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The water bell: An attempt to derive it

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RIASSUNTO: *Il modello di "campana d'acqua", ottenuto da* BOUSSINESQ *in* [6]*, e risolto da* TAYLOR *in* [19]*, è un getto cavo di fluido incomprimibile, non viscoso, nel limite di spessore nullo dello strato di fluido. Discutiamo, attraverso una formulazione variazionale, come si possa ottenere questo modello come limite di soluzioni stazionarie dell'equazione di Eulero con tensione superficiale sui bordi liberi.*

ABSTRACT: *We consider a cave jet of an incompressible non viscous fluid, in the limit of vanishing thickness: the "water bell" model, as obtained by* BOUSSINESQ [6] *and solved by* TAYLOR [19]*. We discuss how to derive this model as a limit of stationary solutions of Euler equation with surface tension on the free boundaries, via a variational principle.*

1 – Introduction

A water bell, *i.e.* a bell like shaped axially symmetric thin cave jet of water, can be obtained by placing a disk shaped obstruction in the path of a vertical cylindrical jet of water, as asserted by SAVART in 1833 [18]. BOUSSINESQ in 1869 [6] gave the mathematical description of it. In 1959 TAYLOR [19] solved the model in the case the effects of the gravity can be neglected, and compared the solution with experimental results. The case

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of a rotating cave jet has been experimentally studied by BARK *et al.* [4], and analytically solved, without gravity, by GASSER and MARTY [9].

The current interest on this subject is in the applied Fluid Mechanics. For example the nebulization of a fluid can be obtained by disintegration of a liquid sheet (see TAYLOR [19], YARIN [20]). Thin sheets of fluid appear also in the splashing of drops on solid surface; this phenomenon appears in a wide variety of applications. For a complete survey on the theory of water sheets and its applications see YARIN [20] and references therein.

In this paper we approach the problem of deriving the water bell model from the Euler equation for an incompressible, non viscous fluid. More precisely we consider an axially symmetric cave jet, with surface tension on the free boundaries, and without external forces. Our strategy is the following. The water bell model is equivalent to a variational principle. The stationary solution of the Euler equations may be obtained formally as an extremum point of a suitable functional. In the limit of vanishing thickness, this functional converges formally to the functional describing the water bell. Our aim would be to find extremal points of the functional for the Euler equations which converge, is some sense, to the solution of water bell model. However, in this paper we are only able to perform this program for a simplified version of the functional for the Euler equation. In this case the problem turns out to be a relaxation problem for a system of ODE.

In Section 2 we give a heuristic derivation of the water bell model. In Section 3 we describe a variational approach for the stationary solution for the Euler equation (which is a jet-cavity problem with surface tension) and we discuss the difficulties that arise in achieving the full program. In Section 4 we introduce a two surface approximation of it and prove (Section 5) that, in the thin limit, it approaches to the model stated in Section 2.

We remark that the problem of finding stationary solutions for the Euler equation is, in this case, a free boundary problem. Many other situation in Fluid Mechanics gives a free-boundary problem, and many approaches are possible. For viscous fluids in an open channel of various geometry see *e.g.* [13] and references therein. The formalism of variational inequalities is used for the filtration problems (see [3] and [8]). The variational approach also describes, in the nonviscous case without surface tension, jets, cavities, and fluids with potentials [8]). Also the case of a fluid at rest with surface tension have a variational formulation: it is the well known problem of minimal surfaces ([17], [11]; see also [7], [12], [1]).

2 – The water sheet limit

We consider an inviscid incompressible fluid in a region Λ on the half-space $\{z > 0\}$ of $\mathbb{R}^3$, axially symmetric with respect to $z-$axis. We indicate with Σ_1, Σ_2 the internal and external boundary of Λ respectively.

The velocity field $\mathbf{u}$ and the pressure P satisfy the stationary Euler equations in the region Λ (see *e.g.* [15], [5],[14]):

$$\begin{aligned} \mathbf{u} \cdot \nabla \mathbf{u} &= -\frac{\nabla P}{\mu}\,, \\ \nabla \cdot \mathbf{u} &= 0\,, \end{aligned} \tag{2.1}$$

with the following conditions on the free-boundaries Σ_i, $i = 1, 2$:

$$\begin{aligned} \mathbf{u}|_{\Sigma_i} \cdot \mathbf{n} &= 0 \\ P\,|_{\Sigma_i} &= P_0 + T\mathcal{H}_i \end{aligned} \tag{2.2}$$

where the constant μ is the density of the fluid, P_0 is the atmospheric pressure, T is the surface tension and $\mathcal{H}_i$ is the mean curvature of the surface Σ_i with respect to $\mathbf{n}$, the outer normal to Λ.

We suppose that the distance between Σ_2 and Σ_1 is small, so that we can represent Σ_2 in terms of $\Sigma_1 = \Sigma$, given by the rotation around the $z - axis$ of a positive function $R(z)$, and of the distance $\lambda(z)$ between Σ_2 and Σ:

$$\begin{aligned} \Sigma &= \{(R(z)\,\cos\,\theta, R(z)\,\sin\,\theta,\, z) :\, 0 \le z \le L,\, 0 \le \theta < 2\pi\}\,, \\ \Sigma_2 &= \{\mathbf{x} + \lambda \mathbf{n}(\mathbf{x}) :\, \mathbf{x} \in \Sigma\}\,, \end{aligned} \tag{2.3}$$

where $\mathbf{n}(\mathbf{x})$ is the outer normal to Σ in $\mathbf{x}$.

In order to describe the water sheet limit, we make the ansatz that the pressure varies linearly between Σ and Σ_2 along $\mathbf{n}$:

$$\nabla P = \frac{2T\mathcal{H}}{\lambda}\mathbf{n} + O(\lambda^0)\,; \tag{2.4}$$

where $\mathcal{H}$ is the mean curvature of Σ.

Let us consider the Lagrangian picture: we consider a particle $\mathbf{x} \in \mathbb{R}^3$ on Σ. The Newton equation reads:

$$\ddot{\mathbf{x}} = -\frac{\nabla P}{\mu}\,. \tag{2.5}$$

In this approximation the gradient of the pressure is orthogonal to the surface Σ. Therefore

$$|\dot{\mathbf{x}}| = v = \text{constant} \tag{2.6}$$

$$\ddot{\mathbf{x}} = -\frac{v^2}{r_c}\mathbf{n} = -\frac{2T\mathcal{H}}{\lambda\mu}\mathbf{n} \tag{2.7}$$

where r_c is the radius of curvature of the trajectory with respect to $\mathbf{n}$.

Let us write $\dot{\mathbf{x}}$ as the sum of its component v_2 on the tangent τ to the curve $(z,\, R(z))$ and v_1 on the orthogonal direction to τ on the tangent plane to the surface Σ. From the Euler formula for the curvature of the normal sections

$$\mathcal{H} = \frac{1}{R_1} + \frac{1}{R_2}, \tag{2.8}$$

where R_1 and R_2 are the principal radius of curvature, we obtain that:

$$\frac{1}{r_c} = \frac{1}{v^2}\Big(\frac{v_1^2}{R_1} + \frac{v_2^2}{R_2}\Big)\,. \tag{2.9}$$

In order to close equations (2.7), (2.9) we need to know v_1 and λ. From the cylindrical symmetry, it follows that the $z-$component of the momentum is conserved, *i.e.*

$$v_1 R = \beta = \text{constant}\,, \tag{2.10}$$

furthermore the rate of flow on the sections of Λ is constant:

$$v_2 \lambda R = \gamma = \text{constant}\,. \tag{2.11}$$

Summarizing:

$$(2.12)_1 \qquad \Big(\frac{v_1^2}{R_1}+\frac{v_2^2}{R_2}\Big)=\frac{2TR}{\gamma\mu}\,v_2\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)\,,$$

$$(2.12)_2 \qquad v_1^2=\frac{\beta^2}{R^2}\,,$$

$$(2.12)_3 \qquad v_2^2=v^2-v_1^2\,,$$

$$(2.12)_4 \qquad \frac{1}{R_1}=\frac{1}{R(1+R'^2)^{\frac{1}{2}}}\,,$$

$$(2.12)_5 \qquad \frac{1}{R_2}=-\frac{R''}{(1+R'^2)^{\frac{3}{2}}}\,;$$

where $' = \frac{d}{dz}$.

In the system (2.12) μ, T are physical parameters of the fluid, while β, γ, v are dynamical parameter.

System (2.12) admits a first integral:

$$(2.13) \qquad I=\frac{1+R'^2}{\Big(\dfrac{2T}{\gamma\mu v}R-\Big(1-\dfrac{\beta^2}{v^2R^2}\Big)^{\frac{1}{2}}\Big)^2}\,,$$

which allows us to find the solutions of (2.12), and to draw the shape of Σ.

In the case $\beta=0$ the level sets of I are hyperbola and two regimes are possible. In the first, $R(0)=R_0<\frac{\gamma\mu v}{2T}$, the jet collapses on the axes $R=0$. If $2R_{\max}$ is the maximum diameter of the surface, the height h of the shape (that is the distance along the $z-axes$ between the point of collapse and the point of maximum diameter) is given by

$$(2.14) \qquad h=\frac{1}{\tau}\,(1-\tau R_{\max})\,\mathrm{arccosh}(1-\tau R_{\max})\,;$$

where

$$(2.15) \qquad \tau=\frac{2T}{\gamma\mu v}\,.$$

In the other case, $R_0>\frac{\gamma\mu v}{2T}$, the trajectories are unbounded.

The case $\beta \neq 0$ is a bit more complicate. In particular by varying β it is possible to individuate different regimes. Here we consider the case $|\beta| < \frac{\gamma\mu v^2}{4T}$, that seems to be the most relevant for the applications (see [6], [19],[20]). In this case the level sets of I show three different regimes separated by $R = R_- = (\frac{1-(1-\frac{4\tau^2\beta^2}{v^2})^{\frac{1}{2}}}{2\tau^2})^{\frac{1}{2}}$, $R = R_+ = (\frac{1+(1-\frac{4\tau^2\beta^2}{v^2})^{\frac{1}{2}}}{2\tau^2})^{\frac{1}{2}}$.

The first, $R < R_-$, and the third, $R > R_+$ correspond qualitatively to the regimes previously described for the case $\beta = 0$. Nevertheless, in this case the conservation of the angular momentum implies that R cannot reach the $z - axis$, and the trajectories for $R < R_-$ collapse at $R = \frac{|\beta|}{v}$.

In the intermediate case, $R_- < R < R_+$, the orbits are periodic; this corresponds, in the real space, to z-periodic surfaces; in particular it exists R_C such that for $R(z) \equiv R_C$ the solution is a cylinder.

An extensive discussion of the case $\beta \neq 0$ can be found in [9].

3 – The variational approach

After having considered, from a heuristic point of view the water sheet limit, we now consider the original problem (2.1-2) in a variational formalism.

Given $R_i : [0, h] \to \mathbb{R}^+$; $i = 1, 2$, let $\Lambda = \Lambda_{R_1,R_2}$ be

$$\Lambda \equiv \{(\rho \cos \theta, \rho \sin \theta, z) : 0 < R_1(z) \leq \rho \leq R_2(z), \\ 0 \leq \theta \leq 2\pi, 0 \leq z \leq h\}, \tag{3.1}$$

and let be Σ_1, Σ_2 defined as

$$\Sigma_i \equiv \{(R_i(z) \cos \theta, R_i(z) \sin \theta, z) : \\ 0 \leq \theta \leq 2\pi,\ 0 \leq z \leq h\} \qquad i = 1, 2\,. \tag{3.2}$$

In order to solve the stationary Euler equation (2.1) in Λ, we remark that

$$\mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{u} \times (\nabla \times \mathbf{u}) + \nabla \frac{\mathbf{u}^2}{2}\,, \tag{3.3}$$

then if $\mathbf{u}$ is an irrotational ($\nabla \times \mathbf{u} = \mathbf{0}$) divergence free field, $\mathbf{u}$ solves the Euler equation, identifying the pressure P in equation (2.1) with $-\mu\frac{\mathbf{u}^2}{2}$ up to a constant.

We can represent a rotating invariant divergence free field $\mathbf{v}$ without swirl, in the plane ρ, z as

$$\mathbf{v} \equiv \frac{1}{\rho}\Big(- \frac{\partial \Phi}{\partial z}, \frac{\partial \Phi}{\partial \rho}\Big) . \tag{3.4}$$

The irrotational condition for $\mathbf{v}$ implies Φ solves the elliptic equation

$$\mathcal{L}\Phi \equiv \rho \nabla \cdot \Big(\frac{\nabla \Phi}{\rho}\Big) = \Big(\frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{\rho}\frac{\partial}{\partial \rho}\Big)\Phi = 0 . \tag{3.5}$$

On the free boundaries we impose:

$$\begin{aligned} \Phi &= 0 \text{ on } \Sigma_1, \\ \Phi &= \Gamma \text{ on } \Sigma_2, \end{aligned} \tag{3.6}$$

which imply that the normal component of $\mathbf{v}$ to the surfaces $\Sigma_{1,2}$ is 0, and that the flux of $\mathbf{v}$ on the sections of Λ is Γ.

We impose also that $\Phi(z, \rho)$ is given at the fixed boundaries $z = 0$ and $z = h$, which correspond to give the z component of the velocity field (also Neumann boundary conditions can be given, which correspond to give the ρ component of $\mathbf{v}$).

The corresponding kinetic energy is given by

$$E[R_1, R_2] \equiv \frac{\mu}{2}\int_\Lambda |\mathbf{v}|^2 = \frac{\mu}{2}\int_\Lambda \frac{|\nabla \Phi|^2}{\rho^2} . \tag{3.7}$$

We can consider a velocity field $\mathbf{u}$ given by the sum of $\mathbf{v}$ and a term of rotation around the $z-$axis, of intensity $\frac{B}{\rho}$. Such a field is a divergence free field, and it is irrotational.

In this way we construct a solution of equation (2.1) satisfying only the first boundary condition of (2.2) for $\mathbf{u}$. Now we have to find $R_{1,2}(z)$ such that also the second condition is satisfied.

Let us define the following functional on $(R_1(\,\cdot\,), R_2(\,\cdot\,))$, with $R_i(0)$ and $R_i(h)$ fixed:

$$F[R_1, R_2] \equiv E[R_1, R_2] - \frac{\mu}{2}\int_\Lambda \frac{B^2}{\rho^2} + P|\Lambda| - T(|\Sigma_1| + |\Sigma_2|) ; \tag{3.8}$$

where P (which will be identified with the atmospheric pressure) is the Lagrange multiplier of the volume $|\Lambda|$, T is the surface tension, and $|\Sigma_i|$ is the area of the surface Σ_i. The kinetic term $\frac{B^2}{\rho^2}$ appear with the minus sign since it plays the role of an effective potential.

The definition of the functional $F[R_1, R_2]$ is motived by the following theorem:

THEOREM 3.1. *If $(R_1(\cdot), R_2(\cdot))$, with $0 < R_1(\cdot) < R_2(\cdot)$ is an extremum point for F, in the space of the function $\mathbf{C}^{2,\alpha}([0,h])$, with $\alpha > 0$, and $R_{1,2}(0)$ and $R_{1,2}(h)$ fixed, then the corresponding velocity field $\mathbf{u}$ is a solution of problem* (2.1), (2.2).

PROOF. We have only to verify the condition (2.2) for the pressure on the free boundaries. First we consider the variation on R_2: let $R_2^\varepsilon(z) = R_2(z) + \varepsilon\delta R(z)$, and let Λ^ε, Σ_2^ε, Φ^ε be the corresponding region occupied by the fluid, the external surface, and the solution of elliptic equation (3.5) respectively.

By explicit calculations

$$\begin{aligned}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Big(\int_{\Lambda^\varepsilon}\Big(-\frac{\mu}{2}\frac{B^2}{\rho^2}+P\Big)-T(|\Sigma_1|+|\Sigma_2|)\Big)=\\ =2\pi\int_0^h dz\Big(-\frac{\mu}{2}\frac{B^2}{R_2(z)^2}+P-T\,\mathcal{H}_2\Big)\,R_2(z)\,\delta R(z)\,,\end{aligned} \tag{3.9}$$

where $\mathcal{H}_2$ is the mean curvature of the surface Σ_2.

To perform the variation on the kinetic term $E[R_1, R_2 + \varepsilon\delta R]$, let us introduce the function $\tilde{\Phi}^\varepsilon$:

$$\tilde{\Phi}^\varepsilon(z,\rho) = \Phi(z, \rho_\varepsilon(z,\rho))\,, \tag{3.10}$$

where Φ is the solution of problem (3.5) in the inperturbed domain Λ, and

$$\rho - R_1(z) = (\rho_\varepsilon(z,\rho) - R_1(z))\frac{R_2(z)-R_1(z)+\varepsilon\delta R(z)}{R_2(z)-R_1(z)}\,. \tag{3.11}$$

The function $\tilde{\Phi}^\varepsilon$ is a regular function in Λ_ε and solves $\mathcal{L}\tilde{\Phi}^\varepsilon + \varepsilon g(z,\rho,\varepsilon) = 0$, where g depends on the seconds derivatives of Φ, R_i, δR. Then g is uniformly bounded in $\mathbf{C}^{\alpha}(\overline{\Lambda}_{\varepsilon})$. The function Φ^{ε} minimize the kinetic energy in Λ_{ε} and solves $\mathcal{L}\Phi^{\varepsilon} = 0$, then, using the Poincaré inequality:

$$0 \leq \int_{\Lambda_{\varepsilon}} \frac{|\nabla\tilde{\Phi}^{\varepsilon}|^2}{2\rho^2} - \int_{\Lambda_{\varepsilon}} \frac{|\nabla\Phi^{\varepsilon}|^2}{2\rho^2} = \int_{\Lambda_{\varepsilon}} \frac{|\nabla(\Phi^{\varepsilon} - \tilde{\Phi}^{\varepsilon})|^2}{2\rho^2} \leq c\varepsilon^2 \,. \tag{3.12}$$

Then

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{\mu}{2} \int_{\Lambda_{\varepsilon}} \frac{|\nabla\Phi^{\varepsilon}|^2}{\rho^2} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{\mu}{2} \int_{\Lambda_{\varepsilon}} \frac{|\nabla\tilde{\Phi}^{\varepsilon}|^2}{\rho^2} \,. \tag{3.13}$$

Performing this derivative we obtain:

$$\begin{aligned} \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{\mu}{2} \int_{\Lambda_{\varepsilon}} \frac{|\nabla\tilde{\Phi}^{\varepsilon}|^2}{\rho^2} &= \pi\mu \int_0^h dz \frac{|\nabla\Phi|^2(z, R_2(z))}{R_2(z)} \delta R(z) + \\ &\quad + \mu \int_{\Lambda} \frac{\nabla\Phi}{\rho^2} \cdot \nabla\Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \tilde{\Phi}^{\varepsilon}\Big) \end{aligned} \tag{3.14}$$

Being

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \tilde{\Phi}^{\varepsilon} = -\partial_{\rho}\Phi \, \delta R \frac{\rho - R_1}{R_2 - R_1} \,, \tag{3.15}$$

integrating by part, and taking into account that $\mathcal{L}\Phi = 0$ and that $\delta R(0) = \delta R(h) = 0$, the second term of equation (3.14) is

$$-2\pi\mu \int_0^h dz \sqrt{1 + R_2'(z)^2} \frac{\nabla\Phi(z, R_2(z)) \cdot \mathbf{n}}{R_2(z)} \partial_{\rho}\Phi(z, R_2(z)) \delta R(z) \,, \tag{3.16}$$

where $\mathbf{n}$ is the external normal to Σ_2 on the plane (z, ρ).
Noting that Φ is constant on Σ_2, and that $|\nabla\Phi|^2(z, R_2(z)) = (\partial_{\rho}\Phi(z, R_2(z)))^2$ $(1 + R_2'(z)^2)$, from (3.14), (3.16) we obtain

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{\mu}{2} \int_{\Lambda_{\varepsilon}} \frac{|\nabla\tilde{\Phi}^{\varepsilon}|^2}{\rho^2} = -\pi\mu \int_0^h dz \frac{|\nabla\Phi|^2(z, R_2(z))}{R_2(z)} \delta R(z) \,. \tag{3.17}$$

Finally, collecting (3.9) and (3.17), and noting that

$$|\mathbf{u}|^2 = \frac{|\nabla\Phi|^2}{\rho^2} + \frac{B^2}{\rho^2} \,, \tag{3.18}$$

we have

$$\mu \frac{|\mathbf{u}(z,\, R_2(z))|^2}{2} + P - T\,\mathcal{H}_2 = 0\,. \tag{3.19}$$

For the variation of R_1 we proceed in the same way, obtaining

$$\mu \frac{|\mathbf{u}(z,\, R_1(z))|^2}{2} - P - T\,\mathcal{H}_1 = 0\,. \tag{3.20}$$

From equation (3.3), we can identify, up to a constant, the term $\frac{-\mu|\mathbf{u}|^2}{2}$ with the pressure in the fluid. Then equations (3.19−20) are the boundary condition (2.2) for the pressure. ∎

Remark 1. We have fixed the value of R_i at the boundary $z = 0$ and $z = h$, and also the value of the $z-$component of the velocity field. In this way we only interpret an irrotational steady solution of Euler equation with surface tension on the free boundaries as extremum point of a functional, and nothing we can say for the problem of finding how the fluid leave the region $z < 0$, where we can assume there are fixed boundaries. This kind of problem for jets and cavities has been considered and solved in [8] in the case of $T = 0$.

Remark 2. The functional F is not bounded from below and from above. In fact, but for non essential terms, it is given by the difference of two unbounded from above functionals: the kinetic energy and the area of the surfaces. Therefore we cannot look, as usual in studying this kind of problems, for an absolute minimum of F (see [16], [2]). Furthermore it is easy to realize that this functional cannot have local minima in functional spaces that are not sufficiently regular. In particular it is possible to increase the area of the surfaces with a $\mathbf{H}^2$ perturbation of the boundary, maintaining the kinetic energy essentially constant. Nevertheless it is possible to find maxima of F, at least for some choice of parameters, for example the cylindrical solution and nearby solutions.

4 – The two surfaces model

In order to avoid the “ultraviolet” problem described above (see Remark 2, at the end of the previous section), we propose an approximate variational problem, for which we prove the existence of local minima.

Let us notice that the functional F is given by the kinetic energy E plus the integral, along z, of a function of R_1, $R_1{}'$, R_2, $R_2{}'$. So, neglecting the energy, it appears as an extremum action problem. Our proposal is to approximate the kinetic term E in such a way that this structure is preserved. A way to do it is the following: given R_1, R_2 let us define $\tilde{\Phi}$ as

$$\tilde{\Phi} \equiv \Gamma \frac{\rho^2 - R_1^2}{R_2^2 - R_1^2}\,, \tag{4.1}$$

and $\widetilde{E}[R_1, R_2]$ as $\frac{\mu}{2}\int_\Lambda \frac{|\nabla\tilde{\Phi}|^2}{\rho^2}$.

Let us notice that the above approximation is exact in the case in which Λ is a cylinder. In general, it gives an estimate from above of the true functional. Furthermore it seems reasonable that the approximation is better and better when the domain Λ is thiner and thiner. With this definitions the variational problem becomes that of finding an extremum for A, where

$$\begin{aligned} A[R_1, R_2] &\equiv \frac{1}{2\pi}\Big(\widetilde{E}[R_1, R_2] - \int_\Lambda \frac{B^2}{\rho^2} + P|\Lambda| - T(|\Sigma_1| + |\Sigma_2|)\Big) = \\ &= \int_0^h dz\, L(R_1, R_1', R_2, R_2')\,. \end{aligned} \tag{4.2}$$

In view of the applications we are interested in, it is convenient to introduce the Lagrangian variables (notice that L has the role of a Lagrangian in an extremum action problem)

$$\begin{aligned} R : R^2 &\equiv \frac{R_1^2 + R_2^2}{2}\,, \\ y : y^2 &\equiv R_2^2 - R_1^2\,. \end{aligned} \tag{4.3}$$

As we shall see in the sequel the thin limit is achieved by the following scaling

$$y = \varepsilon x, \quad P = \frac{p}{\varepsilon^2}, \quad B = \frac{\beta}{\varepsilon}, \quad \Gamma = \varepsilon\gamma, \quad T = T\,, \tag{4.4}$$

when $\varepsilon \to 0$.

It is convenient, to this aim, to introduce the scaled Lagrangian variable $x = \frac{y}{\varepsilon}$. With these choices and by computations, the Lagrangian becomes:

$$L = L_{\text{thin}} + L_1 \ : \tag{4.5}$$

where

$$L_{\text{thin}} = \mu\gamma^2 \frac{1+R'^2}{x^2} - \beta^2 \frac{x^2}{2R^2} + \frac{p}{2}x^2 - 2TR\sqrt{1+R'^2}\,, \tag{4.6}$$

and

$$L_1 = L_\gamma + L_\beta + L_T\,; \tag{4.7}$$

$$L_\gamma = \varepsilon^4 \mu\gamma^2 \frac{(xR' - Rx')^2}{R^4}\Big[\frac{\ln(1+\epsilon_1) + \ln(1-\epsilon_1) - 2\epsilon_1}{(2\epsilon_1)^3}\Big]\,,$$

$$L_\beta = -\varepsilon^4 \beta^2 \frac{x^6}{2R^6}\Big[\frac{\ln(1+\epsilon_1) + \ln(1-\epsilon_1) - 2\epsilon_1}{(2\epsilon_1)^3}\Big]\,, \tag{4.8}$$

$$L_T = TR\sqrt{1+R'^2}\Big[2 - \sqrt{1 + \frac{(\epsilon_2^2 - 2\epsilon_2)R'^2 - \epsilon_1}{1+R'^2}} + \\ - \sqrt{1 + \frac{(\epsilon_2^2 + 2\epsilon_2)R'^2 + \epsilon_1}{1+R'^2}}\Big]\,;$$

where ϵ_1 and ϵ_2 are defined as

$$\epsilon_1 = \frac{x^2}{2R^2}\varepsilon^2, \qquad \epsilon_2 = \frac{xx'}{RR'}\varepsilon^2\,. \tag{4.9}$$

Let us notice that, formally, $L_1 = O(\varepsilon^4)$; therefore we expect that as $\varepsilon \to 0$, the system is described by L_{thin}. In the sequel we show that as ε vanishes the extremum point of A converges in some sense to the solutions of the variational problem associated to L_{thin}. Before to do it we formulate precisely this variational problem and we show that it is equivalent to the thin model discussed in Sections 2. In particular it turns out to be an alternative (variational) formulation of the thin model.

For sake of simplicity we shall consider the case $\beta = 0$; the general case may be tackled in the same way.

THEOREM 4.1. *Let* $r = \frac{\mu\gamma}{2T}\sqrt{\frac{2P}{\mu}}$,

$$\widetilde{\mathcal{M}} = \{(R(\,\cdot\,),\, x(\,\cdot\,)) \in (\mathbf{C}^1[0,h] \times \mathbf{C}[0,h]) :\ R(0) = R_0,\\ R(h) = R_h,\ R(\,\cdot\,) < r\}\,,$$

and let h *be sufficiently small. Then, there exist a local minimum for*

$$A_{\text{thin}}[R,x] = \int_0^h dz\, L_{\text{thin}}(R(z), R'(z), x(z)) \tag{4.10}$$

in $\widetilde{\mathcal{M}}$. *It is exactly the solution of the thin model described in Section* 2.

PROOF. We first minimize on $x(\,\cdot\,)$, obtaining

$$x^2(z) = \gamma\Big(\frac{2\mu}{p}(1 + R'(z)^2)\Big)^{\frac{1}{2}}\,, \tag{4.11}$$

$$\tilde{A}_{\text{thin}}[R] \equiv \min_{x(\,\cdot\,)} A_{\text{thin}}[R,x] = 2T\int_0^h dz\,(r - R(z))\sqrt{1 + R'^2}\,. \tag{4.12}$$

The functional $\tilde{A}_{\text{thin}}$ is the area of the rotation surface $\{(\rho,z) : \rho = \alpha(z)\}$ for $\alpha(z) = r - R(z)$. Its local minimum, which exists and is analytic if h is sufficiently small (see *e.g.* [11]), is described by the first integral

$$\tilde{I} = \frac{r - R(z)}{\sqrt{1 + R'^2}}\,. \tag{4.13}$$

This quantity is exactly $\frac{1}{\sqrt{I}}$, where I, defined in (2.13), is the first integral of the thin model, provided that $v^2 = \frac{2p}{\mu}$. ∎

REMARK. The condition $R(z) < r$ in the definition of $\widetilde{M}$ select the collapsing solution of the equation of the water bell described in Section 2. In the case $R(z) > r$ the functional A_{thin} have local maximum which is the unbounded solution of the water bell equation.

The main result of this section is the following theorem, which is proven in the next section.

Theorem 4.2. *In the hypothesis of Theorem* 4.1, *for* ε *sufficiently small, there exist* $(R^\varepsilon(\,\cdot\,), x^\varepsilon(\,\cdot\,)) \in \widetilde{\mathcal{M}}$, *regular local minimum for the action* (4.2). *As* $\varepsilon \to 0$ *this minimum converges uniformly to the solution of the thin problem.*

5 – The relaxation result

In this section we prove Theorem 4.2, which is a relaxation result. In fact, in the Lagrangian L (see (4.5)) the terms containing x' are multiplied by ε^4, which vanishes in the limit. The proof is based on the fact that $A[R, x]$ is a convex functional of the first derivatives, for all small value of ε, in a neighborhood of the minimum of A_{thin}. We avoid the boundary layer problem in $z = 0$ and $z = h$ preparing the boundary conditions for $x(\,\cdot\,)$ according to the behavior of the limit (see definition (5.13) and Theorem 4.1).

Let us call $\overline{R}$ the minimum of the action $\tilde{A}_{\text{thin}}$ in (4.12) and $\bar{x}$ the corresponding thickness of Λ, via (4.11).

In the sequel, we need an estimate of the difference $\tilde{A}_{\text{thin}}[R]-\tilde{A}_{\text{thin}}[\overline{R}]$.

Lemma 5.1. *Let* $R(0) = \overline{R}(0)$, $R(h) = \overline{R}(h)$, $\delta = R - \overline{R}$, *and let*

$$\mathcal{N}(\delta) = \int_0^h dz\,(\sqrt{1 + \delta'(z)^2} - 1)\,. \tag{5.1}$$

If h *and* $\mathcal{N}(\delta)$ *are sufficiently small then*

$$\tilde{A}_{\text{thin}}[R] - \tilde{A}_{\text{thin}}[\overline{R}] \geq c\mathcal{N}(\delta)\,. \tag{5.2}$$

Proof. $\mathcal{N}(\delta)$ is not a norm, but, nevertheless, it satisfies the following estimates:

$$\mathcal{N}(\delta) \leq \min\Big(\int_0^h |\delta'|,\quad \int_0^h \frac{|\delta'|^2}{2},\quad \int_{|\delta'|\leq 1} \frac{|\delta'|^2}{2} + \int_{|\delta'|>1} |\delta'|\Big)\,, \tag{5.3}$$

and

$$\begin{aligned}\int_0^h |\sqrt{1+R'^2} - \sqrt{1+\overline{R'}^2}| \leq \int_0^h |\delta'| \leq \Big(h\int_{|\delta'|\leq 1} |\delta'|^2\Big)^{\frac{1}{2}} + \\ + \int_{|\delta'|>1} |\delta'| \leq c\sqrt{h}\sqrt{\mathcal{N}(\delta)} + c\mathcal{N}(\delta)\,.\end{aligned} \tag{5.4}$$

(5.3) follows from $\sqrt{1+b^2}-1 \le \min\left(|b|,\, \frac{b^2}{2}\right)$, (5.4) from $\sqrt{1+b^2}-1 \ge c\, b^2$ if $|b| \le 1$ and $\sqrt{1+b^2}-1 \ge c\,|b|$ if $|b|>1$.

By the Lagrange formula, taking into account that the first derivative of $\tilde{A}_{\mathrm{thin}}$ in $\overline{R}$ vanishes, it is

$$\begin{aligned}\tilde{A}_{\mathrm{thin}}[R]-\tilde{A}_{\mathrm{thin}}[\overline{R}] &= \int_0^1 d\xi \int_0^h dz(1-\xi)\frac{2\delta\delta'(\alpha'+\xi\delta')}{\sqrt{1+(\alpha'+\xi\delta')^2}}+\\ &\quad+\int_0^1 d\xi\int_0^h dz\,(1-\xi)\frac{(\alpha+\xi\delta)\,\delta'^2}{(1+(\alpha'+\xi\delta')^2)^{\frac{3}{2}}}\,,\end{aligned}\tag{5.5}$$

where $\alpha(z)=r-\overline{R}(z)$. Moreover

$$\begin{aligned}\int_0^1 d\xi\int_0^h dz\,(1-\xi)\frac{2\delta\delta'(\alpha'+\xi\delta')}{\sqrt{1+(\alpha'+\xi\delta')^2}} &\ge -2\|\delta\|_\infty \int_0^h |\delta'| \ge\\ &\ge -2\Big(\int_0^h |\delta'|\Big)^2 .\end{aligned}\tag{5.6}$$

In order to estimate the second term of (5.6) from below, we first observe that if h and $\mathcal{N}(\delta)$ are sufficiently small, then, because of (5.4), $\|\delta\|_\infty$ is small in such a way that $(\alpha(z)+\xi\delta(z)) \ge c$. By consider separately the contributes $|\delta'| \le 1$ and $|\delta'|>1$ we find, respectively,

$$\begin{aligned}\int_0^1 d\xi \int_{|\delta'|\le 1} dz\,\frac{(1-\xi)\,\delta'^2}{(1+(\alpha'+\xi\delta')^2)^{\frac{3}{2}}} &\ge \int_0^1 d\xi\int_{|\delta'|\le 1}\times\\ &\quad\times dz\,\frac{(1-\xi)\,\delta'^2}{(1+(|\alpha'|+1)^2)^{\frac{3}{2}}} = c\int_{|\delta'|\le 1}|\delta'|^2,\end{aligned}\tag{5.7}$$

$$\begin{aligned}\int_0^1 d\xi\int_{|\delta'|>1} dz\,\frac{(1-\xi)\,\delta'^2}{(1+(\alpha'+\xi\delta')^2)^{\frac{3}{2}}} &\ge \int_{|\delta'|>1} dz\int_0^{\frac{1}{|\delta'|}}\times\\ &\quad\times d\xi\frac{(1-\xi)\delta'^2}{(1+(|\alpha'|+1)^2)^{\frac{3}{2}}} = c\int_{|\delta'|>1} dz\left(|\delta'|-\frac{1}{2}\right)\ge\\ &\ge c\int_{|\delta'|>1} dz\,|\delta'|\,.\end{aligned}\tag{5.8}$$

Finally, by (5.6) – (5.8), using (5.3), (5.4), we find

$$\tilde{A}_{\text{thin}}[R] - \tilde{A}_{\text{thin}}[\overline{R}] \geq (c - c\,h)\,\mathcal{N}(\delta) - c\mathcal{N}(\delta)^2\,, \tag{5.9}$$

that, for h and $\mathcal{N}(\delta)$ sufficiently small implies the thesis. □

Now we shall prove that also the full functional $A[R,x]$ has a local minimum, at least in a neighborhood of the minimum of A_{thin}. To do this, we need some estimate on the residual part of the functional. Let us define $A_1[R,x] = \int_0^h L_1(R, R', x, x')$.

LEMMA 5.2. *Let $a = \frac{\mu\gamma^2}{12T}$, if $x^2\,R < \frac{a}{2}$, $\varepsilon^2 x^2 < 2R^2$, $R(z) > \frac{\min_z \overline{R}(z)}{2}$ and $\mathcal{N}(\delta)$ bounded (where δ and $\mathcal{N}$ are defined as in Lemma 5.1), then*

$$A_1[R,x] \geq -c\varepsilon^4\,; \tag{5.10}$$

where c depends on T, $a, \overline{R}, \overline{R}'$, $\mathcal{N}(\delta)$.

PROOF. First we give an estimate of L_1. Using $1 - \sqrt{1+b} \geq -\frac{b}{2}$ in the expression (4.8) for L_T, and noting that the term on ϵ_1 in L_γ is positive and increasing on $\epsilon_1 \geq 0$, with some tedious calculations it is possible to obtain

$$L_1 \geq \varepsilon^4 \frac{T}{R^4}\Big(a(xR' - Rx')^2 - \frac{R^3 x^2 x'^2}{\sqrt{1+R'^2}}\Big)\,. \tag{5.11}$$

Taking the minimum of (5.11) in x', and imposing $x^2\,R < \frac{a}{2}$, we have

$$L_1 \geq -\varepsilon^4 \frac{a^2 T}{8} \frac{\sqrt{1+R'^2}}{R^5}\,. \tag{5.12}$$

Using the hypothesis $R \geq \frac{\min \overline{R}}{2}$ and that (5.4), we obtain (5.10). □

Now, let us define the space of motions:

$$\begin{aligned}\mathcal{M} =&\{(R(\,\cdot\,),\, x(\,\cdot\,)) \in \big(\mathbf{C}^1[0,h] \times \mathbf{C}^1[0,h]\big) :\\ &R(0) = R_0,\, R(h) = R_h,\, x(0) = \bar{x}(0),\, x(h) = \bar{x}(h),\\ &x^2\,R < \frac{a}{2},\;\; x^2 < 2R^2,\;\; \mathcal{N}(\delta) < \overline{\mathcal{N}}\}\,.\end{aligned} \tag{5.13}$$

where $\varepsilon < 1$, and h, $\overline{\mathcal{N}}$ are small in the sense of Lemma 5.1, and in such a way that $R(z) > \frac{\min \overline{R}}{2}$.

On $\mathcal{M}$, from Lemma 5.1 and Lemma 5.2, we have

$$A[R,x] = A_{\text{thin}}[R,x] + A_1[R,x] \geq \tilde{A}_{\text{thin}}[\overline{R}] - c\varepsilon^4\,. \tag{5.14}$$

Then let us consider a minimizing sequence (R_n, x_n) for A on $\mathcal{M}$.

LEMMA 5.3.

$$\int_0^h (R_n')^2 \leq c \tag{5.15}$$

$$\int_0^h \Big(\frac{d}{dz} x_n^3\Big)^2 \leq c \tag{5.16}$$

PROOF. Without loss of generality we can consider A bounded from above on the minimizing sequence, then, from (5.10)

$$c + c\varepsilon^4 \geq A_{\text{thin}}[R_n, x_n] \geq \int_0^h \Big(\mu\gamma^2 \frac{1 + {R_n'}^2}{x_n^2} - 2TR_n\sqrt{1 + {R_n'}^2}\Big)\,. \tag{5.17}$$

Being $\mathcal{N}(\delta_n)$ bounded, because of (5.13), we have the uniform control of the minimum and the maximum in z of R_n, moreover $x_n^2 < \frac{a}{2R_n}$. From (5.17), estimating the term in $\sqrt{1 + {R_n'}^2}$ with the corresponding term in $\overline{R}$ and $\overline{\mathcal{N}}$ via (5.4) we obtain (5.15).

From (5.11) we have

$$\begin{aligned} L_1 &\geq \varepsilon^4 \frac{T}{R_m^4}(a(x_n R_n' - R_n x_n') - R_n^3 x_n^2 {x_n'}^2) \geq \\ &\geq \varepsilon^4 \frac{aT}{R_n^4}\Big(\frac{{x_n'}^2 {R_n}^2}{4} - 3{x_n}^2 {R_n'}^2\Big)\,. \end{aligned} \tag{5.18}$$

Using $x_n^2 R_n < \frac{a}{2}$ and (5.15)

$$c\varepsilon^4 \int_0^h \Big(\frac{d}{dz} x_n^3\Big)^2 \leq A_1[R_n, x_n] + c\varepsilon^4\,. \tag{5.19}$$

Observing that $A_1[\overline{R},\bar{x}]$ is of order ε^4, that A_{thin} takes minimum in $\overline{R}$, $\bar{x}$ and that for large n

$$A[R_n,x_n] \leq A_{\text{thin}}[\overline{R},\bar{x}] + A_1[\overline{R},\bar{x}]\,, \tag{5.20}$$

we have

$$A_1[R_n,x_n] \leq A_1[\overline{R},\bar{x}] \leq c\varepsilon^4\,. \tag{5.21}$$

Finally collecting (5.19) and (5.21) we obtain (5.16). □

PROOF OF THEOREM 4.2. If h and ε are sufficiently small, then, R_n, x_n^3 converge weakly in $\mathbf{H}^1[0,h]$ to a local minimum of A; namely from (5.15) and (5.16), R_n and x_n^3 are compact in $\mathbf{H}^1$; then R_n and x_n converge uniformly to R_ε, $x_\varepsilon \in \mathbf{C}[0,h]$, respectively. By direct calculation, if R and x verify (5.13) the functional A is convex in R', x'; then it is lower semi-continuous. If h is small enough, from (5.15), (5.16) it follows that the bounds (5.13) are strictly verified by R_ε, x_ε. Using (5.20), (5.10), and (5.21)

$$\begin{aligned}\tilde{A}_{\text{thin}}[R_n] - \tilde{A}_{\text{thin}}[\overline{R}] &\leq A_{\text{thin}}[R_n,x_n] - A_{\text{thin}}[\overline{R},\bar{x}] \leq \\ &\leq A_1[\overline{R},\bar{x}] - A_1[R_n,x_n] \leq c\varepsilon^4\,.\end{aligned} \tag{5.22}$$

Furthermore, using Lemma 5.1, and the fact that $\mathcal{N}(\delta)$ is convex, we have

$$\mathcal{N}(R-R^\varepsilon) \leq c\varepsilon^4 < \overline{\mathcal{N}} \tag{5.23}$$

if ε is small. To conclude the proof of the existence we observe that $\mathcal{M}$, defined in (5.13), is not empty since $(\overline{R},\,\bar{x}) \in \mathcal{M}$, if we choose opportunely $\overline{R}_0$, $\overline{R}_0'$.

The uniform convergence of R_ε to $\overline{R}$ follows from (5.23). For the convergence of x_ε we proceed as follow. Let us denote with w_ε the function which minimizes on $x(\,\cdot\,)$ $A_{\text{thin}}[R_\varepsilon,x]$ with R_ε fixed (see (4.11)). From the positivity of $\tilde{A}_{\text{thin}}[R_\varepsilon] - \tilde{A}_{\text{thin}}[\overline{R}]$, the estimate (5.22), and an explicit calculation:

$$\frac{p}{2}\int_0^h |x_\varepsilon - w_\varepsilon|^2 \leq c\varepsilon^4\,. \tag{5.24}$$

We observe that, from (4.11), (5.4), (5.23)

$$\int_0^h |x_\varepsilon^2 - w_\varepsilon^2| \leq c\varepsilon^2 \,. \tag{5.25}$$

From (5.24), (5.25) and the compactness in $\mathbf{H}^1[0,h]$ of x_ε^3, it follows that x_ε converges uniformly to $\bar{x}$. ∎

REMARK 1. It is not difficult to see that the minimum of A is a weak solution of the associated Lagrange equation, and, by an usual procedure, that it is also a $\mathbf{C}^2$ solution.

REMARK 2. In our opinion it could be possible to prove the existence of extremum points of the functional (3.7) near the minima of the approximate functional, by working in a sufficiently regular space (see also Remark 2 of Section 3). Namely if R_1, $R_2 \in \mathbf{C}^2$ the difference between the true functional (3.7) and the approximate functional (4.2) vanishes as ε^4 when ε goes to 0.

Another possible approach is to construct solutions of the Euler equation (2.1-2) using a fixed point for the free boundary. Namely the inverse of the mean curvature operator is regularizing (see *e.g.* [10]). But in this case the thin limit seems to be a very delicate relaxation problem, in which compactness properties can disappear.

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