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Monomiality and integrals involving Laguerre polynomials

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Riassunto: *Si i ntroduce il concetto di "quasi monomialità" per i polinomi di Laguerre e si studiano i problemi iso-spettrali. Si dimostra che l'uso di questo concetto può essere particolarmente utile nelle applicazioni, poiché fornisce espressioni analitiche per integrali infiniti che coinvolgono prodotti di gaussiane e polinomi di Laguerre.*

Abstract: *We introduce the concept of quasi monomiality for Laguerre polynomials and study the associated iso-spectral problems. We show that the use of this concept may be particularly useful in application, providing e.g. analytical expressions for infinite integrals involving products of Gaussian and Laguerre polynomials.*

1 – Introduction

The concept of quasi-monomiality is often exploited to derive classes of iso-spectral problems [1], [2]. By quasi monomial we mean any expression characterised by an integer n, satisfying the relation

$$\begin{aligned} \widehat{m} f_n &= f_{n+1} \\ \hat{p} f_n &= n f_{n-1}\,, \end{aligned} \tag{1}$$

where $\widehat{m}$ and $\hat{p}$ play the role of multiplication and derivative operators.

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An example of quasi-monomial is provided by

$$
{}_{\delta}x_n = \prod_{m=0}^{n} (x - m\delta)\,, \tag{2}
$$

whose associated multiplication and derivative operators read

$$
\begin{aligned}
\widehat{m} &= x e^{\delta \frac{d}{dx}}\,, \\
\hat{p} &= \frac{e^{\delta \frac{d}{dx}} - 1}{\delta}\,.
\end{aligned} \tag{3}
$$

It is worth noting that when $\delta = 0$, ${}_{\delta}x_n = x^n$, $\widehat{m} = x$ and $\hat{p}$ reduces to the ordinary derivative.

As a further example of q.m. we consider the Hermite polynomials defined by the generating function

$$
e^{xt - \frac{t^2}{2}} = \sum_{n=0}^{\infty} \frac{t^n}{n!} He_n(x) \tag{4}
$$

and satisfying relations of the type (1), where

$$
\begin{aligned}
\widehat{m} &= x - \frac{d}{dx} \\
\hat{p} &= \frac{d}{dx}\,.
\end{aligned} \tag{5}
$$

The same notion has been extended to the Kampé de Fériét polynomials [3] defined by

$$
e^{xt + yt^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, y)
$$

with

$$
\begin{aligned}
\widehat{m} &= x + 2y \frac{\partial}{\partial x} \\
\hat{p} &= \frac{\partial}{\partial x}\,.
\end{aligned} \tag{6}
$$

The use of the concept of quasi-monomiality and of correspondences of the type (3), (5), (6) allows to define classes of isospectral problems.

We can e.g. introduce Hermite-Laguerre polynomials, denoted by ${}_H\mathcal{L}_n(x)$ and specified by the series (1)

$$ {}_H\mathcal{L}_n(x) = \sum_{k=0}^{n} \frac{n!(-1)^k He_k(x)}{(k!)^2(n-k)!}\,, \tag{7} $$

which are easily shown to satisfy a differential equation of the type

$$ [\widehat{m}\hat{p}^2 + (1-\widehat{m})\hat{p} + n]\,{}_H\mathcal{L}_n(x) = 0\,, \tag{8a} $$

which in terms of the differential operators (3) reads

$$ \Big[\frac{d^3}{dx^3} - (1+x)\frac{d^2}{dx^2} - (1-x)\frac{d}{dx} - n\Big]\,{}_H\mathcal{L}_n(x) = 0\,. \tag{8b} $$

As a further example we define the Hermite-Bessel functions which is defined by the series

$$ {}_HJ_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{2^{n+2s}} \frac{He_{n+2s}(x)}{s!(n+s)!} \tag{9} $$

and has been shown to be solution of the fourth order differential equation

$$ \Big[\frac{d^4}{dx^4} - 2x\frac{d^3}{dx^3} + (x^2+1)\frac{d^2}{dx^2} - x\frac{d}{dx} + x^2 - n^2 - 1\Big]\,{}_HJ_n(x) = 0\,. \tag{10} $$

More in general we can associate to any function possessing a series expansion in ordinary monomials, its counter part in a q.m. basis, by replacing x^n with f_n, the use of concept, and of correspondences of the above type may open unsupected areas of research, either in pure and applied mathematics.

The introduction of Kampé de Fériét-Bessel function has opened a new point of view to the theory of generalized Bessel functions. By using the generating function[1]

$$ e^{\frac{\widehat{m}}{2}(t-1/t)} = e^{\frac{1}{2}(x+2y\frac{\partial}{\partial x})(t-\frac{1}{t})} = e^{\frac{x}{2}(t-\frac{1}{t})+\frac{y}{4}(t-\frac{1}{t})^2} e^{y\frac{\partial}{\partial x}(t-1/t)}\,, \tag{11} $$

[1] We have used the Weyl disentanglement rule:

$$ e^{\hat{A}+\hat{B}} = e^{-\frac{K}{2}} e^{\hat{A}} e^{\hat{B}} \quad \text{if} \quad [\hat{A},\hat{B}] = K,\ [K,\hat{A}] = [K,\hat{B}] = 0. $$

we have been able to introduce that two-variable Bessel function,

$$ {}_H\mathcal{J}_n(x,y) = \sum_{s=0}^{\infty} \frac{(-1)^s H_{n+2s}(x,y)}{2^{n+2s} s!(n+s)!}\,, \tag{12} $$

whose properties and whose importance in applications has been discussed in refs. [2], [4]. According to the above examples it is evident that the use of the concept of quasimonomiality is not a formal curiosity without any significant consequence, but may be powerful and flexible tool opening new possibilities in different and apparently uncorrelated field of research.

In this paper we explore the importance of polynomials of the type (7) to derive analytical expressions for infinite integrals involving products of Gaussian functions and Laguerre polynomials.

2 – Quasi-monomials and infinite integrals

Before entering into the specific details of the problem, let us remind a few important points.

The generating function of the ordinary Laguerre polynomials is provided by [5]

$$ \sum_{n=0}^{\infty} t^n \mathcal{L}_n(x) = \frac{1}{(1-t)} \exp\Big(\frac{xt}{1-t}\Big) \qquad |\,t\,| < 1\,. \tag{13} $$

According to (13) and to (4) the generating function of ${}_H\mathcal{L}_n(x)$ is

$$ \sum_{n=0}^{\infty} t^n\, {}_H\mathcal{L}_n(x) = \frac{1}{(1-t)} \exp\Big[-\frac{xt}{1-t} - \frac{1}{2}\frac{t^2}{(1-t)^2}\Big] \qquad |\,t\,| < 1\,, \tag{14} $$

(see Section 3 for further comments), while that associated to the two variable case reads

$$ \sum_{n=0}^{\infty} t^n\, {}_H\mathcal{L}_n(x,y) = \frac{1}{(1-t)} \exp\Big[-\frac{xt}{1-t} + y\frac{t^2}{(1-t)^2}\Big] \qquad |\,t\,| < 1\,. \tag{15} $$

As first example of application we consider the integral

$$ \mathcal{J}_n(a,b,c) = \int_{-\infty}^{+\infty} e^{-ax^2} \mathcal{L}_n(2bx+c)dx\,, \tag{16} $$

by multiplying both sides of (16) by t^n, summing up and exploiting (13) we get

$$\sum_{n=0}^{\infty} t^n \mathcal{J}_n(a,b,c) = \frac{1}{1-t}\int_{-\infty}^{+\infty} e^{-ax^2}\exp\Big[-\frac{(2bx+c)t}{1-t}\Big]dx\,. \tag{17a}$$

Then, exploiting the properties of the Gaussian integral we end up with

$$\sum_{n=0}^{\infty} t^n \mathcal{J}_n(a,b,c) = \frac{1}{1-t}\sqrt{\frac{\pi}{a}}\exp\Big[-\frac{ct}{1-t}+\frac{b^2t^2}{(1-t)^2a}\Big], \tag{17b}$$

which once confronted with (15) yields

$$\mathcal{J}_n(a,b,c) = \sqrt{\frac{\pi}{a}}\,{}_H\mathcal{L}_n\Big(c,\frac{b^2}{a}\Big), \tag{18}$$

where ${}_H\mathcal{L}_n(x,y)$ is specified by the series

$${}_H\mathcal{L}_n(x,y) = \sum_{k=0}^{n}\frac{(-1)^k n! H_k(x,y)}{(k!)^2(n-k)!}\,. \tag{19}$$

We have checked the above results by means of a numerical procedure and found complete agreement between numerical and analytical results.

By using the same method we can easily conclude that

$$\int_{-\infty}^{+\infty} e^{-ax^2+2bx}\mathcal{L}_n(2cx)dx = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{a}}\,{}_H\mathcal{L}_n\Big(\frac{2\cdot bc}{a},\frac{c}{a}\Big)\,. \tag{20}$$

The extension of the previous results to the case including products of Laguerre polynomials needs some comments.

We consider indeed the following bilateral generating function [3]

$$\exp(xt+yu+zt^2+wu^2+kut) = \sum_{m,n}\frac{t^m}{m!}\frac{u^n}{n!}H_{m,n}(x,y,z,k)\,, \tag{21}$$

where the polynomials $H_{m,n}$ are explicitly provided by the series

$$H_{m,n}(x,y,z,w,k) = m!n!\sum_{\ell=0}^{\min(m,n)}\frac{k^\ell H_{m-\ell}(x,z)H_{n-\ell}(y,w)}{\ell!(m-\ell)!(n-\ell)!}\,. \tag{22}$$

We can accordingly define the following polynomials

$$
(23) \qquad {}_H\mathcal{L}_{m,n}(x,y,z,k)=\sum_{k=0}^{m}\sum_{s=0}^{n}(-1)^{k+s}\frac{m!n!H_{k,s}(x,y,z,w,k)}{(k!)^2(s!)^2(m-k)!(n-s)!}\,,
$$

provided by the generating function

$$
(24) \qquad \begin{aligned}&\sum_{m,n} t^m u^n\, {}_H\mathcal{L}_{m,n}(x,y,z,k)=\frac{1}{(1-t)(1-u)}\times\\ &\quad\times\exp\Big(-x\frac{t}{1-t}-y\frac{u}{1-u}+z\Big(\frac{t}{1-t}\Big)^2+\\ &\quad+w\Big(\frac{u}{1-u}\Big)^2+k\frac{ut}{(1-u)(1-t)}\Big) \qquad (|\,t\,|\neq|\,u\,|)<1\,.\end{aligned}
$$

After these remarks we consider the following integral

$$
(25) \qquad \mathcal{J}_{m,n}(a,b,c,d,f)=\int_{-\infty}^{+\infty}e^{-ax^2}\mathcal{L}_n(2bx+c)\mathcal{L}_m(2dx+f)dx\,.
$$

By employing the same technique as before, i.e. by multiplying both sides of (25) by $t^m u^n$, by summing up and by exploiting the properties of the Gaussian integrals we find

$$
(26) \qquad \mathcal{J}_{m,n}(a,b,c,d,f)=\sqrt{\frac{\pi}{a}}\,{}_H\mathcal{L}_{m,n}\Big(c,f,\frac{b^2}{a},\frac{d^2}{a},\frac{2\cdot bd}{a}\Big)\,,
$$

which is a further proof of the flexibility of the method. Further comments will be presented in the forthcoming concluding section, where we will discuss the possibility of extending the method to integrals containing associated Laguerre polynomials.

3 – Concluding remarks

In the previous section we have exploited the quasi-monomiality of Hermite polynomials and applied it to the evaluation of infinite integrals containing Laguerre polynomials. In this section we discuss the quasi-monomiality of Laguerre polynomials and its consequences. According to the recurrence relations, the Laguerre polynomials are q.m. under the action of the operators[(2)]

$$\hat{m} = 1 - \mathcal{D}_x^{-1}$$
$$\hat{p} = -\Big(x\frac{d^2}{dx^2} + \frac{d}{dx}\Big)\,, \tag{27}$$

where $\mathcal{D}_x^{-1}$ denotes the inverse of the ordinary derivative operator[(3)].

According to (27), the Laguerre polynomials can be recursively generated from[(4)]

$$\mathcal{L}_n(x) = (1 - \mathcal{D}_x^{-1})^n\,. \tag{28}$$

We can define several functions, by using the Laguerre polynomials as q.m.; the function

$$_{\mathcal{L}}e(x) = \sum_{s=0}^{\infty} \frac{1}{s!}\mathcal{L}_s(x)\,, \tag{29}$$

plays the role of exponential function and satisfies the differential equation

$$\Big(x\frac{d^2}{dx^2} + \frac{d}{dx}\Big)\,_{\mathcal{L}}e(x) = -\,_{\mathcal{L}}e(x)\,. \tag{30}$$

The behaviour of $\mathcal{L}^{e(x)}$ is provided by fig. 1. The $\mathcal{L}$-counterpart of the Gaussian can be written as

$$_{\mathcal{L}}g(x) = \sum_{n=0}^{\infty}(-1)^n \frac{\mathcal{L}_{2n}(x)}{n!}\,, \tag{31}$$

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(2) Note infact that [2]

$$\Big(x\frac{d^2}{dx^2} + \frac{d}{dx}\Big)\mathcal{L}_n(x) = -n\mathcal{L}_{n-1}(x)$$
$$\frac{d}{dx}\mathcal{L}_{n+1}(x) = \frac{d}{dx}\mathcal{L}_n(x) - \mathcal{L}_n(x)\,.$$

(3) To be more precise being $\mathcal{D}_x^{-1}$ an integral we should specify the integration limit as $_a\mathcal{D}_x^{-1} \to \int_a^x (\cdot)dx'$. In this case it is omitted because $a = 0$.

(4) The operator appearing on the r.h.s. of eq. (28) is intended to act on unity, we therefore get $\mathcal{L}_1(x) = 1 - x$, $\mathcal{L}_2(x) = (1 - \mathcal{D}_x^{-1})(1 - x) = 1 - 2x + \frac{x^2}{2}$ and so on.

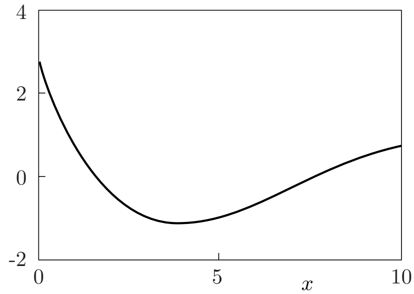

Fig. 1. The Laguerre exponential function *vs.* x.

and it is also easily shown that it is a solution of the third order differential equation

$$\left[x\frac{d^3}{dx^3}+2\frac{d^2}{dx^2}-2\frac{d}{dx}\right]{}_{\mathcal{L}}g(x)=-2{}_{\mathcal{L}}g(x)\,, \tag{32}$$

(for the behaviour of ${}_{\mathcal{L}}g(x)$ *vs.* x see fig. 2). It is evident that the method of q.m. can be exploited for the systematic research of exact solutions of ordinary differential equations.

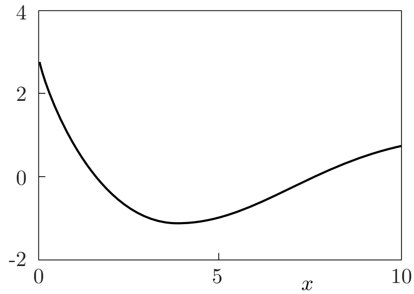

Fig. 2. The Laguerre-Gaussian function *vs.* x.

Further examples of "exotic" functions using the Laguerre polynomials as basis is provided by the Laguerre-Hermite polynomials defined as

$${}_{\mathcal{L}}H_n(x)\sum_{s=0}^{[n/2]}\frac{n!(-1)^s\mathcal{L}_{n-2s}(x)}{s!(n-2s)!} \tag{33}$$

which satisfy recurrences of the type

$$\begin{gathered}\Big(x\frac{d^2}{dx^2}+\frac{d}{dx}+1-\mathcal{D}_x^{-1}\Big)_{\mathcal{L}}H_n(x)={}_{\mathcal{L}}H_{n+1}(x)\\ \Big(x\frac{d^2}{dx^2}+\frac{d}{dx}\Big)_{\mathcal{L}}H_n(x)=-n_{\mathcal{L}}H_{n-1}(x)\,.\end{gathered}\tag{34}$$

Further examples might be discussed, but they will be more systematically investigated in a forthcoming paper.

Let us now come back to the evaluation of integrals of the type (16), (20), (25), by noting that they can be extended to the case of the associated Laguerre polynomials $\mathcal{L}_n^{(m)}(x)$ (see ref. [5]) by performing minor changes. By recalling indeed that

$$\mathcal{L}_n^{(m)}(x)=\sum_{k=0}^{n}\frac{(-1)^k(n+m)!x^k}{(n-k)!k!(m+k)!}\,,\tag{35}$$

we write[5]

$$\mathcal{J}_n^{(m)}(a,b,c,)=\sqrt{\pi a}_H\mathcal{L}_n^{(m)}\Big(c,\frac{b^2}{a}\Big)\,,\tag{36}$$

where

$$_H\mathcal{L}_n^{(m)}(x,y)=\sum_{k=0}^{n}\frac{(-1)^k(n+m)!H_k(x,y)}{k!(m+k)!(n-k)!}\,.\tag{37}$$

It is worth noting that this last result can also be extended to non integer m values provided that

$$_H\mathcal{L}_n^{(\alpha)}(x,y)=\sum_{k=0}^{n}\frac{(-1)^k\Gamma(n+\alpha+1)H_k(x,y)}{k!\Gamma(\alpha+k+1)(n-k)!}\,.\tag{38}$$

The accuracy of the above results has been checked by performing a comparison with a numerical integration.

Before concluding, it is worth clarifying the origin of generating functions of the type (14), (15).

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[5] $\mathcal{J}_n^{(m)}(a,b,c)$ is the integral (16) in which $\mathcal{L}_n$ has been replaced by $\mathcal{L}_n^{(m)}$.

We consider e.g. the case

$$\sum_{n=0}^{\infty} t^n \mathcal{L}_n(\widehat{m}) = \frac{1}{1-t} \exp\Big(-\frac{\widehat{m}t}{1-t}\Big), \tag{39}$$

if $\widehat{m}$ is provided by the first of eqs. (6) we obtain, after using the Weyl rule,

$$\sum_{n=0}^{\infty} t^n \mathcal{L}_n\Big(x+2y\frac{\partial}{\partial x}\Big) = \frac{1}{1-t} \exp\Big[-\frac{xt}{1-t}+\frac{yt^2}{(1-t)^2}\Big] \exp\Big(\frac{2yt}{1-t}\frac{\partial}{\partial x}\Big). \tag{40}$$

If the r.h.s. of (40) is not considered as an operator, the last exponential can be replaced by 1 thus getting eq. (15).

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