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Sharp integrability of nonnegative Jacobians

L. GRECO

Riassunto: *Si mostra che i risultati di migliore sommabilità, noti per lo jacobiano di una applicazione che conserva l'orientamento, sono conseguenza diretta delle stime per l'operatore massimale. In questo modo si estendono alcuni risultati di* [11]. *Si dimostra anche l'ottimalità dei risultati ottenuti.*

Abstract: *We show that improved integrability results for the jacobian of an orientation preserving mapping can be obtained as a consequence of maximal inequalities. With this approach, we extend some results of* [11]. *We also show sharpness of our results.*

1 – Introduction

Let Ω be an open subset of $\mathbb{R}^n$, $n \geq 2$. We shall consider mappings $f = (f^1, \ldots, f^n)$ on Ω in Sobolev classes; the distributional gradient will be denoted by ∇f. For such mappings, the Jacobian determinant

$$J = \det \nabla f = \det(\partial f^i / \partial x_j)$$

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is defined almost everywhere on Ω. Our basic assumption will be $J \geq 0$, that is, f is an orientation preserving mapping.

When studying the Jacobian, the *natural* assumption is $f \in W^{1,n}(\Omega, \mathbb{R}^n)$, as it ensures obviously that $J \in L^1(\Omega)$. Actually, the property of the Jacobian we are mainly concerned with is expressed by the following identity, which holds for any $\varphi \in C_0^\infty(\Omega)$:

$$\int_\Omega \det \nabla(f^1, \dots, f^{n-1}, \varphi f^n)\, dx = 0\,. \tag{1.1}$$

This is easily proved for regular functions f by Stokes' theorem, and then by approximation for functions $f \in W^{1,n}$ or, more generally, $f \in W^{1,\mathbf{p}}$, with $\mathbf{p} = (p_1, \dots, p_n)$ an n-tuple of Hölder conjugate exponents: $p_i \geq 1$, $1/p_1 + \dots + 1/p_n = 1$.

In [17] S. Müller first stressed out the local $L \log L$-integrability of the Jacobian of an orientation preserving mapping, under the natural assumption. This result can be deduced from (1.1). Indeed, choosing φ a cut-off function we obtain the uniform estimate

$$⨍_Q J\, dx \leq C(n) \left(⨍_{2Q} |\nabla f|^{\frac{n^2}{n+1}} dx \right)^{\frac{n+1}{n}}\,, \tag{1.2}$$

for each cube Q such that the double cube $2Q$ is contained in Ω, with $C(n)$ independent of Q. Throughout the paper, the symbol $⨍$ denotes, as usual, the integral mean over the domain of integration. Estimate (1.2) yields a pointwise inequality between maximal functions of J and $|\nabla f|^{\frac{n^2}{n+1}}$ and then Müller's result follows by maximal inequalities, see e. g. [11].

Subsequently to the paper by Müller, in [13] for the first time the natural assumption was relaxed. There, the following estimate was proved for a mapping $f \in W^{1,n-\varepsilon}(\mathbb{R}^n, \mathbb{R}^n)$, $-\infty < \varepsilon \leq 1$ (with no conditions on the sign of the Jacobian)

$$\int_{\mathbb{R}^n} |\nabla f^1|^{-\varepsilon} J\, dx \leq C(n)\, |\varepsilon| \int_{\mathbb{R}^n} |\nabla f|^{n-\varepsilon} dx \tag{1.3}$$

and a local version of this for an orientation preserving mapping $f \in$

$W^{1,n-\varepsilon}(\Omega,\mathbb{R}^n)$

$$\fint_Q |\nabla f^1|^{-\varepsilon} J\,dx \le C(n)\left(\fint_{2Q} |\nabla f|^{\frac{n(n-\varepsilon)}{n+1}} dx\right)^{\frac{n+1}{n}} + C(n)\,|\varepsilon| \fint_{2Q} |\nabla f|^{n-\varepsilon}\,dx\ , \tag{1.4}$$

for $2Q \subset \Omega$. From (1.4), an improved integrability result for the Jacobian, dual to the Müller's one, was deduced: if $|\nabla f|^n$ is in the Zygmund class $L^1 \log^{-1} L(\Omega)$, then $J \in L^1_{\rm loc}(\Omega)$. Furthermore, estimate (1.2) is still valid. The problem in relaxing the natural assumption is that, in general, the Jacobian may not be integrable and (1.1) is meaningless. To prove (1.3), the key ingredient in [13] for overcoming these difficulties was Hodge decomposition to write

$$|\nabla f^1|^{-\varepsilon}\nabla f^1 = \nabla g + h\ ,$$

where $g \in W^{1,\frac{n-\varepsilon}{1-\varepsilon}}(\Omega)$ and $h \in L^{\frac{n-\varepsilon}{1-\varepsilon}}(\Omega,\mathbb{R}^n)$ is a divergence free vector field. Hence (1.1) can be used with $\nabla g = |\nabla f^1|^{-\varepsilon}\nabla f^1 - h$ in place of ∇f^1 and (1.3) follows by estimates for Hodge decomposition, see [12], and [14] for a more recent presentation.

The approach of [13] was then carried on in many papers [3], [6], [7], [9], [11], [18], [19]. In particular, in [11] the phenomenon of the improved integrability of the Jacobian is deeply studied. Essentially, the result of that paper can be stated as follows: for an orientation preserving mapping f on Ω, if $\Phi(|\nabla f|^n) \in L^1_{\rm loc}(\Omega)$, then $\Psi(J) \in L^1_{\rm loc}(\Omega)$, where Ψ and Φ are suitably related Orlicz functions. To present in a clear fashion the tools used there for the proofs, we consider separately the integrability results for the Jacobian which are above L^1-degree and the results which are below L^1-degree, i.e. the convex case and the concave case, respectively, in the terminology of [11].

The convex case, essentially, was proved using (1.2) in conjunction with maximal inequalities. We have to mention here that in the results of [11] a gap was left, concerning the case $t \prec \Psi(t) \prec t\,\log\log t$. This gap was filled then by [16] and [19].

The concave case was treated by a technique (first introduced in [7]) of *averaging* (1.4) with respect to ε, see also [19]. See [10] for further developments of this technique. The drawback of this method is that it seems to yield the result for the Jacobian only for particular functions Φ and Ψ, namely, functions which are somehow Laplace transforms; see [11] for more details.

Our goal in this paper is to show that the results of [11] can be obtained and to some extent generalized as a quite direct consequence of maximal inequality, via the following classical approximation argument.

LEMMA 1.1. *There exists a constant $C = C(n) > 0$ with the following property: for any $u \in W^{1,p}(\Omega, \mathbb{R}^m)$, $1 \leq p < \infty$, and any $t > 0$, there exists $g = g_t \in \mathrm{Lip}(\Omega, \mathbb{R}^m) \cap W^{1,p}(\Omega, \mathbb{R}^m)$ such that $g(x) = u(x)$ for a.e. $x \in \Omega$ verifying $M\,|\nabla u|(x) \leq t$, and $\|\nabla g\|_\infty \leq C\,t$.*

Here Ω is a cube or the whole space $\mathbb{R}^n$, and M denotes the (local to Ω) Hardy-Littlewood maximal operator. A proof of Lemma 1.1 can be inferred from some calculations in [1], see also [8].

To illustrate our approach, we consider a sample case: we derive (1.3) with $0 < \varepsilon < 1$ for a mapping f not necessarily orientation preserving. Notice that the essence of that inequality is the presence of the factor ε, for ε small. To prove it, we apply Lemma 1.1 to the function $u = f^1$ and find $g \in W^{1,n-\varepsilon}(\mathbb{R}^n) \cap \mathrm{Lip}(\mathbb{R}^n)$. Then, by Stokes' theorem $\det \nabla(g, f^2, \dots, f^n)$ has zero integral over $\mathbb{R}^n$, and therefore

$$\int_{M \leq t} J\,dx \leq C\,t \int_{M > t} |\nabla f|^{n-1}\,dx\ .$$

Now we multiply both sides by $t^{-\varepsilon-1}$ and integrate over $(0, \infty)$ with respect to t; notice that it is legitimate to use Fubini theorem to change the order of integration. So we get

$$\int_{\mathbb{R}^n} M^{-\varepsilon} J\,dx \leq C\,\frac{\varepsilon}{1-\varepsilon} \int_{\mathbb{R}^n} M^{1-\varepsilon} |\nabla f|^{n-1}\,dx\ .$$

By Young and Hadamard inequalities we find that

$$|\nabla f^1|^{-\varepsilon} J \leq (1-\varepsilon)\,M^{-\varepsilon} J + \varepsilon\,M^{1-\varepsilon} |\nabla f|^{n-1}$$

Moreover, the integral of $M^{1-\varepsilon}|\nabla f|^{n-1}$ can be estimated first using Hölder inequality and then maximal theorem as follows

$$\leq \Big(\int_{\mathbb{R}^n} M^{n-\varepsilon}\,dx\Big)^{\frac{1-\varepsilon}{n-\varepsilon}} \Big(\int_{\mathbb{R}^n} |\nabla f|^{n-\varepsilon}\,dx\Big)^{\frac{n-1}{n-\varepsilon}} \leq C \int_{\mathbb{R}^n} |\nabla f|^{n-\varepsilon}\,dx\ .$$

Hence we conclude with (1.3).

Now we come to present our results. We consider a nonnegative increasing function $\Phi \in C^1(0,\infty)$. Even though some of our results can be stated in greater generality, for simplicity we assume that

$$t \mapsto \Phi(t)\, t^{-\frac{n}{n+1}} \quad \text{is increasing.} \tag{1.5}$$

Also, since we are interested in integrability properties of local character, if necessary, we modify Φ in order to guarantee that $\Phi'(t)/t$ is integrable near the origin.

First, we consider the convex case, that is,

$$\int_1^\infty \frac{\Phi'(\tau)}{\tau}\, d\tau = \infty\ . \tag{1.6}$$

The following result fills the mentioned gap in [11].

THEOREM 1. *Let Q be a cube of $\mathbb{R}^n$ and $f = (f^1, \ldots, f^n)\colon 2Q \to \mathbb{R}^n$ be an orientation preserving mapping with $\Phi(|\nabla f|^n) \in L^1(2Q)$. Then $J \in L^1(Q)$ and*

$$⨍_Q J\, dx \le C(n) \left(⨍_{2Q} |\nabla f|^{\frac{n^2}{n+1}} dx \right)^{\frac{n+1}{n}}\ . \tag{1.7}$$

As already remarked, estimate (1.7) can be used in conjunction with maximal inequalities, to obtain improved integrability results for the Jacobian in Orlicz spaces. Here we do not examine this topic and refer to [11] for more details.

Next, we consider the concave case:

$$\int_1^\infty \frac{\Phi'(\tau)}{\tau}\, d\tau < \infty\ . \tag{1.8}$$

Now we define

$$\Psi(t) = t \int_t^\infty \frac{\Phi'(\tau)}{\tau}\, d\tau\ . \tag{1.9}$$

The following result generalizes analogous results of [11] for the concave case.

THEOREM 2. *Let Q be a cube and $f\colon 2Q \to \mathbb{R}^n$ be an orientation preserving mapping with $\Phi(|\nabla f|^n) \in L^1(2Q)$. Then $\Psi(J) \in L^1(Q)$ and*

$$\fint_Q \Psi(J)\,dx \le C(n,\Phi)\Big(1+\fint_{2Q} \Phi(|\nabla f|^n)\,dx\Big)^{\frac{n+1}{n}} \tag{1.10}$$

In this paper we are mainly concerned with orientation preserving mappings. It is however appropriate to mention here that regularity properties of Jacobians and other non-linear quantity for general mappings are studied in [4], [15].

2 – Notation and preliminary results

We begin by introducing the (local) maximal operator. For simplicity, we assume that Ω is a cube of $\mathbb{R}^n$. For $0 \le h \in L^1(\Omega)$, the Hardy-Littlewood maximal function of h is defined by

$$Mh(x) = \sup\Big\{\fint_Q h\,; \quad x \in Q \subset \Omega\Big\}\,,$$

the supremum being taken over all subcubes of Ω containing the given point $x \in \Omega$. We recall the following result from [11].

LEMMA 2.1. *If $\Gamma \in C^1(0,\infty)$ is a nonnegative function such that $\Gamma(t)/t^p$ is increasing for some $p>1$, then*

$$\int_\Omega \Gamma(Mh)\,dx \le \frac{3^n p}{p-1}\int_\Omega \Gamma(2h)\,dx \tag{2.1}$$

By (1.5), $\Gamma(t) = \Phi(t^n)$ verifies assumptions of Lemma 2.1 with $p = n^2/(n+1)$.

In the sequel, we shall use the

LEMMA 2.2. *If $u \in W^{1,1}(\Omega)$, then for a. e. $x \in \Omega$ we have*

$$\Big|u(x) - \fint_\Omega u(y)\,dy\Big| \le \operatorname{diam}(\Omega)\, M|\nabla u|(x)\,.$$

Proof. The inequality is proved easily for a function of class $C^1(\Omega)$, and then for a general function in $W^{1,1}(\Omega)$ by approximation.

We shall need to study formula (1.9). For notational simplicity, we introduce the operator

$$\mathcal{F}\Phi(t) = t \int_t^\infty \frac{\Phi'(\tau)}{\tau}\, d\tau \tag{2.2}$$

on the set $\mathcal{L}$ of nonnegative increasing functions $\Phi \in C^1(0,\infty)$ such that $\Phi'(t)/t$ is integrable in a neighborhood of ∞. We collect some elementary properties of $\mathcal{F}$ in the following lemma. These properties are intended for t large.

Lemma 2.3. *For Φ and $\Phi_1 \in \mathcal{L}$, we have:*

i) *if $\Phi(t) = t^\alpha$, with $0 < \alpha < 1$, then $\mathcal{F}\Phi(t) = \frac{\alpha}{1-\alpha}\, t^\alpha$;*

ii) *the map $\Phi \mapsto \mathcal{F}\Phi$ is "monotonic", in the sense that, if $\Phi_1 = \delta\,\Phi$ with δ decreasing, then $\mathcal{F}\Phi_1 \le \delta\,\mathcal{F}\Phi$;*

iii) *if $t \mapsto \Phi(t)\, t^{-\alpha}$ is decreasing, with $0 < \alpha < 1$, then $\mathcal{F}\Phi(t) \le \frac{\alpha}{1-\alpha}\,\Phi(t)$; if $t \mapsto \Phi(t)\, t^{-\alpha}$ is increasing, then $\mathcal{F}\Phi(t) \ge \frac{\alpha}{1-\alpha}\,\Phi(t)$.*

Proof. Assertion i) is trivial. To prove ii), we see that $\Phi_1' = \delta'\,\Phi + \delta\,\Phi' \le \delta\,\Phi'$ and hence $\mathcal{F}\Phi_1 \le \delta\,\mathcal{F}\Phi$. Finally, iii) is an easy consequence of i) and ii).

Notice that iii) and (1.5) imply that $\Psi = \mathcal{F}\Phi$ is "larger" than Φ, hence Theorem 2 exhibits an integrability improvement for J.

We prove now another inequality relating Φ and Ψ.

Lemma 2.4. *If (1.8) holds and Ψ is given by (1.9), then for any t, $s > 0$ we have*

$$\Psi(t) \le t\, \frac{\Psi(s)}{s} + \Phi(s) \tag{2.3}$$

Proof. As $\tau \mapsto \Psi(\tau)/\tau$ is decreasing, inequality (2.3) is trivial if $t > s$. Assuming $t \leq s$, we then compute

$$\Psi(t) = t \int_t^\infty \frac{\Phi'(\tau)}{\tau}\, d\tau = t \int_t^s \frac{\Phi'(\tau)}{\tau}\, d\tau + t \int_s^\infty \frac{\Phi'(\tau)}{\tau}\, d\tau \leq$$

$$\leq \int_0^s \Phi'(\tau)\, d\tau + t \int_s^\infty \frac{\Phi'(\tau)}{\tau}\, d\tau \leq \Phi(s) + t\, \frac{\Psi(s)}{s}\ .$$

3 – Proof of Theorem 1

Let R denote the edge of Q. We pick $\varphi \in C_0^\infty(2Q)$, $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on Q and $|\nabla\varphi| \leq C(n)/R$, and set $F = \big(f^1, \dots, f^{n-1}, \varphi\,(f^n - f^n_{2Q})\big)$, where f^n_{2Q} is the integral mean of f^n over the cube $2Q$. On the other hand, applying Lemma 1.1 to $f \in W^{1,1}(2Q, \mathbb{R}^n)$, for each $t > 0$ we find $g = g_t \in \mathrm{Lip}(2Q, \mathbb{R}^n)$ such that $g(x) = f(x)$ if $M(x) := M|\nabla f|(x) \leq t$, and $|\nabla g| \leq C\,t$. We set $G = G_t = \big(g^1, \dots, g^{n-1}, \varphi\,(g^n - f^n_{2Q})\big)$. Clearly, by Stokes' theorem $\det \nabla G$ has zero integral on $2Q$, that is, by the properties of g

$$\int_{M \leq t} \det \nabla F\, dx = -\int_{M > t} \det \nabla G\, dx \tag{3.1}$$

Moreover

$$\int_0^\infty \frac{\Phi'\big((t/2)^n\big)}{t} \Big| \int_{M>t} \det \nabla G\, dx \Big|\, dt < \infty \tag{3.2}$$

To not distract ourself from the main course of the proof, we postpone the verification of (3.2) until the end of this Section. By (3.1) and (3.2), we see that the function

$$t \mapsto \frac{\Phi'\big((t/2)^n\big)}{t} \int_{M \leq t} \det \nabla F\, dx$$

is integrable over $(0, \infty)$. As by assumption (1.6) the function $\Phi'\big((t/2)^n\big)/t$ is not integrable, then the limit

$$\lim_{t \to \infty} \int_{M \leq t} \det \nabla F\, dx \tag{3.3}$$

must to be zero, if it exists. Let us compute the limit. Obviously, we have the following point-wise equality

$$\det \nabla F = \varphi \det \nabla f + (f^n - f^n_{2Q}) \det \nabla (f^1, \dots , f^{n-1}, \varphi) \,. \tag{3.4}$$

The integral of the second term in the right hand side is easily bounded by Hölder and Sobolev-Poincaré inequalities

$$\begin{aligned} &⨍_{2Q} \big|(f^n - f^n_{2Q}) \det \nabla (f^1, \dots , f^{n-1}, \varphi)\big| \, dx \leq \\ &\leq \frac{C(n)}{R} ⨍_{2Q} |f^n - f^n_{2Q}| \, |\nabla f|^{n-1} \, dx \leq \\ &\leq \frac{C(n)}{R} \left(⨍_{2Q} |f^n - f^n_{2Q}|^{n^2} \, dx \right)^{\frac{1}{n^2}} \left(⨍_{2Q} |\nabla f|^{\frac{n^2}{n+1}} \, dx \right)^{\frac{n^2-1}{n^2}} \leq \\ &\leq C(n) \left(⨍_{2Q} |\nabla f|^{\frac{n^2}{n+1}} \, dx \right)^{\frac{n+1}{n}} \leq \\ &\leq C(n, \Phi) \left(1 + ⨍_{2Q} \Phi(|\nabla f|^n) \, dx \right)^{\frac{n+1}{n}} \,. \end{aligned} \tag{3.5}$$

Last inequality is a consequence of (1.5), since this implies $t^{\frac{n}{n+1}} \leq \big(1 + \Phi(t)/\Phi(1)\big)$. As also $J \geq 0$, it is now clear that we can pass to the limit in (3.3) to get

$$\int_{2Q} \det \nabla F \, dx = 0 \tag{3.6}$$

which, in conjunction with (3.4) and (3.5), immediately implies (1.7).

Now we prove (3.2). First, notice that in (3.1) the integral in the left hand side, and thus the one in the right hand side, is a measurable function of t. Moreover, if $M|\nabla f|(x) > t$ for a.e. x, then the integral is zero. In the other case, we take x_0 such that $M|\nabla f|(x_0) \leq t$ and $g^n(x_0) = f^n(x_0)$ so that, by the properties of g and φ and also using Lemma 2.2, we have

$$|\det \nabla G| \leq C(n) \, (t^n + t^{n-1}(|g^n - g^n(x_0)| + |f^n(x_0) - f^n_{2Q}|) \, |\nabla \varphi|) \leq C(n) \, t^n \,.$$

Therefore, by maximal inequality (2.1)

$$
\begin{aligned}
&\int_0^\infty \frac{\Phi'\big((t/2)^n\big)}{t}\Big|\int_{M>t}\det\nabla G\,dx\Big|\,dt \le \\
&\le C(n)\int_0^\infty \Phi'\big((t/2)^n\big)\,dt^n\int_{M>t}dx \le \\
&\le C(n)\int_{2Q}\Phi((M/2)^n)\,dx \le C(n)\int_{2Q}\Phi\big(|\nabla f|^n\big)\,dx
\end{aligned}
\tag{3.7}
$$

concluding the proof.

Remark. Equality (3.6) means that integration by parts can be carried on, that is, distributional and point-wise determinants coincide, see [2], [5]. This generalizes a result of [6].

4 – Proof of Theorem 2

We define F and G as in the proof of Theorem 1, thus we have (3.1), (3.4) and hence

$$
\begin{aligned}
\int_{M\le t}\varphi\,J\,dx \le \frac{C(n)}{R}\int_{M\le t}&|f^n-f^n_{2Q}|\,|\nabla f|^{n-1}\,dx+ \\
&+\Big|\int_{M>t}\det\nabla G\,dx\Big|\,.
\end{aligned}
\tag{4.1}
$$

Now we multiply both sides by $\Phi'\big((t/2)^n\big)/t$ and integrate over $(0,\infty)$ with respect to t; we examine each term of (4.1) in this process. Using Fubini theorem to change the order of integration ($J\ge 0$), by the definition of Ψ we get

$$
\int_0^\infty \frac{\Phi'\big((t/2)^n\big)}{t}\,dt\int_{M\le t}\varphi\,J\,dx = \frac{1}{n}\int_{2Q}\varphi\,J\,\frac{\Psi\big((M/2)^n\big)}{(M/2)^n}\,dx
\tag{4.2}
$$

and similarly

$$
\begin{aligned}
&\int_0^\infty \frac{\Phi'((t/2)^n)}{t}\,dt\int_{M\le t}|f^n-f^n_{2Q}|\,|\nabla f|^{n-1}\,dx = \\
&\qquad= \frac{1}{n}\int_{2Q}|f^n-f^n_{2Q}|\,|\nabla f|^{n-1}\,\frac{\Psi((M/2)^n)}{(M/2)^n}\,dx\,.
\end{aligned}
\tag{4.3}
$$

Finally, the integral of last term is estimated in (3.7).

Recall that $\Psi(t)/t$ is bounded, so the right hand side of (4.3) can be controlled by

$$\leq C(n,\Phi)\int_{2Q} |f^n - f^n_{2Q}|\,|\nabla f|^{n-1}\,dx$$

and then we can estimate further as in (3.5). On the other hand, inequality (2.3) yields

$$\Psi(J) \leq J\,\frac{\Psi\big((M/2)^n\big)}{(M/2)^n} + \Phi\big((M/2)^n\big)\,. \tag{4.4}$$

Using (4.4), (4.1), (4.2), (4.3), (3.7), (3.5) and maximal inequality (2.1), we prove (1.10).

5 – An example

We show now that Theorem 2 is optimal, in the sense that for any function Θ growing faster than Ψ at infinity, there exists a mapping f whose Jacobian J does not change sign and $\Phi(|\nabla f|^n)\in L^1$, but $\Theta(|J|)\notin L^1$. The results of this section complement Section 7 of [11].

We produce a mapping on the unit ball $B=\{x\,:\,r=|x|\leq 1\}$, which is locally Lipschitz on $B\backslash\{0\}$. Precisely, we prove the following

Proposition 5.1. *Under condition* (1.8)*, let the function* Ψ *defined by* (1.9) *verify* $\Psi(t)\geq C\,t^{1/n}$*, for large* t*. Then, for every function* $\Theta\in C^1(0,\infty)$ *such that*

$$\lim_{t\to\infty}\frac{\Theta(t)}{\Psi(t)}=\infty\,, \tag{5.1}$$

there exists a mapping f *on* B *such that* $J\leq 0$, $\Phi(|\nabla f|^n)\in L^1(B)$, $\Psi\big(|J|\big)\in L^1(B)$*, but* $\Theta\big(|J|\big)\notin L^1_{\mathrm{loc}}(B)$.

We consider a mapping

$$f(x) = \gamma(|x|)\, x\ ,$$

where $\gamma \in \mathrm{Lip}_{\mathrm{loc}}(]0,1])$ is positive. The following point-wise equality holds

$$\nabla f(x) = \gamma'(|x|)\, \frac{x \otimes x}{|x|} + \gamma(|x|)\, \mathcal{I} \tag{5.2}$$

and so

$$\begin{gathered} |\nabla f|^2 = (n-1)\,\gamma^2 + (\gamma + r\,\gamma')^2\ , \\ J = \det \nabla f = \gamma^{n-1}(\gamma + r\,\gamma')\ . \end{gathered} \tag{5.3}$$

We assume $J \le 0$, that is $\gamma + r\,\gamma' \le 0$. From (5.3) we get

$$(r\,\gamma)^n = \gamma(1)^n - n \int_r^1 \rho^{n-1} J(\rho)\, d\rho \tag{5.4}$$

Moreover, if the function $r \mapsto -r^p J(r)$ is increasing for some $p > n$, then $\gamma(1) > 0$ can be chosen so that

$$0 \le -\frac{\gamma + r\,\gamma'}{\gamma} \le \frac{p-n}{n} \tag{5.5}$$

and $|\nabla f| \sim \gamma$.

Lemma 5.1. *Under assumption* (1.8)*, if* Ψ *is defined by* (1.9) *and inequality* (5.5) *holds, then* $\Psi(-J) \in L^1 \implies \Phi(|\nabla f|^n) \in L^1$*.*

Proof. As $t \mapsto \Psi(t)/t$ is decreasing, (5.5) implies for any $k > (p-n)/n$

$$\Psi(-J) = \Psi\Big(-\frac{\gamma + r\,\gamma'}{\gamma}\,\gamma^n\Big) \ge -\frac{\gamma + r\,\gamma'}{\gamma}\,\frac{\Psi(k\,\gamma^n)}{k}$$

and thus, integrating by parts twice yields, for $r > 0$

$$\begin{aligned} n\,k \int_r^1 \Psi(-J)\,\rho^{n-1}\, d\rho &\ge -\int_r^1 \frac{\Psi(k\,\gamma^n)}{\gamma^n}\, \big((\rho\,\gamma)^n\big)'\, d\rho \ge \\ &\ge -\Psi(k\,\gamma(1)^n) - \int_r^1 \big(\Phi(k\,\gamma^n)\big)'\,\rho^n\, d\rho \ge \\ &\ge -\Psi(k\,\gamma(1)^n) - \Phi(k\gamma(1)^n) + n \int_r^1 \Phi(k\gamma^n)\,\rho^{n-1} d\rho\,. \end{aligned}$$

To conclude the proof, it suffices to let r go to 0.

To construct our example, for a given Θ verifying (5.1), we only need to show a function $J \le 0$ such that $r \mapsto -r^p J$ is increasing and

$$\int_0^1 \Psi(-J)\, r^{n-1}\, dr < \infty \ , \ \text{but} \ \int_0^1 \Theta(-J)\, r^{n-1}\, dr = \infty \ .$$

The following lemma deals with the existence of such a function.

Lemma 5.2. *Let Ψ, Θ be continuous functions on $[0,\infty[$ verifying (5.1) and $\Psi(0)=0$, $\Psi(t) \ge C\, t^{n/p}$, for t large. Then there exists an increasing function $g\colon]0,1] \to [0,\infty[$ so that*

$$\int_0^1 \Psi\big(g(r)\, r^{-p}\big)\, r^{n-1}\, dr < \infty \ , \ \textit{but} \ \int_0^1 \Theta\big(g(r)\, r^{-p}\big)\, r^{n-1}\, dr = \infty \ .$$

Proof. We are going to construct $g = \sum g_k\, \chi_{I_k}$, with g_k constants and I_k pair-wise disjoint subintervals of $]0,1]$. Let us pick up a sequence (b_k) of positive numbers such that $\sum b_k < \infty$, but $\sum k\, b_k = \infty$. By (5.1), for each $k \in \mathbb{N}$ we can find $T_k > 0$ so that $t > T_k \implies \Theta(t)/\Psi(t) \ge k+1$. If we define $M_k = \max\big\{\Psi(t)\,,\ 0 \le t \le T_k\big\}$, then

$$\Psi(t) > M_k \implies t > T_k \implies \frac{\Theta(t)}{\Psi(t)} \ge k+1$$

We want to define g_k and $I_k =]a_{k+1}, a_k]$ recursively. Let us start with $a_0 = 1$, set $b_0 = 1$ and consider the equation

$$\int_{a_1}^{a_0} \Psi\big(g_1\, r^{-p}\big)\, r^{n-1}\, dr = b_0 \tag{5.6}$$

in the two unknowns $a_1 \in (0, a_0)$ and $g_1 > 0$. For every fixed g_1, we have

$$\int_{a_0}^{a_0} \Psi\big(g_1\, r^{-p}\big)\, r^{n-1}\, dr = 0 \ , \int_0^{a_0} \Psi\big(g_1\, r^{-p}\big)\, r^{n-1}\, dr \ge C\, g_1^{n/p} \int_0^{\tilde a} \frac{dr}{r} = \infty \ .$$

Therefore we can solve (5.6). Moreover, as $\Psi(0)=0$ and Ψ is continuous, choosing g_1 small enough, we can find a_1 as close to 0 as we like. Then we select a solution (a_1, g_1) to (5.6) so that

$$\int_0^{a_1} r^{n-1}\, dr = \frac{a_1^n}{n} < \frac{b_1}{2M_1} \ .$$

It is clear that we can iterate the above argument, to construct (g_k) and (a_k) decreasing sequences verifying $\forall k \in \mathbb{N}$

$$\int_{a_{k+1}}^{a_k} \Psi\big(g_{k+1}\, r^{-p}\big)\, r^{n-1}\, dr = b_k \text{ and } \int_0^{a_k} r^{n-1}\, dr < \frac{b_k}{(k+1)\, M_k} \ .$$

Let us now show that

$$\int_{a_{k+1}}^{a_k} \Theta\big(g_{k+1}\, r^{-p}\big)\, r^{n-1}\, dr > k\, b_k \ . \tag{5.7}$$

Set $A_k = \big\{r \in (a_{k+1}, a_k) \ : \ \Psi\big(g_{k+1}\, r^{-p}\big) > M_k\big\}$ and $B_k = (a_{k+1}, a_k)\backslash A_k$. Then

$$\int_{B_k} \Psi\big(g_{k+1}\, r^{-p}\big)\, r^{n-1}\, dr \le M_k \int_0^{a_k} r^{n-1}\, dr < \frac{b_k}{k+1}$$

and hence

$$\int_{A_k} \Psi\big(g_{k+1}\, r^{-p}\big)\, r^{n-1}\, dr > b_k - \frac{b_k}{k+1} = \frac{k}{k+1}\, b_k \ .$$

From this, the definitions of A_k and M_k we get (5.7) and by the properties of the sequence (b_k) we conclude the proof.

Now we choose $p = n^2$; we remark that condition $\Psi(t) \ge C\, t^{1/n}$ of Proposition 5.1 holds by iii) of Lemma 2.3 under the assumption that $t \mapsto \Phi(t)\, t^{-1/n}$ is increasing, which is weaker than (1.5).

In conclusion, we set $-J = r^{-n^2} g$, with g increasing, and define γ by means of (5.4). Notice that $\Phi\big(|\nabla f|^n\big) \in L^1 \implies |\nabla f| \sim \gamma \in L^1(B)$. Moreover by (5.4) we easily get $f \in L^{\frac{n}{n-1}}(B)$ and therefore equality (5.2) holds also in the sense of distributions.

Proposition 5.1 shows also optimality of Theorem 1, in particular of condition (1.6) for (1.7). Actually, if (1.6) is false, that is, (1.8) holds, $\Psi(t)$ given in (1.9) measures the best degree of integrability of J, for general f, and it is essentially smaller than $\Theta(t) = t$.

Concerning optimality of formula (1.9) and significance of Theorem 2 itself, the following remark is of some interest.

Remark. Let $\Phi \in \mathcal{L}$, that is, (1.8) holds, and let $\Psi = \mathcal{F}\Phi$. Lemma 2.3 implies that, if $\Phi(t)\, t^{-\alpha}$ is decreasing for some $\alpha \in (0,1)$, e. g. $\alpha = n/(n+1)$, then $\Psi(t) \le \alpha\, \Phi(t)/(1-\alpha) \sim \Phi(t)$. By Hadamard inequality, $\Phi\big(|\nabla f|^n\big) \in L^1$ trivially implies $\Psi\big(|J|\big) \in L^1$ and this is optimal, as shown by the above example.

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Indirizzo dell'Autore:

L. Greco – Dipartimento di Matematica e Applicazioni "R. Caccioppoli" – Università degli Studi di Napoli "Federico II" – Via Cintia – 80126 Napoli, Italy
E-mail: greco@matna2.dma.unina.it