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A Hausdorff HS-space which is not regular

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RIASSUNTO: *In questo lavoro si dimostra, costruendo un controesempio, che il teorema di* H.-J. SCHMIDT [6]*, che afferma che ogni HS-spazio di Hausdorff è regolare, è falso.* M. PAOLI *e* E. RIPOLI [4] *avevano notato che la dimostrazione del teorema non era corretta, ma avevano lasciato aperta la questione sulla verità dell'enunciato.*

ABSTRACT: *In this paper I disprove, with a counterexample, a theorem of* H.-J. SCHMIDT [6]*, which states that each Hausdorff HS-space is regular.*
M. PAOLI *and* E. RIPOLI [4] *noted that the proof of this theorem is incorrect, but they left the statement open.*

1 – Introduction

A topological space X is called a HS-space if, for every subspace A of X the map $i_A : \mathcal{C}(A) \longrightarrow \mathcal{C}(X)$, defined by $i_A(B) = \mathrm{cl}_X(B)$, for each $B \in \mathcal{C}(A)$, is a continuous map, where we denote with $\mathcal{C}(X)$ the set of all non-empty closed subsets of X, with the Tychonoff topology, which is generated by the sets $\mathcal{C}(X,U) = \{F \in \mathcal{C}(X) : F \subset U\}$, for each U open subset of X.

In [6] H.-J.SCHMIDT gave it as a theorem that if a HS-space is Hausdorff then it is necessarily regular. M. PAOLI and E. RIPOLI in [4] and [5]

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noted that the proof of this theorem is incorrect, but the question of the correctness of the statement remained open.

S. Barov, G. Dimov and St. Nedev in [1], [2] gave a partial proof of the theorem, in particular they showed that the theorem of Schmidt is true for all spaces with countable character. About HS-space see also [7] and [3]. In this paper I show, by constructing a counterexample, that the theorem of H.J. Schmidt is false.

2 – The construction of the space X

Our space X will be the set of all finite sequences $p = (x_0, x_1, \dots, x_n)$ with $x_i \in \omega_i$, where ω_i is the $(i+1)$-st infinite cardinal and $n \in \omega$.

The topology of X will be defined in several steps.

Given $p, q \in X$ we put $p \leq q$ if p is a restriction of q (i.e. $p = (x_0, x_1, \dots, x_n) \leq q = (y_0, y_1, \dots, y_m)$ if $n \leq m$ and $x_i = y_i \ \forall\, i \leq n$). If $p \leq q$ or $q \leq p$, we say that p and q are comparable. The empty sequence p_0 is hence the initial point of X.

Note that X is a tree, each $\{x : x < a\}$ for each $a \in X$ is a finite totally ordered set and hence each non empty set has minimal elements.

Given $p \in X$,

Notation 1. $X_p = \{q \in X : q \geq p\}$.

Note that p and q are not comparable if and only if $X_p \cap X_q = \emptyset$.

Put

$$\Lambda = \prod_{n \in \omega} \omega_n \,.$$

If $\lambda \in \Lambda, n \in \omega$ put $\lambda_n \in \omega_n$ so that $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_n, \dots)$

For each λ we define

$$W_\lambda = \{p \in X : \quad p = (x_0, x_1, \dots, x_n) \Longrightarrow (x_i \geq \lambda_i \ \forall\, i \leq n)\}\,.$$

Notation 2. $\mathcal{W} = \{W_\lambda : \lambda \in \Lambda\}$.

Note that the correspondence $\lambda \longmapsto W_\lambda$ is one to one and that if $W \in \mathcal{W}$ then

a) $\forall\, r \in W \ \exists\, s \in W,\ s > r$;

b) $s \leq r \in W \Longrightarrow s \in W$.

Given $p \in X$,

Notation 3. $\mathcal{W}_p = \{W \cap X_p : p \in W \in \mathcal{W}\}$.

Clearly $X_p \in \mathcal{W}_p$.

Remark 1. $\mathcal{W}$ and $\mathcal{W}_p$ are closed under finite intersections.

Given $Z \in \mathcal{W}_p$

Notation 4. Denote by Z^* the biggest $W \in \mathcal{W}$ such that $W \cap X_p = Z$. Note that if $p = (x_0, x_1, \dots, x_n)$ and $Z^* = W_\lambda$ then $\lambda_0 = \lambda_1 = \dots = \lambda_n = 0$.

Notation 5. Denote by $\mathcal{V}_p$ the family of all sets V such that:

$$1) \quad V = \bigcup_{r \in R} X_r \text{ for some } R \subset X$$
$$2) \quad q \in X_p,\ Z \in \mathcal{W}_q \Longrightarrow V \cap Z \neq \emptyset\,.$$

(Note that we can say that $V \in \mathcal{V}_p$ if and only if V is open in the topology generated by the X_q's and $V \cap X_p$ is dense in X_p with the topology generated by the traces on X_p of the elements of all $\mathcal{W}_q$'s.)

Note also that

a) property 1 is equivalent to $t \geq s \in V \Longrightarrow t \in V$
b) X_p, $X_p \setminus \{p\} \in \mathcal{V}_p$
c) the union in property 1 can be taken disjoint, (by taking the minimal elements of V).

Remark 2. $V \in \mathcal{V}_p$, $q \geq p \Longrightarrow V \in \mathcal{V}_q$.

Proposition 1. *$\mathcal{V}_p$ is closed under finite intersections.*

Proof. Let $V_1, V_2 \in \mathcal{V}_p$ and $V = V_1 \cap V_2$. Clearly V satisfies property 1. Let $q \in X_p$ and $Z \in \mathcal{W}_q$; as a consequence of property 2 for V_1, there exists $s \in V_1 \cap Z$. From $s \in Z \in \mathcal{W}_q$ it follows that $Z \cap X_s \in \mathcal{W}_s$; hence, by property 2 for V_2, there exists $t \in V_2 \cap (Z \cap X_s)$. By property 1 for V_1 we infer that $t \geq s \in V_1 \Longrightarrow t \in V_1$ so $t \in V \cap Z$. ∎

Notation 6. Denote by $\mathcal{F}$ the family of all the sets $F \subset X$ which have only finite chains.

Notation 7. Let $\mathcal{U}_p$ the family of all the sets U having the form:

$$U = X_p \setminus \bigcup_{x \in F} (X_x \setminus V_x) \quad \text{with } F \in \mathcal{F},\ V_x \in \mathcal{V}_x \text{ for all } x \in F.$$

(Note also here that $X_x \setminus V_x$ is closed in X_x in the topology generated by the X_q's and closed nowhere dense in the topology generated by the by the traces on X_p of the elements of all $\mathcal{W}_q$'s.)

Remark 3. $U \in \mathcal{U}_p$, $q \geq p \Longrightarrow U \cap X_q \in \mathcal{U}_q$.

Proposition 2. *$\mathcal{U}_p$ is closed under finite intersections.*

Proof. Let

$$U_1 = X_p \setminus \bigcup_{x \in F_1} (X_x \setminus V_x^1) \quad , \quad U_2 = X_p \setminus \bigcup_{x \in F_2} (X_x \setminus V_x^2) .$$

It is enough to note that $F_1 \cup F_2 \in \mathcal{F}$ and that if $x \in F_1 \cap F_2$ then

$$(X_x \setminus V_x^1) \cup (X_x \setminus V_x^2) = X_x \setminus (V_x^1 \cap V_x^2). \qquad \square$$

Remark 4. $U \in \mathcal{U}_p$, $F \in \mathcal{F} \Longrightarrow U \setminus F \in \mathcal{U}_p$.

In fact it is enough to take $V_x = X_x \setminus \{x\}$ for all $x \in F$ to see that $X_p \setminus F \in \mathcal{U}_p$ and intersect it with U.

Notation 8. Let $\tau_p =$ the family of all the sets T having the form: $T = U \cap Z$ with $U \in \mathcal{U}_p$, $Z \in \mathcal{W}_p$.

Remark 5. τ_p is closed under finite intersections.

Proposition 3. *If $T = U \cap Z$ with $U \in \mathcal{U}_p$, $Z \in \mathcal{W}_p$, and if $q \in Z$ then $T \cap X_q \in \tau_q$.*

Proof. By Remark 3 and Notation 3. $\square$

3 – The topology τ of X

Definition 1. Given $p \in X$, a neighborhood base of p will be given by the sets having the form:

$$\{p\} \cup T \qquad \text{with } T \in \tau_p.$$

By Proposition 3, this gives really a topology τ on X and each $\{p\} \cup T$ is open, in particular each element of $\mathcal{W}_p$ is open, (in fact clopen).

Each $\bigcup_{x\in F}(X_x \setminus V_x)$ with $F \in \mathcal{F}$, $V_x \in \mathcal{V}_x$ is a closed set in (X,τ) and the topology τ is the coarsest one in which these sets are closed and the elements of $\mathcal{W}_p$ are open. In particular the elements of $\mathcal{F}$ are closed.

4 – Clopen partition associated to an element of $\mathcal{W}$

Let $W = W_\lambda \in \mathcal{W}$. Given $p = (x_0, \dots, x_n)$, let $W(p) \in \mathcal{W}_p$ be defined by

Notation 9. $W(p)=\{(y_0,\dots,y_m)\in X_p : y_{n+1}\ge\lambda_{n+1},\dots,y_m\ge\lambda_m\}$.

Notation 10. Denote by $\mathcal{P}_W$ the family of all maximal (with respect to the inclusion) elements of the set $\{W(p) : p \in X\}$.

Proposition 4. *$\mathcal{P}_W$ is a clopen partition.*

Proof. Since $\mathcal{P}_W$ is an open cover of X, it is enough to show that its elements are disjoint.

Let $p = (x_0, x_1, \dots, x_n)$, $q = (y_0, y_1, \dots, y_m)$ and suppose that there exists $r \in W(p) \cap W(q)$. Hence $r \ge p$ and $r \ge q$, and so p and q are comparable.

Suppose $p \le q$. Then $n \le m$ and $r \ge q \ge p$. Hence $y_i \ge \lambda_i$, for all i with $n < i \le m$. Hence $W(q) \subset W(p)$. □

Clearly

a) $W_1 \subset W_2 \Longrightarrow W_1(p) \subset W_2(p)$,
b) $p \in W \Longrightarrow W(p) = W \cap X_p$.
c) $p \in W(q) \Longrightarrow W(q) \cap X_p = W(p)$.

5 – Some lemmas

Lemma 1. $T \in \tau_p \Longrightarrow T \ne \emptyset$.

Proof. Let $T = Z \setminus \bigcup_{x\in F}(X_x \setminus V_x)$ with $Z \in \mathcal{W}_p$, $F \in \mathcal{F}$, $V_x \in \mathcal{V}_x$, $V_x \subset X_x$ for all $x \in F$.

Since $Z \subset X_p$ if x is not comparable with p, we can erase it from F; since, if $x < p$ then $(X_x \setminus V_x) \cap X_p = X_p \setminus V_x$ and $V_x \in \mathcal{V}_p$, we can assume that $F \subset X_p$ and even that $F \subset Z$.

Let $x_1 < x_2 < \ldots < x_n$ be a maximal chain of F. By Remark 2 we have $V_{x_1}, V_{x_2}, \ldots , V_{x_n} \in \mathcal{V}_{x_n}$; so

$$V = \bigcap_{i=1}^{n} V_{x_i} \in \mathcal{V}_{x_n} \,.$$

Being $x_1, \ldots , x_n$ the only elements of F comparable with x_n we have that, if $x \in F$ is different from them, then $X_x \cap X_{x_n} = \emptyset$ and hence $V \cap (\bigcup_{x \in F}(X_x \setminus V_x)) = \emptyset$, i.e. $Z \cap V \subset T$.

Being $Z \cap X_{x_n} \in \mathcal{W}_{x_n}$ and $x_n \in X_{x_n}$, by property 2 for V we conclude that $T \supset V \cap Z \neq \emptyset$. ∎

LEMMA 2. *Let $U_s \in \mathcal{U}_s$, for all $s \in S$. Then $\{p\} \cup \bigcup_{s \in S} U_s$ is a neighborhood of p in $\{p\} \cup \bigcup_{s \in S} X_s$.*

PROOF. We can suppose that $S \subset X_p$ and $p \notin S$. By taking the minimal elements of S we can suppose that $\bigcup_{s \in S} X_s$ is a disjoint union.

Let $U_s = X_s \setminus \bigcup_{x \in F_s}(X_x \setminus V_x)$ with $F_s \subset X_s$. We have $F = \bigcup_{s \in S} F_s \in \mathcal{F}$.

If we take $U = X_p \setminus \bigcup_{x \in F}(X_x \setminus V_x) \in \mathcal{U}_p$ it results that

$$\Big(U \cup \{p\}) \cap (\{p\} \cup \bigcup_{s \in S} X_s\Big) = \{p\} \cup \bigcup_{s \in S} U_s. \qquad ∎$$

LEMMA 3. *Let $T_p \in \tau_p$ for all $p \in A \subset X$. There exists a clopen partition $\mathcal{P}$ of X so that if $p \in A$, $p \in M \in \mathcal{P}$, then there exists $U_p \in \mathcal{U}_p$:*

$$T_p \cap M = U_p \cap M \,.$$

PROOF. Let $T_p = U_p \cap Z_p$ with $U_p \in \mathcal{U}_p, Z_p \in \mathcal{W}_p$.

Let, for each $p \in A$, $\lambda^p = (\lambda^p_i)_{i \in \omega} \in \Lambda$ be such that $Z^*_p = W_{\lambda_p}$ (see Notation 4 for Z^*_p). Then $Z_p = Z^*_p \cap X_p$ for each $p \in A$ and if $p = (x_0, \ldots , x_n)$ then $\lambda^p_i = 0$ for each $i \leq n$.

Let

$$W = \bigcap_{p \in A} Z^*_p \,.$$

We have that $W \in \mathcal{W}$, since $\sup_{p \in A} \lambda^p_i \in \omega_i$ because $\lambda^p_i \neq 0 \Longrightarrow p = (x_0, \ldots , x_n)$ with $n < i$. (i.e. $\lambda^p_i \neq 0$ for a number $\leq \omega_{i-1}$ of p's).

Let $\mathcal{P}_W$ be the clopen partition associated to W, hence $\mathcal{P}_W \subset \bigcup_{p\in X} \mathcal{W}_p$; since $W \subset Z_p^*$ we have that $W(p) \subset Z_p^*(p) = Z_p^* \cap X_p = Z_p$.

If $p \in M \cap A$ with $M \in \mathcal{P}_W$, there exists q so that $M = W_q$, then $X_p \cap M = W(p) \subset Z_p$ hence $Z_p \cap M = X_p \cap M$, then

$$T_p \cap M = U_p \cap Z_p \cap M = U_p \cap X_p \cap M = U_p \cap M. \qquad \square$$

Lemma 4. *$H \cap X_d \subset \bigcup_{r\in R} X_r \subset X_d$, and $r \notin \mathrm{cl}(H)$ for all $r \in R \Longrightarrow d \notin \mathrm{cl}(H)$.*

Proof. We can suppose $d \notin R$.

For each $r \in R$, let $T_r \in \tau_r$ such that $T_r \cap H = \emptyset$. Let $T_r = U_r \cap Z_r = U_r \cap Z_r^*$ with $U_r \in \mathcal{U}_r, Z_r \in W_r$ and $Z_r^* \in \mathcal{W}$ given by Notation 4. Since $d < r$ for all $r \in R$, $d \in W = \bigcap_{r\in R} Z_r^* \in \mathcal{W}$ (as in Lemma 3). Hence $U_r \cap W \subset T_r$. Since W is an open neighborhood of d, we have, by Lemma 2, that $(\{d\} \cup \bigcup_{r\in R} U_r) \cap W$ is a neighborhood of d in $(\{d\} \cup \bigcup_{r\in R} X_r)$. Hence $d \notin \mathrm{cl}(H \cap X_d)$, so $d \notin \mathrm{cl}(H)$. $\square$

6 – Properties of the space X

Theorem 1. *X is a Hausdorff, non-regular space.*

Proof. Since all X_p's are clopen, the space X is Hausdorff.

Let p_0 be the first point of X and $C = \{(n) : n \in \omega\}$, so that $X = \{p_0\} \cup \bigcup_{x\in C} X_c$.

Since $C \in \mathcal{F}$, C is a closed set.

For each $c \in C$ let $\{c\} \cup T_c$ be a neighborhood of c with $T_c = U_c \cap Z_c$, $U_c \in \mathcal{U}_c$, $Z_c \in \mathcal{W}_c$.

Obviously, $Z_c^* \supset C$ for each $c \in C$ (see Notation 4 for Z_c^*). Further, as in the proof of Lemma 3, we have that

$$\bigcap_{c\in C} Z_c^* = W \in \mathcal{W} \quad \text{and} \quad T_c \supset U_c \cap W\,.$$

By Lemma 2, $\{p_0\} \cup \bigcup_{x\in C} U_c$ is a neighborhood of p_0 in X and hence $(\{p_0\} \cup \bigcup_{x\in C} U_c) \cap W$ is a neighborhood of p_0 in X contained in $\{p_0\} \cup \bigcup_{x\in C} T_c$. We infer that, to be separated from C, p_0 must be isolated, contrarily to Lemma 1. $\square$

Theorem 2. *X is a HS-space.*

Proof. To show that X is a HS-space we need to show that if Y is a subspace of X, C is a closed subset of Y and D is a closed subset of X with $D \cap \mathrm{cl}(C) = \emptyset$, then there exists an open neighborhood U of C in Y so that if K is a closed subset of U in Y then $D \cap \mathrm{cl}(K) = \emptyset$.

Since D is closed, for each $c \in C$ there exists $T_c \in \tau_c$ not intersecting D.

By Lemma 3 there is a clopen partition $\mathcal{P}$ of X so that, if $c \in C, c \in M \in \mathcal{P}$, then there exists $U_c \in \mathcal{U}_c$ with $T_c \cap M = U_c \cap M$.

Since M is clopen we can assume that $M \supset Y$, so that U_c does not intersect D for each $c \in C$.

Let C^* be the set of all minimal elements of C. For each $c \in C^*$ let

$$U_c = X_c \setminus \bigcup_{x \in F_c} (X_x \setminus V_x)$$

with $F_c \in \mathcal{F}$, $V_x \in \mathcal{V}_x$ for all $x \in F_c$, $c \in F_c \subset X_c$.

For each $x \in F_c$ consider the set R_c^x of all minimal elements of $X_x \cap U_c \cap Y \setminus C$. Put

$$R_c = \bigcup_{x \in F_c} R_c^x \,.$$

We have that $R_c \in \mathcal{F}$. In fact if $z_1 \in R_c^{x_1}$, $z_2 \in R_c^{x_2}$, with $z_1 < z_2$, then $x_1 \leq z_1 < z_2$, $x_2 \leq z_2$. Hence x_1 and x_2 are comparable. We cannot have $x_2 \leq x_1 \leq z_1 < z_2$, by the minimality of z_2 and so $x_1 < x_2$.

Clearly $R = \bigcup_{c \in C^*} R_c \in \mathcal{F}$. By Remark 4

$$U_c \setminus R \in \mathcal{U}_c \,,$$

and, by Remark 3, $\{p\} \cup (U_c \setminus R)$ is an open neighborhood of p for all $p \geq c$. Hence

$$U = C \cup \bigcup_{c \in C^*} (U_c \cap Y \setminus R)$$

is an open neighborhood of C in Y, disjoint from D.

Note that

$$U \setminus C \subset \bigcup_{r \in R} X_r \,.$$

Let $K \subset U$, K closed in Y. Call $H = K \setminus C$.

We have $H \subset U \setminus C \subset \bigcup_{r\in R} X_r$.

Since $R \subset Y \setminus U$, $K \subset U$ and K is closed in Y, we have that $r \notin \mathrm{cl}_X(K)$ for all $r \in R$. Hence $r \notin \mathrm{cl}_X(H)$ for all $r \in R$.

We will show that $d \notin \mathrm{cl}_X(H)$ for each $d \in D$. This fact together with the equality $D \cap \mathrm{cl}_X(C) = \emptyset$ will imply that $D \cap \mathrm{cl}_X(K) = \emptyset$. So, let $d \in D$.

i) If there exists $c \in C^*$ with $d < c$, since

$$H \cap X_d \subset \bigcup_{r\in R\cap X_d} X_r \subset X_d\,,$$

we obtain, by Lemma 4, that $d \notin \mathrm{cl}_X(H)$.

ii) If d is comparable with no $c \in C^*$, then $X_d \cap (\bigcup_{c\in C^*} X_c) = \emptyset$, hence $X_d \cap (\bigcup_{r\in R} X_r) = \emptyset$, therefore $d \notin \mathrm{cl}_X(H)$.

iii) If there exists $c \in C^*$ with $c < d$, (i.e. $d \in X_c$), then $H \cap X_d \subset U_c \cap Y \setminus R$ and also $H \cap X_d \subset U_c \cap Y \setminus C$.
Since

$$d \notin U_c = X_c \setminus \bigcup_{x\in F_c} (X_x \setminus V_x)$$

there is an $x \in F_c$ with $d \in X_x \setminus V_x$, and so

$$H \cap X_d \subset X_x \cap (U_c \cap Y \setminus C)\,.$$

We have that $H \cap X_d \subset X_d \cap \cup\{X_r : r \in R_c^x\}$. Let's prove that there exists no $r \in R_c^x$ such that $r < d$. Indeed, we have that $d \notin V_x$. The set V_x is of the form $V_x = \cup\{X_s : s \in S\}$. Since $X_x \setminus V_x \neq \emptyset$, we obtain that $s \not\leq x$ for each $s \in S$. Hence $d \notin \cup\{X_s : s \in S \cap X_x\}$. Since $x \in F_c \subset X_c$, we have that $R_c^x \subset X_x \cap U_c \subset X_x \setminus \cup\{X_y \setminus V_y : y \in F_c\} = \cap\{(X_x \setminus X_y) \cup (X_x \cap V_y) : y \in F_c\} \subset X_x \cap V_x = \cup\{X_s : s \in S \cap X_x\}$. Suppose that there exists $r \in R_c^x$ such that $r < d$. Then there exsists $s' \in S \cap X_x$ such that $r \in X_{s'}$. Therefore, $d \in X_{s'}$. This is a contradiction. So there is no $r \in R_c^x$ such that $r < d$. This imply that

$$H \cap X_d \subset \bigcup_{r\in R_c^x\cap X_d} X_r \subset X_d\,.$$

Again by Lemma 4, we have $d \notin \mathrm{cl}(H)$. □

It remains open the

QUESTION. Is there a regular HS-space which is not normal?

The reason of this question is Theorem 2.14 from [BD2N2] where it is proved that Schmidt's conjecture is equivalent to the following one: any Hausdorff HS-space is normal. So, it is natural to ask whether a regular HS-space is normal. This question is raised in the cited above paper [3] and an approach to its solution is proposed there.

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