$, $\frac{1}{\theta} = \frac{1}{q} - \frac{1}{p}$ and $q' = \frac{q}{q-1}$, inequality* (2.1) *holds if and only if the following two inequalities hold:*

$$\int_0^\infty \Big[\Big(\int_0^r \overline{u}(t)dt\Big)^{\frac{1}{p}}\Big(\int_0^r \overline{v}(t)dt\Big)^{-\frac{1}{p}}\Big]^\theta \overline{u}(r)dr < \infty \tag{2.5}$$

$$\int_0^\infty \Big[\Big(\int_r^\infty t^{-q}\overline{u}(t)dt\Big)^{\frac{1}{q}}\Big(\int_0^r \Big[\int_0^t \overline{v}(s)ds\Big]^{-p'} t^{p'}\overline{v}(t)dt\Big)^{\frac{1}{q'}}\Big]^\theta \times \Big[\int_0^r \overline{v}(s)ds\Big]^{-p'} r^{np'}\overline{v}(r)dr \ < \infty. \tag{2.6}$$

Part B and also Part C with $1 < q < p$ were first due to SAWYER ([7] Theorem 2, p.148). Part C was proved by STEPANOV ([8] Theorem 3, p. 175). Part A with $p \neq 1$ is also contained in this last paper. The full case of Part A, was found by HEINIG and MALIGRANDA ([3] Corollary 3.5, p. 150).

Assuming for the moment the validity of Proposition 1, the theorem follows once we prove that conditions (2.2), (2.3), (2.4), (2.5) and (2.6) are equivalent to (1.2), (1.3), (1.4), (1.5) and (1.6) respectively. The conclusion is just based on the following computations:

$$\int_0^R \overline{u}(t)dt \approx \int_{|x|<R^{\frac{1}{n}}} u(x)dx, \quad \int_0^R \overline{v}(t)dt \approx \int_{|x|<R^{\frac{1}{n}}} v(x)dx,$$
$$\int_R^\infty t^{-q}\overline{u}(t)dt \approx \int_{R^{\frac{1}{n}}<|x|} |x|^{-nq}u(x)dx,$$
$$\int_0^R \Big[\int_0^t \overline{v}(s)ds\Big]^{-p'} t^{p'}\overline{v}(t)dt \approx \int_{|x|<R^{\frac{1}{n}}} \Big[\int_{|y|<|x|} v(y)dy\Big]^{-p'} |x|^{np'}v(x)dx.$$

PROOF OF PROPOSITION 1. In fact we will prove that the boundedness $M : L^p_v(\mathcal{R}.D.) \to L^q_u$ holds if and only if for all $f(x) = \varphi(|x|) \geq 0$ with $\varphi(.) \searrow$, the following is true:

$$\Big(\int_{\mathbb{R}^n} \Big[\int_{|y|<|x|} f(y)dy\Big]^q |x|^{-nq}u(x)dx\Big)^{\frac{1}{q}} \leq C\Big(\int_{\mathbb{R}^n} f^p(x)v(x)dx\Big)^{\frac{1}{p}}. \tag{2.7}$$

And this last inequality is also equivalent to

$$\Big(\int_0^\infty [H(\varphi\circ\Phi)]^q(t)\overline{u}(t)dt\Big)^{\frac{1}{q}} \leq C\Big(\int_0^\infty (\varphi\circ\Phi)^p(t)\overline{v}(t)dt\Big)^{\frac{1}{p}} \tag{2.8}$$

for all $\varphi(.) \searrow$, where $\Phi(t) = t^{\frac{1}{n}}$. So the conclusion will follow by the immediate equivalence between (2.1) and (2.8).

The equivalence between (2.8) and (2.7) holds because

$$\int_{\mathbb{R}^n} f^p(x)v(x)dx \approx \int_0^\infty (\varphi \circ \Phi)^p(t)\overline{v}(t)dt$$

$$\int_{\mathbb{R}^n} \Big[\int_{|y|<|x|} f(y)dy\Big]^q |x|^{-nq}u(x)dx \approx \int_0^\infty [H(\varphi \circ \Phi)]^q(t)\overline{u}(t^{\frac{1}{n}})dt.$$

Finally it remains to see the equivalence between inequality (2.7) and $M : L^p_v(\mathcal{R}.D.) \to L^q_u$. Since $|x|^{-n}\int_{|y|<|x|} f(y)dy \le c(Mf)(x)$ for all $f(.) \ge 0$, then clearly the boundedness implies (2.7). For the converse it is sufficient to find a fixed constant $C > 0$ such that

$$(Mf)(x) \le C|x|^{-n}\int_{|y|<|x|} \varphi(|y|)dy \qquad (2.9)$$
$$\text{for all } f(x) = \varphi(|x|) \ge 0 \text{ with } \varphi(.) \searrow.$$

We have just to bound the quantities $\mathcal{I}(x,r) = r^{-n}\int_{|x-y|<r} \varphi(|y|)dy$, $r > 0$, by the right member of (2.9). First consider $|x| \le 2r$. Then

$$\begin{aligned}\mathcal{I}(x,r) &\le c_1|x|^{-n}\int_{|y|<|x|} \varphi(|y|)dy + c_2\varphi(|x|) \;\text{ since } |x| \le 2r \text{ and } \varphi(.) \searrow\\ &\le c_1|x|^{-n}\int_{|y|<|x|} \varphi(|y|)dy + c_3|x|^{-n}\int_{\frac{1}{2}|x|<|y|<|x|} \varphi(|x|)dy\\ &\le C|x|^{-n}\int_{|y|<|x|} \varphi(|y|)dy.\end{aligned}$$

Next take $|x| > 2r$. Then

$$\begin{aligned}\mathcal{I}(x,r) &\le r^{-n}\int_{|x-y|<r;\,\frac{1}{2}|x|<|y|<|x|} \varphi(|y|)dy \le c_2\varphi\Big(\frac{1}{2}|x|\Big)\\ &\le C|x|^{-n}\int_{\frac{1}{4}|x|<|y|<\frac{1}{2}|x|} \varphi\Big(\frac{1}{2}|x|\Big)dy \le C|x|^{-n}\int_{|y|<|x|} \varphi(|y|)dy.\end{aligned}$$

Proof of the Corollary. Since $0 < \beta < nq$, then

$$\int_{|x|<R} u(x)dx \approx R^\beta \quad \text{for all } R > 0 \qquad (2.10)$$

$$\int_{R<|x|} |x|^{-nq}u(x)dx \approx R^{\beta-nq} \quad \text{for all } R > 0. \qquad (2.11)$$

The hypothesis $0 < \gamma < np$ implies for all $R > 0$

$$\int_{|x|<R} v(x)dx \approx R^{\gamma} \tag{2.12}$$

$$\int_{|x|<R} \Big[\int_{|y|<|x|} v(y)dy\Big]^{-p'} |x|^{np'} v(x)dx \approx R^{p'[n-\frac{\gamma}{p}]} . \tag{2.13}$$

By (2.10) and (2.12), condition (1.2) is satisfied if and only if $\frac{\beta}{q} = \frac{\gamma}{p}$. And by (2.11) and (2.12), condition (1.3) is equivalent to this last equality. It is also the case for (1.4) in view of (2.11) and (2.13).

For $q < p$ and $\frac{\beta}{q} = \frac{\gamma}{p}$, condition (1.5) is false since the integral in (1.5) becomes equivalent to $\int_0^\infty r^{[\beta\frac{\theta}{p}-\gamma\frac{\theta}{p}+\beta]} r^{-1}dr = \int_0^\infty r^{-1}dr = \infty$. Consequently the boundedness $M : L^p_v(\mathcal{R}.D.) \to L^q_u$ does not hold.

As in (2.10), then

$$\int_{|x|<r} w(x)dx \approx r^{\beta} \qquad \text{for } 0 < r \le 1 \tag{2.14}$$

$$\int_{|x|<r} w(x)dx = \int_{|x|\le 1} |x|^{\beta-1}dx + \int_{1<|x|<r} |x|^{\delta-1}dx \le cr^{\delta} \text{ for } r > 1. \tag{2.15}$$

Calling $\mathcal{I}$ the integral in (1.5) [with $u(.) = w(.)$] and using (2.14), (2.15) and (2.12) then

$$\mathcal{I} \le c_1 \int_0^1 r^{\beta\frac{\theta}{p}-\gamma\frac{\theta}{p}} \times r^{\beta-1}dr + c_1 \int_1^\infty r^{\delta\frac{\theta}{p}-\gamma\frac{\theta}{p}} \times r^{\delta-1}dr =$$
$$= c_1 \int_0^1 r^{\theta[\frac{\beta}{q}-\frac{\gamma}{p}]} \times r^{-1}dr + c_1 \int_1^\infty r^{\theta[\frac{\delta}{q}-\frac{\gamma}{p}]} \times r^{-1}dr.$$

This inequality leads to the finitness of $\mathcal{I}$ since $\frac{\beta}{q} - \frac{\gamma}{p} > 0$ and $\frac{\delta}{q} - \frac{\gamma}{p} < 0$. So condition (1.5) is satisfied. As in (2.11), for $\delta - nq < 0$

$$\int_{r<|y|} |y|^{-nq} w(y)dy \approx r^{\delta-nq} \quad \text{for all } r > 1. \tag{2.16}$$

And $\delta - nq < 0 < \beta - nq$ implies, for $0 < r \le 1$:

$$\int_{r<|y|} |y|^{-nq} w(y)dy = \int_{r<|y|\le 1} |y|^{-nq+\beta}dy + \int_{1<|y|} |y|^{-nq+\delta}dy \le c_2 . \tag{2.17}$$

If $\mathcal{J}$ denotes the integral in (1.6) [with $u(.) = w(.)$] then $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2$, where $\mathcal{J}_1$ is the integral corresponding to $|x| \leq 1$ and $\mathcal{J}_2$ for $|x| > 1$. Using (2.17), (2.12) and (2.13) then

$$\mathcal{J}_1 \leq c_3 \int_0^1 r^{p' \frac{\theta}{q'}(n - \frac{\gamma}{p})} \times r^{-\gamma p' + np'} \times r^{\gamma - 1} dr = \\ = c_3 \int_0^1 r^{\theta(n - \frac{\gamma}{p})} \times r^{-1} dr < \infty\,, \quad \text{since } \Big(n - \frac{\gamma}{p}\Big) > 0.$$

From (2.16), (2.12) and (2.13) then

$$\mathcal{J}_2 \leq c_4 \int_1^\infty r^{\frac{\theta}{q}(\delta - nq)} \times r^{p' \frac{\theta}{q'}(n - \frac{\gamma}{p})} \times r^{-\gamma p' + np'} \times r^{\gamma - 1} dr = \\ = c_4 \int_1^\infty r^{\theta[\frac{\delta}{q} - \frac{\gamma}{p}]} \times r^{-1} dr < \infty\,, \quad \text{since } \Big(\frac{\delta}{q} - \frac{\gamma}{p}\Big) < 0.$$

These computations show that $\mathcal{J} < \infty$. So the condition (1.6) is satisfied and consequently $M : L^p_v(\mathcal{R}.D.) \to L^q_w$.

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