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Refinement masks of Hurwitz type in the cardinal interpolation problem

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Riassunto: *Si studiano le proprietà di una particolare classe di funzioni di raffinamento simmetriche, a supporto compatto e totalmente positive. Si dimostra che tali funzioni di raffinamento possono essere usate nel problema dell'interpolazione cardinale generalizzata poiché esiste un unico valore eccezionale che viene qui calcolato esattamente. Vengono presentati alcuni esempi numerici riguardanti l'interpolazione e la costruzione di wavelets semi-ortogonali tramite tali funzioni di raffinamento.*

Abstract: *We analyse the properties of a particular class of symmetric, compactly supported, totally positive refinable functions. We show that these refinable functions can be used in the generalized cardinal interpolation problem for which there exists a unique exceptional value which can be evaluated exactly. Some numerical examples concerning interpolation and construction of semi-orthogonal wavelets by means of these refinable functions are displayed.*

1 – Cardinal interpolation by refinable functions

In relation with the cardinal splines $M_n(x)$, of order n, Schoenberg proposed [12], [13] the *cardinal interpolation problem* consisting of seeking a function $S \in \text{span}\{M_n(\cdot - k)\}$, i.e.

$$S(x) = \sum_{k\in\mathbb{Z}} b_k M_n(x-k)\,, \tag{1.1}$$

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satisfying the interpolation conditions

$$S(\alpha+j) = y_j, \quad j \in \mathbb{Z} \tag{1.1'}$$

where $\mathbf{y} = \{y_j\}_{j\in\mathbb{Z}}, y_j \in \mathbb{R}$, is a given sequence and $0 \le \alpha < 1$.

This problem was solved under various conditions on $\mathbf{y}$; in particular, it was proved that, if $\mathbf{y} \in l^1(\mathbb{Z})$, the cardinal interpolation problem has a unique solution for each $\alpha \in [0,1)$ different from an exceptional value α_0.

The value of α_0 depends on the order of the cardinal splines in consideration; in particular, $\alpha_0 = 0$ for the cardinal splines of odd order and $\alpha_0 = \frac{1}{2}$ for the cardinal splines of even order.

The observation that the cardinal splines are refinable functions [1], has suggested the following *Generalized Cardinal Interpolation Problem* (GCIP) [7].

Let φ be a solution (called a *refinable function*) of the *refinement equation*

$$\varphi(x) = \sum_{j\in\mathbb{Z}} a_j \varphi(2x-j), \quad x \in \mathbb{R} \tag{1.2}$$

where the *mask* $\mathbf{a} = \{a_j\}_{j\in\mathbb{Z}}$ satisfies the condition

$$\sum_{j\in\mathbb{Z}} a_{2j+1} = \sum_{j\in\mathbb{Z}} a_{2j} = 1\,, \tag{1.3}$$

and let $\mathbf{y} = \{y_j\} \in l^1(\mathbb{Z})$ be a sequence of real data. We seek a function $F \in \operatorname{span}\{\varphi(\cdot - k)\}$, i.e.

$$F(x) = \sum_{k\in\mathbb{Z}} c_k \varphi(x-k)\,, \tag{1.4}$$

interpolating the data y_j at the points $\alpha + j$, with α fixed in $[0,1)$, that is

$$F(\alpha+j) = y_j, \quad j \in \mathbb{Z}\,. \tag{1.5}$$

The property of total positivity of the B-splines plays a crucial role in interpolation problems, thus it is important to construct refinable functions enjoying the same property.

Let us introduce the discrete Fourier transform $A(z)$ of the mask, also termed the *symbol* of the sequence $\{a_j\}_{j\in\mathbb{Z}}$:

$$A(z) = \sum_{j\in\mathbb{Z}} a_j z^j, \quad z = e^{i\omega}. \tag{1.6}$$

In the B-splines case, which corresponds to the mask

$$\begin{cases} a_j = \dfrac{1}{2^{n-1}}\dbinom{n}{j} & j = 0, \dots, n \\ a_j = 0 & \text{otherwise} \end{cases}$$

($n \in \mathbb{N}$ fixed), the symbol is a left-half plane stable polynomial (Hurwitz polynomial) having the form:

$$B_n(z) = \frac{1}{2^{n-1}}(z+1)^n, \quad z = e^{i\omega}.$$

In [4] it has been proved that if the symbol is a Hurwitz polynomial, the corresponding φ is a *ripplet*, that is, φ is totally positive:

$$\begin{aligned} \det_{l,j=1,\dots,r} \varphi(x_l - i_j) \geq 0 \qquad \forall\ x_1 < \dots < x_r,\ i_1 < \dots < i_r \\ x_l \in \mathbb{R},\ i_j \in \mathbb{Z} \end{aligned} \tag{1.7}$$

with strict positivity holding if and only if $i_l < x_l < i_l + n - 1$, $l = 1, \dots, r$.

Moreover, in this case, φ enjoys the variation diminishing property

$$S^-\Big(\sum c_j \varphi^{(l)}(\cdot - j)\Big) \leq S^-(\Delta^{(l)} c), \tag{1.8}$$

where $S^-(\mathbf{b})$ denotes the number of (strict) sign changes in the sequence $\mathbf{b} = \{b_j\}_{j\in\mathbb{Z}}$, and

$$(\Delta^l c)_j = (\Delta^{l-1} c)_{j+1} - (\Delta^{l-1} c)_j$$

and $(\Delta^0 c)_j = c_j$, as usual.

In [13, Lecture 4] Schoenberg proved that the solution of the cardinal interpolation problem exists and is unique provided that the Euler-Frobenius polynomial

$$\Pi(z;\alpha) = \sum_{j\in\mathbb{Z}} M_n(\alpha + j) z^j$$

does not vanish on the unit circle $|z| = 1$.

The proof of Schoenberg is mainly based on the total positivity of the B-splines. Because the φ we are considering are ripplets, the line of reasoning developed in [13] can be extended to show that also the solution of the generalized cardinal interpolation problem exists and is unique provided that the Euler-Frobenius polynomial

$$\Pi(z;\alpha) = \sum_{j\in\mathbb{Z}} \varphi(\alpha+j) z^j \tag{1.9}$$

does not vanish on the unit circle $|z| = 1$.

In Section 2 we analyze the properties of a particular class of symmetric refinable functions totally positive and compactly supported on $[0, n]$.

In Section 3 we study its behaviour in the cardinal interpolation showing that there exists a unique value α_0 such that $\Pi(z;\alpha_0) = 0$ on $|z| = 1$. We display also some numerical examples.

In Section 4 we construct the *semi-orthogonal* wavelets, here called *pre-wavelets*, associated with these refinable functions.

2 – A particular class of symmetric masks

In [5] the following class of positive symmetric masks compactly supported on $[0, n], n \geq 3$, depending on the real parameter $h > n - 2$, has been introduced:

$$a_{j,n}^{(h)} = \frac{1}{2^h}\left[\binom{n}{j} + 4(2^{h-n+1}-1)\binom{n-2}{j-1}\right], \quad j = 0, 1, \dots, n \tag{2.1}$$

(assume $\binom{l}{i} = 0$ for $i < 0$ or $i > l$). The masks (2.1) satisfy also the conditions (1.3).

Note that any sequence $\mathbf{a}^{(h,n)} = \{a_{j,n}^{(h)}\}$ is *bell-shaped*, because it satisfies the relations

$$a_{j,n}^{(h)} < a_{j+1,n}^{(h)}, \quad j = 0, 1, \dots, [n/2]\,. \tag{2.2}$$

The function $\varphi_{h,n}$, solution to the refinement equation

$$\varphi_{h,n}(x) = \sum_{j=0}^{n} a_{j,n}^{(h)} \varphi_{h,n}(2x - j)\,, \tag{2.3}$$

is positive, compactly supported on $[0, n]$, centrally symmetric, that is

$$\varphi_{h,n}(x) = \varphi_{h,n}(n-x), \quad \forall\, x \in (0,n)\,, \tag{2.4}$$

and such that

$$\sum_{j\in\mathbb{Z}} \varphi_{h,n}(x-j) = 1, \quad \forall\, x \in \mathbb{R} \tag{2.5}$$

(see [11, Corollary 5.1]). For the sake of simplicity, in the following we shall use symmetric instead of centrally symmetric.

Let us denote by Φ the set

$$\Phi := \{\varphi_{h,n} : n \geq 3, h > n-2, n \in \mathbb{N}, h \in \mathbb{R}\}\,.$$

The symbol of the masks (2.1) has the form

$$A_{h,n}(z) = \sum_{j=0}^{n} a_{j,n}^{(h)} z^j = (z+1)^{n-2}\frac{1}{2^h}[z^2 + 2(2^{h-n+2}-1)z + 1] \tag{2.6}$$

from which it immediately follows that, for $h > n-2$, $A_{h,n}(z)$ is a Hurwitz polynomial, thus $\varphi_{h,n}$ is a ripplet and (1.7) and (1.8) hold. Moreover, $\varphi_{h,n} \in C^{n-3}$ (see [4, Theorem 4.2]).

As a consequence of (2.2) and (1.8), $\varphi_{h,n}$ is *bell-shaped*, in the sense that it is increasing on $[0, n/2)$ and decreasing on $(n/2, n]$.

Remark. We observe that choosing $h = n-1$ in (2.1), we obtain the mask of the B-spline of order n.

Let us write the symbol $A_{h,n}(z)$ as the product of $B_1(z)$, the symbol associated to the B-spline of order 1, and $A_{h-1,n-1}(z)$, the symbol associated to $\varphi_{h-1,n-1}$, that is

$$A_{h,n}(z) = B_1(z)\frac{1}{2}A_{h-1,n-1}(z). \tag{2.7}$$

Denoting by $\hat{f}(\omega)$ the Fourier transform of a function $f(x)$, i.e.

$$\hat{f}(\omega) = \int_R f(x)e^{-i\omega x}dx\,,$$

and taking the Fourier transform of the refinement equation (2.3), we have

$$
\begin{aligned}
\hat{\varphi}_{h,n}(\omega) &= \frac{1}{2} A_{h,n}(z) \hat{\varphi}_{h,n}\Big(\frac{\omega}{2}\Big) = \\
&= \frac{1}{2} B_1(z) \frac{1}{2} A_{h-1,n-1}(z) \hat{\varphi}_{h,n}\Big(\frac{\omega}{2}\Big), \quad z = e^{i\omega}
\end{aligned} \tag{2.8}
$$

from which it follows

$$
\hat{\varphi}_{h,n}(\omega) = \widehat{M_1}(\omega) \hat{\varphi}_{h-1,n-1}(\omega)\,. \tag{2.9}
$$

In [4, proof of Theorem 4.2] it has been shown that the convolution between a refinable function corresponding to a symbol of Hurwitz type and the B-spline M_l is a refinable function corresponding to a symbol of Hurwitz type, too.

It is worth noting that (2.9) gives us an additional information. In fact, we have the following proposition, which can be proved by a recursive procedure.

PROPOSITION 2.1. *If $\varphi \in \Phi$, then also the convolution $\varphi * M_l \in \Phi$; in particular,*

$$
\varphi_{h,n}(x) = (\varphi_{h-l,n-l} * M_l)(x)\,. \tag{2.10}
$$

From (2.6) and some results in [4] it follows that the effect of the convolution with M_1 is to increase the smoothness of the refinable functions. This is also true in the case of the well known Daubechies refinable functions of compact support [3] whose symbol is not a Hurwitz polynomial.

From (2.10) it is easy to prove, by induction, the following corollary.

COROLLARY 2.2. *The derivatives of order m of $\varphi_{h,n}(x)$ can be expressed as*

$$
\varphi_{h,n}^{(m)}(x) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \varphi_{h-m,n-m}(x-j), \qquad m \le n-3\,. \tag{2.11}
$$

3 – The exceptional value for the GCIP

Here we are interested in finding the value of α for which the Euler-Frobenius polynomial $\Pi(z;\alpha)$ vanishes on $|z|=1$, when $\varphi \in \Phi$.

Due to the compact support of φ, $\Pi(z;\alpha)$ is a polynomial of degree $n-1$.

We recall that when the sequence of the coefficients of a polynomial is totally positive, the polynomial has only real negative zeros [6], thus the exceptional values of the generalized cardinal interpolation problem are the roots of the equation

$$\Pi(-1;\alpha) = \sum_{j=0}^{n-1} \varphi(\alpha+j)(-1)^j = 0\,. \tag{3.1}$$

It is worth noting that such a value of α depends only on the refinable function φ. In [7] we have shown that in the case $n=3$ the exceptional value is unique and its value is 0 if and only if the refinable functions are symmetric. In the case $n \geq 3$ the following theorem holds.

THEOREM 3.1. *If φ is a totally positive, compactly supported, symmetric refinable function whose support is $[0,n]$, then the values $\alpha_0 = 0$ for n odd and $\alpha_0 = \frac{1}{2}$ for n even are exceptional values with respect to the generalized cardinal interpolation problem.*

PROOF. Consider the case n odd.
Due to (3.1), we are interested in finding the roots of the equation

$$\sum_{j=0}^{n-1} \varphi(\alpha+j)(-1)^j = 0\,. \tag{3.2}$$

For $\alpha = 0$ (3.2) reduces to

$$\begin{aligned}\Pi(-1;0) = \sum_{j=0}^{n-1} \varphi(j)(-1)^j = \varphi(0)+ \\ + \sum_{j=1}^{(n-1)/2} \varphi(j)(-1)^j + \sum_{j=(n-1)/2+1}^{n-1} \varphi(j)(-1)^j\,.\end{aligned}$$

Observing that the first term is zero and using the symmetry property $\varphi(x) = \varphi(n-x)$, $\forall\, x \in \mathbb{R}$, in the last sum, we obtain

$$\Pi(-1;0) = \sum_{j=1}^{(n-1)/2} \varphi(j)(-1)^j + \sum_{j=1}^{(n-1)/2} \varphi(j)(-1)^{n-j} = 0\,.$$

The case n even can be proved in a similar way. ∎

REMARK. This result generalizes the analogous one concerning the $\mathcal{L}$-splines [9].

In the proof of Theorem 3.1 we used only the symmetry of φ on the integers. On the other hand, the symmetry on the integers implies that φ is symmetric for any $x \in \mathbb{R}$, as proved in the following theorem.

THEOREM 3.2. *The conditions*

$$\varphi(i) = \varphi(n-i), \quad \forall\, i \in \mathbb{N} \tag{3.3}$$

hold if and only if φ is symmetric $\forall\, x \in \mathbb{R}$.

PROOF. If φ is symmetric, then (3.3) holds. Suppose now that (3.3) holds.

Using the refinement equation (1.2) with $j \in [0,n]$, and (3.3), we obtain

$$\sum_{j=0}^{n} a_j \varphi(2i-j) = \sum_{j=0}^{n} a_{n-j} \varphi(2i-j), \quad \forall\, i \in \mathbb{N}$$

from which it follows that $a_j = a_{n-j}$, which implies that φ is symmetric $\forall\, x \in \mathbb{R}$ (see [11, p. 207]). ∎

From the previous theorems we conjecture that the only refinable functions having exceptional values

$$\begin{cases} \alpha_0 = 0 & \text{for} \quad n \quad \text{odd} \\ \alpha_0 = \dfrac{1}{2} & \text{for} \quad n \quad \text{even} \end{cases}$$

are the symmetric ones. For instance, in fig. 1 we display the zeros $l_1(\alpha)$, $l_2(\alpha)$, $l_3(\alpha)$ of the Euler-Frobenius polynomial in a non-symmetric case,

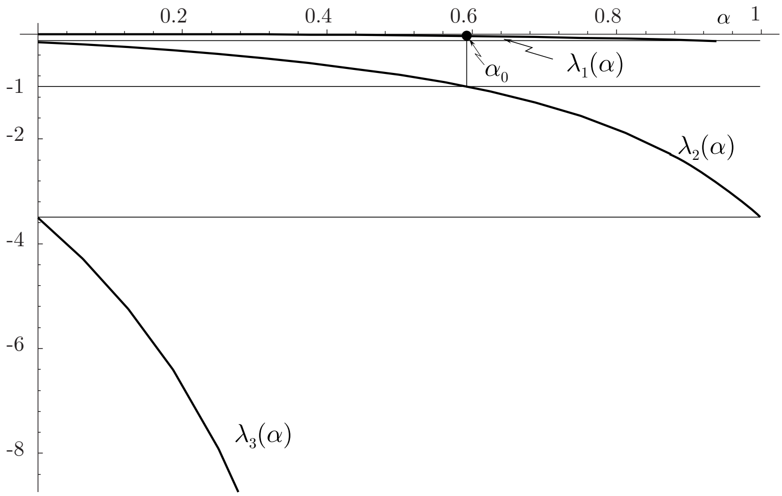

Fig. 1. – Zeros of the Euler-Frobenius polynomial in a non-symmetric case: $a_0 = \frac{1}{12}$, $a_1 = \frac{11}{24}$, $a_2 = \frac{19}{24}$, $a_3 = \frac{13}{24}$, $a_4 = \frac{1}{8}$.

corresponding to the mask $a_0 = \frac{1}{12}, a_1 = \frac{11}{24}, a_2 = \frac{19}{24}, a_3 = \frac{13}{24}, a_4 = \frac{1}{8}$ whose symbol is still a Hurwitz polynomial. The graph shows that, in this case, the unique exceptional value is near 0.6.

On the other hand, total positivity enables us to prove that the exceptional value is unique.

First of all we need the following results.

Lemma 3.3. *Let $t \in \mathbb{R}$ be fixed. If $t < 0$, the Euler-Frobenius polynomial $\Pi(t;\alpha)$ has exactly one simple zero for any $\alpha \in [0,1)$.*

Proof. It is easy to prove that

$$\Pi(t;\alpha + k) = t^{-k}\Pi(t;\alpha)\,. \tag{3.4}$$

For $\alpha = 0$ and $k = 1$ one has

$$\Pi(t;1) = t^{-1}\Pi(t;0)\,, \tag{3.5}$$

thus, if $t < 0$, $\Pi(t;1)\Pi(t;0) < 0$.

Now, suppose that $\Pi(t;\alpha)$ has $2l+1$, $l>0$, sign changes in $[0,1)$. Then, from (3.4) it follows that $g(x)=\Pi(t;x)$, $x\in\mathbb{R}$, has at least $(2l+1)L$ sign changes in the interval $[0,L]$. On the other hand, since φ is compactly supported, we have

$$g(x)=\Pi(t;x)=\sum_{j\in\mathbb{Z}}\varphi(x+j)t^j=\sum_{j=-L}^{n}\varphi(x+j)t^j$$

and, due to the total positivity of φ,

$$Z\big(g(x),[0,L)\big)\le n+L+1\,,$$

where Z denotes the number of zeros of $g(x)$ counting their multiplicities. Thus, we have the inequality $(2l+1)L\le n+L+1$ that is false for L large. It follows that $\Pi(t;\alpha)$ has a unique simple zero for $\alpha\in[0,1)$. ∎

Let us label the zeros of $\Pi(t;\alpha)$ as $\lambda_1(\alpha)\ge\ldots\ge\lambda_{n-1}(\alpha)$. Now, following the line of reasoning outlined in [9] and [14] in the case of $\mathcal{L}$-splines, we can prove the following lemma.

LEMMA 3.4. *The functions $\lambda_1(\alpha),\ldots,\lambda_{n-1}(\alpha)$ are continuous and strictly decreasing for $\alpha\in[0,1)$, and $\Pi(t;0)$ has exactly $n-1$ negative zeros. Moreover,*

$$\lim_{\alpha\to1^-}\lambda_i(\alpha)=\lim_{\alpha\to0^+}\lambda_{i+1}(\alpha)\quad i=1,\ldots,n-2$$

and

$$\lim_{\alpha\to0^+}\lambda_1(\alpha)=0,\qquad \lim_{\alpha\to1^-}\lambda_{n-1}(\alpha)=-\infty\,.$$

PROOF. The continuity of $\lambda_i(\alpha), i=1,\ldots,n-1$, is an immediate consequence of the continuity of $\varphi(j+\alpha)$ (recall (1.9)).

Suppose that there exist two different values of $\alpha\in[0,1)$, say $\overline{\alpha}$ and $\widetilde{\alpha}$, such that

$$\lambda_i(\overline{\alpha})=\lambda_j(\widetilde{\alpha})$$

for some indices i and j (which can be also equal) and let us reach a contradiction.

In this case $\Pi(\lambda_i(\overline{\alpha});\overline{\alpha}) = \Pi(\lambda_j(\widetilde{\alpha});\widetilde{\alpha}) = 0$ and the function $g(\alpha) = \Pi(t;\alpha)$ with $t = \lambda_i(\overline{\alpha}) = \lambda_j(\widetilde{\alpha})$ has two zeros, which contradicts Lemma 3.3. Thus, $\lambda_1(\alpha), \dots, \lambda_{n-1}(\alpha)$ are strictly monotonous functions of $\alpha \in [0,1)$.

Let us denote by $\Lambda(0)$ and $\Lambda(1)$ the sets of the values

$$\lambda_\mu(0^+) = \lim_{\alpha\to 0^+} \lambda_\mu(\alpha), \quad \lambda_\mu(1^-) = \lim_{\alpha\to 1^-} \lambda_\mu(\alpha), \quad \mu = 1,\dots,n-1\,,$$

respectively.

Due to Lemma 3.3, all the values in each set are distinct, that is $\lambda_\mu(0^+) \neq \lambda_\nu(0^+)$ and $\lambda_\mu(1^-) \neq \lambda_\nu(1^-)$ for all $\mu \neq \nu$. In particular, each set can contain the values 0 and $-\infty$ only once. Thus, the two sets $\Lambda(0)$ and $\Lambda(1)$ have at least $n-2$ distinct elements, whose values are finite.

It is easy to show, by direct evaluation, that the value 0 belongs to $\Lambda(0)$ and the value $-\infty$ belongs to $\Lambda(1)$. Moreover, from (3.5), it follows that the $n-2$ zeros of $\Pi(t;1)$ and $\Pi(t;0)$ are equal, thus

$$\Lambda(0) = \{\lambda_1(0^+) = 0, \lambda_2(0^+), \dots, \lambda_{n-1}(0^+)\}\,,$$

$$\Lambda(1) = \{\lambda_1(1^-) = \lambda_2(0^+), \dots, \lambda_{n-2}(1^-) = \lambda_{n-1}(0^+), \lambda_{n-1}(1^-)\}\,,$$

and the claim follows. □

As a consequence of the previous lemma, it easily follows:

Theorem 3.5. *Let $\varphi \in \Phi$ having support $[0,n]$. Then the unique exceptional value $\alpha_0 \in [0,1)$ with respect to the generalized cardinal interpolation problem is*

$$\begin{cases} \alpha_0 = 0 & \textit{for} \quad n \quad \textit{odd} \\ \alpha_0 = \dfrac{1}{2} & \textit{for} \quad n \quad \textit{even}\,. \end{cases} \tag{3.6}$$

The behaviour of the zeros of $\Pi(z;\alpha)$ are displayed in fig. 2 in the case $n=3$, $h=4$ and in fig. 3 in the case $n=4$, $h=5$.

A procedure to construct the interpolating function $F(x)$ can be found in [7]. Here this procedure has been used to construct the function $F(x)$ interpolating the following test functions:

$$f_1(x) = \begin{cases} 1 & x \in [-3.5, 3.5] \\ 0 & \text{otherwise} \end{cases} \qquad f_2(x) = \begin{cases} \sin^2\left(\dfrac{\pi}{5}x\right) & x \in [0,10] \\ 0 & \text{otherwise} \end{cases}$$

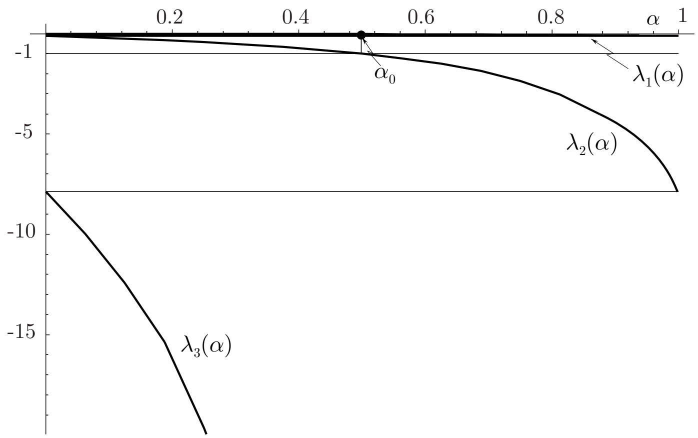

Fig. 2. – Zeros of the Euler-Frobenius polynomial corresponding to the choise $n = 4$ and $h = 5$ in the mask (2.1) (the values of $\lambda_1(\alpha)$ are very near to 0).

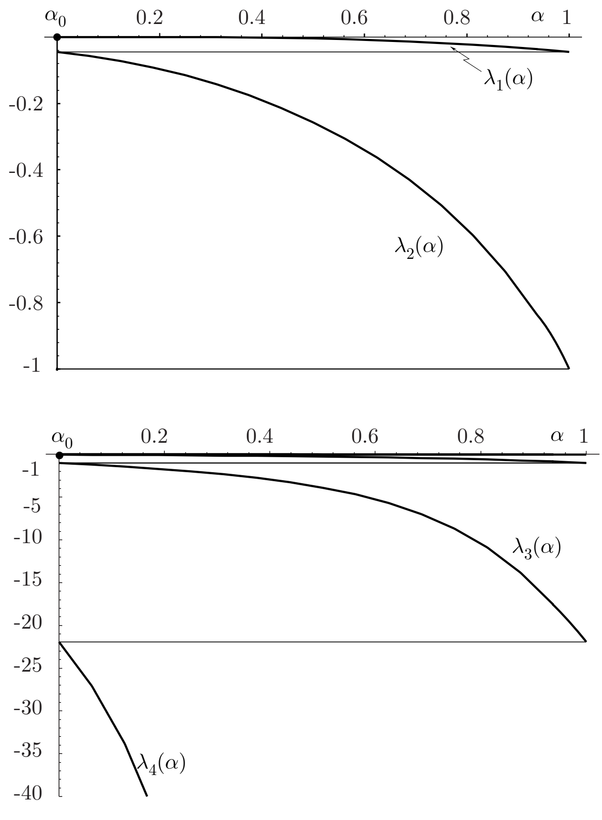

Fig. 3. – Zeros of the Euler-Frobenius polynomial corresponding to the choise $n = 3$ and $h = 4$ in the mask (2.1).

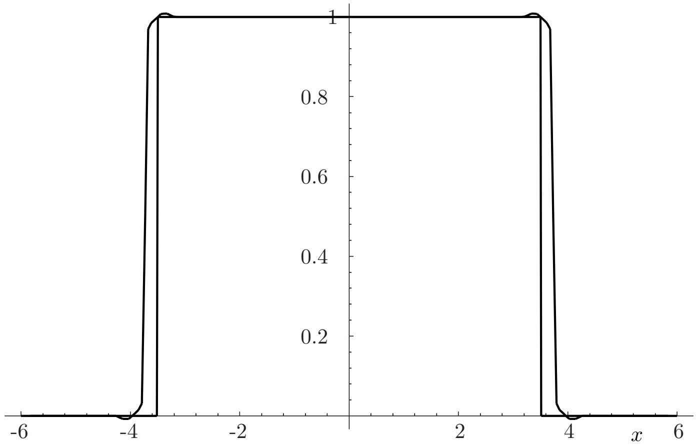

Fig. 4. – Graphs of $f_1(x)$ and $F_{6,3}(x)$ for $\alpha = \frac{1}{2}$.

In fig. 4 the graphs of $f_1(x)$ and of the interpolating function $F_{6,3}(x)$, belonging to span$\{\varphi_{6,3}(\cdot - k)\}$, are displayed. We observe that the use of a refinable function with a high value of h enables us to smooth the oscillations due to the Gibbs phenomenon.

In fig. 5 the graphs of $f_2(x)$ and of the interpolating function $F_{5,4}(x)$, belonging to span$\{\varphi_{5,4}(\cdot - k)\}$, are displayed. They differ slightly only near the zeros of $f_2(x)$.

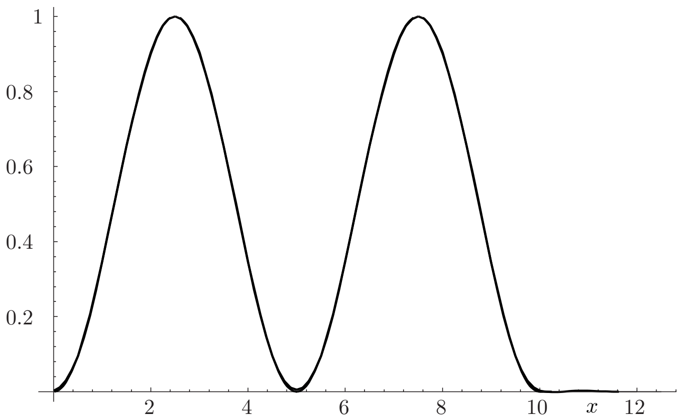

Fig. 5. – Graphs of $f_2(x)$ and $F_{5,4}(x)$ for $\alpha = 0$.

4 – Wavelets and dual bases

As a consequence of the fact that $\varphi_{h,n}$ is a ripplet, the integer translates of $\varphi_{h,n}(x)$ form a Riesz basis [5], thus the function

$$\psi_{h,n}(x) = \sum_{j\in\mathbb{Z}} (-1)^j \mu^{(h)}_{j-1,n} \varphi_{h,n}(2x-j) \tag{4.1}$$

where

$$\mu^{(h)}_{j,n} = \int_R \varphi_{h,n}(x)\varphi_{h,n}(2x+j)dx, \quad j\in\mathbb{Z} \tag{4.2}$$

is a *pre-wavelet*, that is $\psi_{h,n}(2^r x - l)$ is orthogonal to $\psi_{h,n}(2^s x - m)$ for all $m, l, r, s \in \mathbb{Z}$, with $r \neq s$ [10].

Moreover, $\psi_{h,n}$ form a Riesz basis too and can be used in the wavelet decompositon.

REMARK. For $h = n-1$, $\psi_{h,n}$ is the same as the *semi-orthogonal wavelet* constructed in [2].

To give an idea of the behaviour of the pre-wavelets, the graph of $\psi_{5,4}(x)$ is displayed in fig. 6.

It is known that in the wavelet decomposition one needs also the dual bases of both the refinable functions and the wavelets. The dual bases of $\varphi_{h,n}$ and $\psi_{h,n}$ have been constructed in [8], following the procedure given in [1].

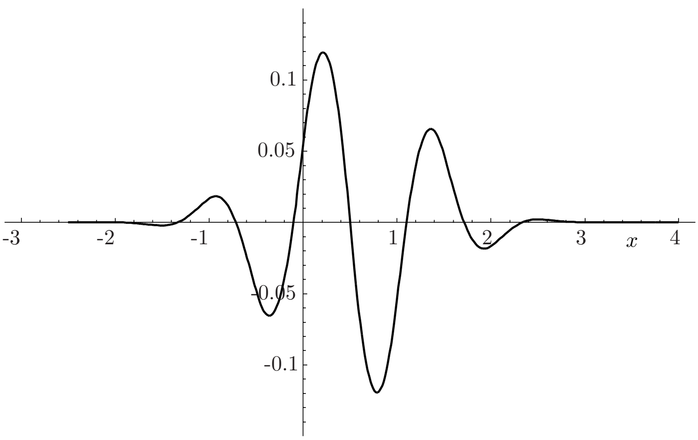

Fig. 6. – Graph of $\psi_{5,4}(x)$.

REFERENCES

[1] C.K. Chui: *Introduction to wavelets*, (Academic Press, New York) 1992.

[2] C.K. Chui – J.Z. Wang: *On compactly supported spline wavelets and a duality principle*, Trans. Amer. Math. Soc., **330** (1992), 903-916.

[3] I. Daubechies: *Ten lectures on wavelets*, SIAM, Philadelphia, Pennsylvania,1992.

[4] T.N.T. Goodman – C.A. Micchelli: *On refinement equations determined by Pólya frequency sequence*, SIAM J. Math. Anal., **23** (1992), 766-784.

[5] L. Gori – F. Pitolli: *Multiresolution analysis based on certain compactly supported refinable functions*, Proceeding of ICAOR: International Conference on Approximation and Optimization (Romania) (G. Coman, W.W. Breckner, P. Blaga, eds.), Transilvania Press, 1997, pp. 81-90.

[6] S. Karlin: *Total positivity*, Stanford Univ. Press, Stanford, 1968.

[7] M.L. Lo Cascio – F. Pitolli: *Generalized cardinal interpolation by refinable functions: some numerical results*, Note di Matematica, **15** (1995), 191-201.

[8] M.L. Lo Cascio – F. Pitolli: *A class of symmetric compactly supported wavelets and associated dual bases*, Series on Advances in Mathematics for Applied Sciences, **45** Advanced Mathematical Tools in Metrology III (P. Ciarlini, M.G. Cox, F. Pavese, D. Richter, eds.), World Scientific, Singapore, 1997 pp. 270-273.

[9] C.A. Micchelli: *Cardinal $\mathcal{L}$-splines*, Studies in spline functions and approximation theory (S. Karlin, C.A. Micchelli, A. Pinkus, eds.), Academic Press , New York, 1976, pp. 203-250.

[10] C.A. Micchelli: *Using the refinement equation for the construction of pre-wavelets*, Numer. Alg., **1** (1991), 75-116.

[11] C.A. Micchelli – H. Prautzsch: *Refinement and subdivision for spaces of integer translates of a compactly supported function*, Num. Anal. 1987 (D.F. Griffiths, G.A. Watson eds), Longman Sc. & Tech., Harlow, 1988, pp. 192-222.

[12] I.J. Schoenberg: *Cardinal interpolation and spline functions I*, J. Approx. Th., **2** (1969), 167-206.

[13] I.J. Schoenberg: *Cardinal spline interpolation*, SIAM Monograph, **12**, Philadelphia, 1973.

[14] I.J. Schoenberg: *On Micchelli's theory of cardinal $\mathcal{L}$-splines*, Studies in spline functions and approximation theory (S. Karlin, C.A. Micchelli, A. Pinkus, eds.), Academic Press, New York, 1976, pp. 251-276.

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