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Evans-Vasilesco theorem in Dirichlet spaces

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Riassunto: *Si dimostra il teorema di Evans-Vasilesco per una classe generale di forme di Dirichlet di tipo diffusione. Questo risultato è utilizzato per provare che ogni misura di Radon diffusa e nulla sugli insiemi di capacità nulla può essere espressa come il prodotto di una funzione boreliana e di una misura di Kato.*

Abstract: *The Evans-Vasilesco Theorem is proved for a general class of Dirichlet forms of diffusion type. This result is applied to show that every diffuse Radon measure vanishing on all sets of capacity zero can be expressed as the product of a Borel function and a Kato measure.*

1 – Introduction

The purpose of this paper is to extend a classical potential theoretic result, the Evans-Vasilesco Theorem, to the case of a Dirichlet-Poincaré form of diffusion type on a connected, locally compact, separable Hausdorff space X. An interesting application of this result is the decomposition of a non-negative diffuse Radon measure, vanishing on all sets of capacity zero, as the product of a non-negative Borel function and a non-negative Kato measure. This result permits to reduce some problems for arbitrary Radon measures to the more regular case of Kato measures, which have been introduced in [6] in the framework of Dirichlet spaces to

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study the continuity of the solutions to some equations associated with a Dirichlet form. Our decomposition theorem is used in [8] to study the asymptotic behaviour of a sequence of Dirichlet problems on varying domains for Dirichlet forms of diffusion type.

The classical Evans-Vasilesco Theorem on a bounded open subset Ω of $\mathbb{R}^n$ asserts that the potential $G_\Omega\mu$ of a non-negative Radon measure μ on Ω is continuous at a point ξ of the support F of μ in Ω if and only if its restriction $(G_\Omega\mu)|_F$ is continuous at ξ (see, e.g., [10, 1.V.8]). The proof is based on the standard estimates of the Green's function.

By using this theorem, the decomposition mentioned above has been proved in [2, Proposition 2.5] for all uniformly elliptic operators in $\mathbb{R}^n$, using also a previous representation result in terms of measures in the class $H^{-1}(\mathbb{R}^n)$ proved in [9, Lemma 4.15].

In this paper we extend the Evans-Vasilesco Theorem and the decomposition result to the more general context of Dirichlet-Poincaré forms. More precisely, we consider a strongly local regular Dirichlet form $(a[\cdot,\cdot], D[a])$ on $L^2(X,m)$ in the sense of [11] and we require that the measure m on X satisfies a doubling property with respect to the *intrinsic* balls, *i.e.*, the balls in the metric induced by the Dirichlet form; moreover we assume that the Dirichlet form $(a[\cdot,\cdot], D[a])$ satisfies a Poincaré inequality on the intrinsic balls.

In some recent papers ([3]-[6] and [15]) M. Biroli and U. Mosco developed a theory which extends the classical results of the variational theory of (second order) uniformly elliptic equations to the more general context of Dirichlet-Poincaré forms, which includes a wide class of degenerate elliptic operators with discontinuous coefficients, such as weighted and sub-elliptic operators. In [5] and [6] the notions of Green's functions and Kato measures relative to a Dirichlet form have been introduced and some classical properties have been established in this more general framework. In particular an estimate of the Green's function has been proved in [5], which depends on the intrinsic structure of the space.

The form of this estimate induces some difficulties in the proof of the Evans-Vasilesco Theorem in this context (Theorem 4.1), which are solved by using the doubling property of the measure m with respect to the intrinsic balls.

To prove the Evans-Vasilesco Theorem it is useful to introduce the notion of *weak solution* (in the duality sense of [14] and [16]) for a Dirichlet problem with an arbitrary bounded Radon measure μ in the right hand side, and to extend to these solutions a representation formula by means of the Green's function proved in [6] only for variational solutions corresponding to Radon measures which belong to the dual of the space $D[a]$.

By using Theorem 4.1 we shall prove (Theorem 4.3) that every diffuse non-negative Borel measure μ vanishing on all sets of capacity zero is equivalent (in the sense of Definition 2.2) to a measure that can be written as the product of a non-negative Borel function and a non-negative Kato measure. In the proof of this result we follow the outlines of the proof in [2] and we use a preliminary decomposition analogous to Lemma 4.15 in [9], stated in our general context in [17, Remark 4(2)]. A different proof can be obtained by using Theorem 2.4 in [1] (see also [12, Theorem 2.7]).

We remark that in the classical case of the Laplace operator on the Euclidean space $\mathbb{R}^n$ all measures μ vanishing on all sets of capacity zero are diffuse (*i.e.*, $\mu(\{x\}) = 0$ for every x), since all singletons have capacity zero. On the contrary in the context of Dirichlet forms a singleton can have a strictly positive capacity. We note that, as proved in [6], every Kato measure is diffuse, so that a measure which charges a singleton with positive capacity can not be absolutely continuous with respect to a Kato measure.

The plan of the paper is the following. In Section 2 we fix the notation and we recall some notions and some results proved in [5], [6] and [11]. In Section 3 we introduce the notion of weak solution corresponding to a measure and we study some properties of these solutions. Moreover we recall the notions of Green's function and Kato measure. In Section 4 we prove our main results: the extension of the Evans-Vasilesco Theorem and the representation result for a diffuse measure vanishing on all sets of capacity zero.

2 – Preliminaries on Dirichlet forms

Let X be a *connected, locally compact, separable Hausdorff space* and let m be a positive Radon measure on X, with $\operatorname{supp} m = X$.

Let us consider a *strongly local regular Dirichlet form* $a[\cdot,\cdot]$ on the Hilbert space $L^2(X,m)$, whose domain will be denoted by $D[a]$. For the definition and the standard properties of Dirichlet forms we refer to the book by Fukushima [11] and to the recent papers [3]-[6] and [15]. The form $a[\cdot,\cdot]$ admits the representation

$$a[u,v] = \int_X \alpha[u,v](dx)$$

for u, $v \in D[a]$, where $\alpha[\cdot,\cdot]$ is a Radon-measure-valued non-negative definite symmetric bilinear form, which is called the *energy measure* of $a[\cdot,\cdot]$. For the definition and the main properties of $\alpha[\cdot,\cdot]$ we refer to [13] and [5].

Since the form $a[\cdot,\cdot]$ is *regular*, there exists a core C which is dense both in the space $C_c(X)$ of continuous function with compact support, endowed with the uniform norm, and in $D[a]$, endowed with the intrinsic norm $\|u\|_a^2 = a[u,u] + \int_X u^2 m(dx)$. We assume that C is an *m-separating* core, *i.e.*, for every x, $y \in X$ with $x \neq y$ there exists $\phi \in C$ such that $\alpha(\phi,\phi) \leq m$ on X and $\phi(x) \neq \phi(y)$.

For every open subset Ω of X, the closure of $C_c(\Omega) \cap D[a]$ in $D[a]$ for the intrinsic norm is denoted by $D_0[a,\Omega]$, while $D_{\rm loc}[a,\Omega]$ is the space of all functions u defined in Ω such that for every $U \subset\subset \Omega$ there exists $w \in D[a]$ with $u = w$ m-a.e. in U.

The dual of $D_0[a,\Omega]$ is denoted by $D_0'[a,\Omega]$ and the corresponding duality pairing by $\langle\cdot,\cdot\rangle$. The positive cone $D_0'[a,\Omega]_+$ is the set of all $T \in D_0'[a,\Omega]$ such that $\langle T,u\rangle \geq 0$ for every $u \in D_0[a,\Omega]$ with $u \geq 0$ m-a.e. in Ω.

The intrinsic distance $d\colon X \times X \to [0,+\infty]$ associated with the Dirichlet form $a[\cdot,\cdot]$ is defined by

$$d(x,y) = \sup\{\varphi(x) - \varphi(y) : \varphi \in C, \alpha[\varphi,\varphi] \leq m \text{ on } X\}\,.$$

We assume that the metric topology is equivalent to the given topology of X. In the following, we denote by $B(x,r)$ the *intrinsic ball* centred at x with radius r, *i.e.*, $B(x,r) = \{y \in X : d(x,y) < r\}$. We assume that the measure m on X satisfies the duplication condition with respect to the intrinsic balls, *i.e.*, there exist two constants $R_0 > 0$ and $C_0 > 1$ such that

$$0 < m(B(x,2r)) \leq C_0 m(B(x,r)) < +\infty$$

for every $x \in X$ and for every $0 < r < R_0$.

We assume also that the following *Poincaré inequality* holds on the balls $B(y,r)$ with $0 < r < R_0$: for every $u \in D[a]$ we have

$$\int_{B(y,r)} |u - u_{B(y,r)}|^2 m(dx) \leq C_1 r^2 \int_{B(y,kr)} \alpha[u,u](dx)\,,$$

where $k \geq 1$ and $C_1 > 0$ are constants independent of y and r, and $u_{B(y,r)}$ denotes the average of u on $B(y,r)$.

Following [11, Section 3.1], we consider a notion of capacity in X relative to the form $a[\cdot,\cdot]$, and denoted by cap^a. The expression quasi everywhere (abbreviated as q.e.) means "except on a set of capacity zero".

Let Ω be an open set in X.

Definition 2.1. By $\mathcal{M}_0^a(\Omega)$ we denote the space of all non-negative Borel measures μ on Ω which are absolutely continuous with respect to cap^a, *i.e.*, $\mu(B) = 0$ for every Borel set $B \subset \Omega$ with $\mathrm{cap}^a(B) = 0$.

We introduce an equivalence relation in the class $\mathcal{M}_0^a(\Omega)$.

Definition 2.2. We say that two non-negative measures μ and λ belonging to $\mathcal{M}_0^a(\Omega)$ are *equivalent* (and we write $\mu \simeq \lambda$) if $\int_\Omega u^2 \mu(dx) = \int_\Omega u^2 \lambda(dx)$ for every $u \in D_0[a,\Omega]$. Here and in the rest of the paper we identify the function u with its quasi-continuous representative (see [11, Chapter 3]).

Remark 2.3. As one can see in [11, Section 3.2], for every functional $T \in D_0'[a,\Omega]_+$ there exists a non-negative Radon measure $\mu \in \mathcal{M}_0^a(\Omega)$ such that $\langle T, u\rangle = \int_\Omega u\,\mu(dx)$ for every $u \in D_0[a,\Omega]$. We shall always identify T with μ.

With the form $a[\cdot,\cdot]$ we associate the linear operator $A\colon D_0[a,\Omega] \to D_0'[a,\Omega]$ defined by $\langle Au,\, v\rangle = a[u,v]$, for every $u,\, v \in D_0[a,\Omega]$.

3 – Weak solutions, Green's functions, and Kato measures

Let Ω be an open set in X. Suppose that Ω is contained in a ball $B(x_0,R)$ such that $B(x_0,2R)$ is relatively compact and different from X. By Theorem 1.1 of [4] there exists a constant $c > 0$ such that

$$\int_\Omega u^2 m(dx) \leq c\,a[u,u]$$

for every $u \in D_0[a,\Omega]$. This implies that for every $f \in D_0'[a,\Omega]$ the problem

$$\begin{cases} Av = f & \text{in } D_0'[a,\Omega], \\ v \in D_0[a,\Omega], \end{cases} \tag{3.1}$$

has a solution by the Lax-Milgram Lemma. By the definition of A, a function $v \in D_0[a,\Omega]$ is a solution of (3.1) if and only if

$$a[v,z] = \langle f, z\rangle \qquad \forall z \in D_0[a,\Omega]\,.$$

We refer to [5] for the main properies of the solution v.

Given a bounded Radon measure μ on Ω we shall consider also the Dirichlet problem

$$\begin{cases} Au = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Following [14, Sections 5 and 8] and [16, Section 9] we introduce the notion of weak solution to this problem and we prove a representation formula for this weak solution by means of the Green's function $G_\Omega(x,y)$ corresponding to A.

Definition 3.1. Let μ be a bounded Radon measure. We say that u is a *weak solution* of the Dirichlet problem (3.2) if $u \in L^1(\Omega,m)$ and if

$$\int_\Omega u(x) f(x)\, m(dx) = \int_\Omega v(x)\, \mu(dx) \qquad \forall f \in C_c(\Omega)\,,$$

where v is the solution to (3.1).

We notice that the solution v is continuous in Ω by Theorem 5.13 in [5]. Thanks to the L^∞-estimate for v proved in that paper (Theorem 4.1), the proof of existence and uniqueness of the solution of (3.2) can be obtained by adapting the arguments in [14] and [16]. Moreover, using the same technique as in [14], one can prove that there exists a constant $p > 1$ such that we have the estimate

$$\|u\|_{L^p(\Omega,m)} \leq C\ |\mu|(\Omega), \tag{3.3}$$

where C depends only on Ω and C_0, C_1, R_0. This implies that, if (μ_k) converges to μ weakly in the sense of measures, then the corresponding solutions of (3.2) converge weakly in $L^p(\Omega, m)$ to the solutions corresponding to μ. It is easy to see that, if the measure μ belongs to $D'_0[a,\Omega]$, then u coincides with the solution v of problem (3.1) with $f = \mu$.

Definition 3.2. Given $x \in \Omega$, we define $G_\Omega(x,\cdot)$ as the unique solution of (3.2) with $\mu = \delta_x$, the Dirac mass at x. The function $G_\Omega(\cdot,\cdot)$ is called the Green's function of the form $a[\cdot,\cdot]$ in the open set Ω.

In the following proposition we recall the estimate of the Green's function proved in [5, Theorem 1.3].

Proposition 3.3. *Assume that with $20r < R_0$ and that the ball $B(x, 40r)$ is relatively compact and different from X. Then for every $y \in X$ with $0 < d(x,y) < r/16$ we have*

$$\frac{1}{c}\int_{d(x,y)}^{r} \frac{s\ ds}{m(B(x,s))} \le G_{B(x,r)}(x,y) \le c\int_{d(x,y)}^{r} \frac{s\ ds}{m(B(x,s))}. \tag{3.4}$$

The constant c depends only on the constants C_0, C_1, and R_0.

Notice that the above estimate, together with the doubling property for the measure m, implies

$$G_{B(x,r)}(x,y) \le c' \int_{4d(x,y)}^{4r} \frac{s\,ds}{m(B(x,s))} \tag{3.5}$$

for every y such that $0 < d(x,y) < r/16$, where $c' = c\,C_0^2/16$.

Moreover, for every $\mu \in D'_0[a,\Omega]_+$ the solution u of (3.2) satisfies

$$u(x) = \int_\Omega G_\Omega(x,y)\,\mu(dy)$$

for m-a.e. $x \in \Omega$ (see [6, Proposition 3.2]). By repeating the same proof of Theorem 6.1 in [14] this representation holds also if μ is an arbitrary bounded Radon measure.

For every non-negative bounded diffuse Radon measure μ on Ω the potential of μ relative to the Dirichlet form $a[\cdot,\cdot]$ is the function $G_\Omega\mu : \Omega \to [0,+\infty]$ defined by

$$(G_\Omega\mu)(x) = \int_\Omega G_\Omega(x,y)\,\mu(dy)$$

for every $x \in \Omega$. It coincides m-a.e. with the solution u of (3.2).

Proposition 3.4. *Suppose that Ω_1 and Ω_2 satisfy the conditions assumed for Ω at the beginning of this section. Let μ_1 and μ_2 be two non-negative bounded diffuse Radon measures in Ω_1 and Ω_2 respectively, and let $u_1 = G_{\Omega_1}\mu_1$ and $u_2 = G_{\Omega_2}\mu_2$. Suppose that μ_1 and μ_2 coincide on an open set $U \subset \Omega_1 \cap \Omega_2$. Then $u_1 - u_2$ is continuous in U.*

Proof. Let (f_k^1) and (f_k^2) be two sequences in $L^2(\Omega_1, m)$ and $L^2(\Omega_2, m)$ respectively, which converge to μ_1 and μ_2 weakly in the sense of measures, and let $u_k^1 \in D_0[a, \Omega_1]$ and $u_k^2 \in D_0[a, \Omega_2]$ be the corresponding solutions of (3.1). By (3.3) the sequences (u_k^1) and (u_k^2) converge weakly to u_1 and u_2 in $L^p(\Omega_1, m)$ and $L^p(\Omega_2, m)$ respectively.

Let V be a relatively compact open subset of U. Since μ_1 and μ_2 coincide on U, we may suppose that f_k^1 and f_k^2 coincide m-a.e. on V. In this case the function $u_k = u_k^1 - u_k^2$ belongs to $D_{\mathrm{loc}}[a, V]$ and is a local solution of the equation $Au = 0$ in V.

By applying Caccioppoli's inequality (see [5, Propositions 5.1 and 5.3]) for every open set $W \subset\subset V$ we obtain that $\int_W \alpha[u_k, u_k](dx)$ is bounded uniformly with respect to k. This implies that $u_1 - u_2 \in D_{\mathrm{loc}}[a, W]$, and since W and V are arbitrary we have also $u_1 - u_2 \in D_{\mathrm{loc}}[a, U]$. Moreover, passing to the limit, we obtain that $u_1 - u_2$ is a local solution of the equation $Au = 0$ in U, and by the regularity results (see [5, Corollary 1.2 and Theorem 5.4]) this implies that $u_1 - u_2$ is continuous on U. □

Following [6], we recall the notion of Kato measure.

Definition 3.5. Let Ω be a relatively compact open subset of X. Let us assume that $\mathrm{diam}(\Omega) = R/2$, with $R < R_0$, and that there exists $x_0 \in \Omega$ such that $B(x_0, 4R)$ is relatively compact in X and $B(x_0, 4R) \neq X$. We say that μ is a *Kato measure on* Ω if μ is a Radon measure on Ω such that

$$\lim_{r \downarrow 0} \sup_{x \in \Omega} \int_{\Omega \cap B(x,r)} \left[\int_{d(x,y)}^{R} \frac{s\, ds}{m(B(x,s))} \right] |\mu|(dy) = 0\,,$$

where $|\mu|$ denotes the total variation of the measure μ. The space of the Kato measures is denoted by $K(\Omega)$, while $K^{\mathrm{loc}}(\Omega)$ indicates the space of all Radon measures μ on Ω such that $\mu \in K(\Omega')$ for every open set

$\Omega' \subset\subset \Omega$. The sets of non-negative elements of $K(\Omega)$ and $K^{\rm loc}(\Omega)$ are denoted by $K_+(\Omega)$ and $K_+^{\rm loc}(\Omega)$ respectively.

REMARK 3.6. $K(\Omega)$ is a Banach space with the norm

$$\|\mu\|_{K(\Omega)} = \sup_{x\in\Omega} \int_\Omega \left[\int_{d(x,y)}^R \frac{s\,ds}{m(B(x,s))} \right] |\mu|(dy)$$

(see [6, Theorem 2.7]). Every $\mu \in K(\Omega)$ is a diffuse measure, *i.e.*, $\mu(\{x\}) = 0$ for every $x \in \Omega$ (see [6, Proposition 2.3]). Moreover, if $\mu \in K(\Omega)$ and g is a bounded Borel function, then $g\mu \in K(\Omega)$.

PROPOSITION 3.7. *Let Ω be as in Definition 3.5 and let μ be a non-negative bounded Radon measure on Ω. Then μ belongs to $K_+^{\rm loc}(\Omega)$ if and only if μ is diffuse and $G_\Omega\mu$ is finite and continuous on Ω.*

PROOF. If μ belongs to $K_+^{\rm loc}(\Omega)$, then μ is diffuse (see [6, Proposition 2.3]) and $G_\Omega\mu$ is continuous (see [6, Theorem 4.1]).

Conversely, suppose that μ is diffuse and $G_\Omega\mu$ is finite and continuous. Let Ω' be a relatively compact open subset of Ω and let $R' = 2\,\mathrm{diam}(\Omega')$. Let us fix a constant $0 < r < R'/20$ such that $B(x,r) \subset \Omega$ for every $x \in \Omega'$. If $x \in \overline{\Omega}'$, $y \in \Omega$, and $0 < d(x,y) < r/16$, then by (3.4)

$$\int_{d(x,y)}^r \frac{s\,ds}{m(B(x,s))} \le c\,G_{B(x,r)}(x,y) \le c\,G_\Omega(x,y)\,, \tag{3.6}$$

where the last inequality follows from the comparison principle. By the doubling property there exist two constants $k > 0$ and $p > 1$ such that $m(B(x,s)) \ge k\,s^p$ for every $x \in \overline{\Omega}'$ and for every $0 < s < R_0$. Therefore

$$\int_{r/16}^{R'} \frac{s\,ds}{m(B(x,s))} \le \frac{1}{k} \int_{r/16}^{R'} s^{1-p} ds = c(r,R') < +\infty\,. \tag{3.7}$$

From (3.6) and (3.7) we obtain

$$\int_{d(x,y)}^{R'} \frac{s\,ds}{m(B(x,s))} \le c\,G_\Omega(x,y) + c(r,R')$$

for every x, $y \in \overline{\Omega}'$, hence

$$\sup_{x\in\Omega'} \int_{\Omega'} \left[\int_{d(x,y)}^{R'} \frac{s\, ds}{m(B(x,s))}\right] \mu(dy) \leq \sup_{x\in\Omega'} G_\Omega\mu(x) + c(r,R')\,\mu(\Omega') < +\infty\,.$$

Therefore μ belongs to $K(\Omega')$ by Proposition 2.3 in [6]. ∎

4 – The main results

In this section we prove an extension of the Evans-Vasilesco Theorem to the case of Dirichlet-Poincaré forms. We follow the outlines of the proof in [10, 1.V.8], using the estimate of the Green's function and the doubling property of the intrinsic balls. Then we apply this theorem to establish that every diffuse measure belonging to $\mathcal{M}_0^a(\Omega)$ is equivalent to a Borel measure that can be written as the product of a non-negative Borel function and a non-negative Kato measure.

THEOREM 4.1. *Let Ω be an open set in X. Suppose that Ω is contained in a ball $B(x_0,R)$ such that $B(x_0,2R)$ is relatively compact and different from X. Let μ be a non-negative bounded diffuse Radon measure on Ω, let F be the support of μ in Ω, and let $\xi \in F$. If $(G_\Omega\mu)|_F$ is continuous at ξ in F and $(G_\Omega\mu)(\xi) < +\infty$, then $G_\Omega\mu$ is continuous at ξ in Ω.*

PROOF. Let us fix $0 < r < R_0/160$, with $B(\xi,9r) \subset \Omega$, such that $B(\xi,321r)$ is relatively compact and different from X. For every $\rho > 0$ let $\mu^{\xi,\rho}$ be the restriction of the measure μ to the ball $B(\xi,\rho)$, *i.e.*, $\mu^{\xi,\rho}(B) = \mu(B \cap B(\xi,\rho))$ for every Borel set $B \subset \Omega$. Since $\mu(\{\xi\}) = 0$ and $(G_\Omega\mu)(\xi) < +\infty$, we have

$$\lim_{\rho\to 0}\,(G_\Omega\mu^{\xi,\rho})(\xi) = 0\,,$$

hence for every $\varepsilon > 0$ there exists δ, with $0 < \delta < r$, such that

$$(G_\Omega\mu^{\xi,\delta})(\xi) < \varepsilon\,. \tag{4.1}$$

Now we note that, since $\mu^{\xi,\delta}$ coincides with μ on $B(\xi,\delta)$, the function $G_\Omega\mu^{\xi,\delta} - G_\Omega\mu$ is continuous on $B(\xi,\delta)$ by Proposition 3.4; hence

$(G_\Omega \mu^{\xi,\delta})|_F$ is continuous at ξ in F. By (4.1) and by the positivity of $G_\Omega(x',y)$, there exists σ, with $0<\sigma<\delta$, such that

$$(G_\Omega \mu^{\xi,\sigma})(x') \le (G_\Omega \mu^{\xi,\delta})(x') < \varepsilon$$

for every $x' \in B(\xi,\sigma)\cap F$. This implies that

$$\lim_{\rho\to 0} \sup_{x'\in F\cap B(\xi,\rho)} (G_\Omega \mu^{\xi,\rho})(x') = 0\,. \tag{4.2}$$

For every $x \in B(\xi,r)$, let x' be a point of F at minimum distance from x; for every $y\in F$ we have

$$d(x,x') \le d(x,y)\,, \tag{4.3}$$

$$d(x',y) \le d(x',x) + d(x,y) \le 2d(x,y)\,. \tag{4.4}$$

Step 1. If $0<\rho<r/8$, then for every $x\in B(\xi,\rho)$ and for every $y\in B(x,\rho)\cap F$ we have $G_{B(x,2r)}(x,y) \le K\, G_{B(x',8r)}(x',y)$, where K is a constant which depends only on C_0, C_1, and R_0.

By (3.5) for every $x\in B(\xi,\rho)$ we have

$$G_{B(x,2r)}(x,y) \le c' \int_{4d(x,y)}^{8r} \frac{s\; ds}{m(B(x,s))}$$

for every $y\in B(x,\rho)\cap F$. Let us remark that $B(x', s-d(x,x'))\subset B(x,s)$ for $s\ge 4d(x,y)$. Since $4d(x,y)-d(x,x')\ge 3d(x,y)$ by (4.3), we obtain that

$$\begin{aligned} G_{B(x,2r)}(x,y) &\le c' \int_{4d(x,y)}^{8r} \frac{s\; ds}{m(B(x', s-d(x,x')))} \le \\ &\le c' \int_{3d(x,y)}^{8r} \frac{s+d(x,x')}{m(B(x',s))}\; ds\,. \end{aligned}$$

By (4.3) for $s\ge 3d(x,y)$ we have $d(x,x')\le s/3$; thus

$$G_{B(x,2r)}(x,y) \le \frac{4}{3}\; c' \int_{3d(x,y)}^{8r} \frac{s\; ds}{m(B(x',s))}\,.$$

Since, by (4.4), $3\,d(x,y) \geq d(x',y)$, the conclusion of Step 1 follows from (3.4) in Proposition 3.3.

STEP 2. If $0 < \rho < r/8$, then $G_{B(\xi,r)}(x,y) \leq K\, G_\Omega(x',y)$ for every $x \in B(\xi,\rho)$ and for every $y \in B(x,\rho) \cap F$.

This property follows from Step 1, observing that for every $x \in B(\xi,\rho)$ we have $B(\xi,r) \subset B(x,2r)$ and $B(x',8r) \subset B(x,9r) \subset \Omega$.

STEP 3. The function $G_{B(\xi,r)}\mu$ is continuous at ξ.

If $0 < \rho < r/8$, by Step 2 for every $x \in B(\xi,\rho)$ we have

$$\int_{B(\xi,\rho)} G_{B(\xi,r)}(x,y)\,\mu(dy) \leq K \int_{B(\xi,\rho)} G_\Omega(x',y)\,\mu(dy)\,.$$

As $x' \in F \cap B(\xi,\rho)$ for every $x \in B(\xi,\rho/2)$, the previous inequality implies that

$$\sup_{x \in B(\xi,\rho/2)} (G_{B(\xi,r)}\mu^{\xi,\rho})(x) \leq K \sup_{x' \in F \cap B(\xi,\rho)} (G_\Omega \mu^{\xi,\rho})(x')\,.$$

Since $G_{B(\xi,r)}\mu = G_{B(\xi,r)}(\mu - \mu^{\xi,\rho}) + G_{B(\xi,r)}\mu^{\xi,\rho}$ and $G_{B(\xi,r)}(\mu - \mu^{\xi,\rho})$ is continuous at ξ by Proposition 3.4, the conclusion of Step 3 follows from (4.2).

STEP 4. The function $G_\Omega\mu$ is continuous at ξ.

The function $G_\Omega\mu - G_{B(\xi,r)}\mu$ is continuous at ξ by Proposition 3.4. As $G_{B(\xi,r)}\mu$ is continuous at ξ by Step 3, we conclude that $G_\Omega\mu$ is continuous at ξ. □

We shall use the following representation result, proved in [9, Lemma 4.15] in the classical case of the Laplace operator on the Euclidean space $\mathbb{R}^n$, and by Ancona (unpublished) and Stollman (see [17, Remark 4(2)]) in the more general framework considered in the present paper.

PROPOSITION 4.2. *For every $\mu \in \mathcal{M}_0^a(\Omega)$ there exist a Radon measure $\lambda \in D_0'[a,\Omega]_+$ and a Borel function $g : \Omega \to [0,+\infty]$ such that $\mu \simeq g\lambda$.*

We are now in a position to prove our decomposition result.

THEOREM 4.3. *Let* Ω *be as in Definition* 3.5 *and let* μ *be a diffuse measure belonging to* $\mathcal{M}_0^a(\Omega)$. *Then there exist a Borel function* $g\colon\Omega\to[0,+\infty]$ *and a Radon measure* $\lambda\in K_+(\Omega)$ *such that* $\mu\simeq g\lambda$.

PROOF. By Proposition 4.2 we can assume that $\mu\in D_0'[a,\Omega]_+$. In particular, we can assume that μ is a Radon measure on Ω. In this case we have to prove that $\mu=g\lambda$, since two equivalent measures are equal if one of them is a Radon measure.

It is enough to prove the theorem locally. Indeed, assume that every point of Ω has a neighbourhood U such that $\mu=g_U\lambda_U$ on U, where $g_U\colon\Omega\to[0,+\infty]$ is a Borel function and $\lambda_U\in K_+(\Omega)$. By Lindelöf Theorem there exists a countable covering (U_n) composed of these neighbourhoods. Let (E_n) be a Borel partition of Ω such that $E_n\subset U_n$ for every n, let $c_n=2^n\|\lambda_{U_n}\|_{K(\Omega)}$, let $\lambda_n(B)=\lambda_{U_n}(B\cap E_n)/c_n$ for every Borel set $B\subset\Omega$, and let $g_n(x)=c_n g_{U_n}(x)$ for every $x\in E_n$. Then $\mu=g_n\lambda_n$ in E_n and $\|\lambda_n\|_{K(\Omega)}\le 2^{-n}$. Let λ be the Borel measure on Ω defined by $\lambda(B)=\sum_n\lambda_n(B)$ for every Borel set $B\subset\Omega$, and let g be the Borel function on Ω such that $g(x)=g_n(x)$ for every $x\in E_n$. Since $K(\Omega)$ is a Banach space (see [6, Theorem 2.7]), we obtain that $\lambda\in K_+(\Omega)$, and, since $\mu=g_n\lambda_n=g\lambda$ in E_n for every n, we conclude that $\mu=g\lambda$ in Ω.

Therefore it is not restrictive to assume that the support F of μ in Ω is compact. In the rest of the proof we follow the outlines of Proposition 2.5 in [2]. Let $u=G_\Omega\mu$. Since $\mu\in D_0'[a,\Omega]_+$, we have $u(x)<+\infty$ for q.e. (and hence for μ-a.e.) $x\in\Omega$. As in [10, 1. V. 9] we approximate the potential u by continuous potentials u_n by applying Lusin Theorem. More precisely, there exists a disjoint sequence F_n of compact subsets of F such that $\mu(F\setminus\bigcup_n F_n)=0$ and the restrictions $u|_{F_n}$ are finite and continuous on F_n. Let μ_n be the restrictions of μ to F_n, *i.e.*, $\mu_n(B)=\mu_n(B\cap F_n)$ for every Borel set $B\subset\Omega$, and let $u_n=G_\Omega\mu_n$. We claim that $u_n|_{F_n}$ is continuous on F_n. In fact u_n is lower semicontinuous on Ω (and hence on F_n); since $u_n|_{F_n}=u|_{F_n}-(G_\Omega(\mu-\mu_n))|_{F_n}$, and $u|_{F_n}$ is continuous on F_n, while $G_\Omega(\mu-\mu_n)$ is lower semicontinuous on Ω, we conclude that $u_n|_{F_n}$ is upper semicontinuous on F_n. Since $u_n|_{F_n}$ is continuous and μ_n is diffuse and bounded, we can apply Theorem 4.1 and we obtain that u_n is continuous and finite at every point of F_n. Since the continuity of u_n at every point of $\Omega \setminus F_n$ follows from Proposition 3.4, we conclude that u_n is continuous on Ω. By Proposition 3.7 the measures μ_n belong to $K_+^{\rm loc}(\Omega)$. As the support of μ_n in Ω is compact, we conclude that $\mu_n \in K_+(\Omega)$.

Since the sets F_n are disjoint and $\mu(F \setminus \bigcup_n F_n) = 0$, we have $\mu = \sum_n \mu_n$. Let $k_n = 2^{-n}\|\mu_n\|_{K(\Omega)}^{-1}$ and let λ be the Borel mesasure defined by $\lambda(B) = \sum_n k_n \mu_n(B)$ for every Borel set $B \subset \Omega$. As $K(\Omega)$ is a Banach space (see [6, Theorem 2.7]), λ is a Kato measure on Ω. Since μ is absolutely continuous with respect to λ, by the Radon-Nikodym Theorem there exist a Borel function $g\colon \Omega \to [0, +\infty[$ such that $\mu = g\lambda$. □

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