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Intersections of Riemannian submanifolds. Variations on a theme by T. J. Frankel

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RIASSUNTO: *Sia M^n una varietà riemanniana completa e connessa di dimensione n e siano V^r e W^s due sottovarietà compatte e totalmente geodetiche di dimensione r ed s. T. Frankel nel 1961 ha dimostrato che le due varietà si intersecano se è: $r+s \geq n$. Risultati analoghi sono stati ottenuti successivamente da S. I. Goldberg e S. Kobayashi per le varietà kähleriane, da A. Gray per quelle quasi-kähleriane, da S. Marchiafava per le varietà quaternioniche, da L. Ornea per quelle localmente conformi, da S. Tanno e Y. -B. Baik per le varietà sasakiane regolari e compatte. Recentemente K. Kenmotsu e C. Xia hanno ottenuto risultati simili per le varietà con curvatura positiva.*

In questo articolo si dimostrano dapprima risultati dello stesso tipo per le ipersuperficie minimali di una varietà riemanniana. Si presenta poi un metodo per costruire varietà sasakiane con curvatura k-positiva e, sulla falsariga del caso kähleriano, si costruisce un sistema ortonormale di vettori tangenti, invarianti per parallelismo lungo una geodetica di uno spazio sasakiano. Utilizzando questo sistema di vettori si riconosce la validità di un teorema del tipo di Frankel, in uno spazio sasakiano, per due sottovarietà compatte invarianti con curvatura bisezionale positiva.

ABSTRACT: *Let M^n be an n-dimensional complete connected Riemannian manifold with positive sectional curvature, and let V^r and W^s be two compact, totally geodesic submanifolds of dimensions r and s. In 1961 T. Frankel proved that V^r and W^s are always intersecting provided $r + s \geq n$. Later a number of analogous results were achieved: for Kähler manifolds by S. I. Goldberg and S. Kobayashi, for nearly Kähler ones by A. Gray, for quaternionic Kähler manifolds by S. Marchiafava, for locally conformal Kähler manifolds by L. Ornea and for compact regular Sasakian manifolds by S. Tanno and Y. -B. Baik. Recently K. Kenmotsu and C. Xia, have obtained similar results on manifolds with partially positive curvature.*

In this paper we prove results of similar type in a Riemannian space for two minimal hypersurfaces. We present a construction of Sasakian manifolds with k-positive *bisectional curvature. In analogy to the Kähler case, an orthonormal vector system is obtained in a Sasakian manifold, which is parallel along a geodesic. Using this vector system, we prove an intersection theorem of Frankel type for two compact invariant submanifolds of a Sasakian space with k-positive bisectional curvature.*

1 – Introduction

Consider an n-dimensional complete, connected Riemannian manifold M^n and two complete totally geodesic submanifolds V^r and W^s of dimensions r and s, respectively. These submanifolds need not intersect, even if $r + s \geq n$, for example two parallel hypersurfaces of a euclidean space or two deviating hyperplanes in a hyperbolic M^n. However, if moreover M^n has positive sectional curvature K and V^r and W^s $(r + s \geq n)$ are compact, then $V^r \cap W^s \neq 0$, as it was shown by T. Frankel [3]. (The statement is clear on the sphere S^n.)

Later a number of analogous results were achieved by different authors using basically the same method of which we will make use too. This method is the following: Assume on the contrary that $V^r \cap W^s = \emptyset$. Because of the compactness there exists a minimal geodesic between V^r and W^s, such that its second variation L'' is positive. Then using an integral formula for L'' which involves the positive sectional curvature K one arrives to a contradiction proving that $V^r \cap W^s \neq \emptyset$. We mention here, only a few among these results. S. I. Goldberg and S. Kobayashi [5] proved that the compact complex submanifolds V^r and W^s $(r + s \geq n)$ of a compact connected Kähler manifold M^n with positive bisectional curvature (the definition of this notion appears at the beginning of Section 3) have nonempty intersection. A. Gray [4] extended these results on nearly Kähler ambient spaces. S. Marchiafava [9] obtained similar results on quaternionic Kähler manifolds. L. Ornea [11] investigated the problem on locally conformal Kähler manifolds. S. Tanno and Y.-B. Baik [13] proved intersection theorems for two compact invariant submanifolds with positive special bisectional curvature of a regular compact Sasakian space. Recently K. Kenmotsu and C. Xia [7], [8] have achieved similar results on manifolds with partially positive curvature (or in other words, with k-positive sectional curvature), a weaker condi-

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tion than positive sectional curvature. (This notion is explained at the beginning of the next section.)

In this paper we prove results of similar type in Riemannian and Sasakian spaces. We present a construction of Sasakian manifolds with k-positive bisectional curvature and we construct a special orthonormal vector system in this space which is parallel along a geodesic.

2 – Intersection of minimal hypersurfaces in a Riemannian manifold

Let $M^n = (M, g)$ be a Riemannian manifold of dimension n, $p \in M^n$ and $e, e_1, \dots, e_k \in \mathfrak{X}(U_p)$, $(k \leq n-1)$ a system of orthonormal tangent vectors of M in a neighbourhood $U_p \subset M$. $K(e \wedge e_i)$ $i = 1, 2, \dots, k$ denotes the sectional curvature of M^n belonging to the plane spanned by e and e_i. If

$$\text{(1)} \qquad \mathrm{Ric}_{(k)}(p) := \sum_{i=1}^{k} K(e \wedge e_i) > 0 \qquad (\text{respectively} \geq 0),$$

for any orthonormal $e, e_1, \dots, e_k$ and for every $p \in M$, then M^n is said to be of *k-positive* (k-nonnegative) *Ricci curvature* (see [7], p. 130). It is clear that for $k = 1$ (1) means positive (nonnegative) sectional curvature, for $k = n - 1$ it means positive (nonnegative) Ricci curvature $\mathrm{Ric}(M^n)$; and that $\mathrm{Ric}_{(k)}(M^n) \geq 0 \Rightarrow \mathrm{Ric}_{(k+1)}(M^n) \geq 0$ $(k \leq n-2)$.

THEOREM 1. *Let $M^n = (M, g)$ be a complete, connected Riemannian manifold with $\mathrm{Ric}(M^n) \geq 0$, V and W two complete minimal hypersurfaces of M^n immersed as closed submanifolds into M. Let one of V and W be compact and $\mathrm{Ric}(M^n) > 0$ either at all points of V or at all points of W.*

Then V and W are intersecting: $V \cap W \neq \emptyset$.

This theorem is analogous to Theorem 2.1 of [7] and to Theorem 1 of [8]. There V and W are two totally geodesic submanifolds V^r and W^s with dimensions r and s, respectively, of an M^n with k-positive Ricci curvature, such that $r + s \geq n + k - 1$. In [8] also an example is given showing that the positivity of Ric_k on V or W is necessary (Example 2 on page 248). The proof of our Theorem 1 runs similar to that of Theorem 2.1 in [7]. However, the fact that $\dim V = \dim W = n-1$ and that V and W are not totally geodesic, but minimal surfaces only, makes throughout certain differences.

PROOF. Our proof is indirect. Assume that $V \cap W = \emptyset$. Since V and W are closed and one of them is compact, there is a pair of points $p \in V$ and $q \in W$ such that the distance $d(p,q)$ is a smallest among $\{d(a,b) \mid a \in V,\ b \in W\}$. Since M^n is complete there exists a geodesic $\gamma : [0,l] \to M$ such that $L(\gamma) = d(p,q) = l$, where $L(\gamma)$ is the length of γ. We choose the parametrization $\gamma(t)$ of γ in such a way that the length $\|\dot\gamma(t)\| = 1$ (i.e. that $\gamma(t)$ is a normal geodesic). We know that γ meets both V and W orthogonally. Now consider an orthonormal basis $E_1, E_2, \dots, E_{n-1}$ of T_pV and translate it parallel along $\gamma(t)$ to $\gamma(l) = q \in W$. The resulting vector fields $E_i(t)$ $i = 1, 2, \dots, n-1$ are perpendicular to $\dot\gamma(t)$ and hence $E_i(l) \in T_qW$.

Then (see [12] Chap. III., §2, p. 88–91) there exists to each vector field $E_i(t)$ a variation $\alpha_i : [0,l] \times (-\varepsilon, \varepsilon) \to M$ of γ with curves in the variation $\alpha_i(t,s)$, such that $\alpha_i(0,s) \subset V$, $\alpha_i(l,s) \subset W$ and $E_i(t)$ is the variational field $\frac{\partial \alpha_i}{\partial s}(t,0) = E_i(t)$ of α_i. Moreover, $\frac{\partial \alpha_i}{\partial s}(t,s) =: E_i(t,s)$ is an extension of $E_i(t) = E_i(t,0)$ to the range of α_i. By fixing $s \in (-\varepsilon, \varepsilon)$ we obtain the curves ${}_ic_s(t) := \alpha_i(t,s)$ whose arc length is denoted by $L_{E_i}(s)$ or simply by $L_i(s)$. So $\frac{d}{ds} L_i \mid_{s=0} \equiv L_i'(0) = 0$, for $\gamma(t) = {}_ic_0(t)$ is a shortest curve between V and W, and hence also among the curves in the variation α_i.

According to the second variational formula for L (see e.g. [12], p. 91 formula (2.9)) we have

$$\frac{d^2}{ds^2}\bigg|_{s=0} L(s) = \frac{1}{l}\left[\int_0^l \Big(\langle \nabla_{\dot\gamma} X^\perp, \nabla_{\dot\gamma} X^\perp\rangle + - \langle R(X^\perp, \dot\gamma)\dot\gamma, X^\perp\rangle\Big) dt + \langle A_{\dot\gamma} X, X\rangle \mid_p^q \right], \tag{2}$$

where $X(t)$ is the variational field along the variational curve $\gamma(t)$, and A is the shape operator of V, respectively W, with respect to the corresponding normal vectors $\dot\gamma(0), \dot\gamma(l)$. $X^\perp(t)$ denotes the vertical component of $X(t)$ with respect to $\dot\gamma(t) : X^\perp = X - \langle X, \dot\gamma\rangle\dot\gamma$. ∇ is the Riemannian connection in M^n, R is the curvature tensor and $\langle , \rangle$ denotes scalar product in M^n. We apply this for $X(t) = E_i(t)$. Then $E_i^{\perp}(t) = E_i(t)$, for the parallel translated $E_i(t)$ $\quad \nabla_{\dot\gamma} E_i = 0$, $\langle A_{\dot\gamma} E_i, E_i\rangle \,|_p^q = \langle \nabla_{E_i} E_i, \dot\gamma\rangle \,|_p^q$, and $\langle R(E_i, \dot\gamma)\dot\gamma, E_i\rangle = K(\dot\gamma \wedge E_i)$. Thus we obtain

$$\frac{d^2}{ds^2}\bigg|_{s=0} L_i(s) \equiv L_i''(0) = \frac{1}{l}\left[\langle \nabla_{E_i} E_i, \dot\gamma\rangle \,|_p^q - \int_0^l K(\dot\gamma \wedge E_i)dt\right]. \tag{3}$$

Since the E_i restricted to the transversal curve $\alpha_i(0, s)$ through p are tangent to V, we can apply the relation

$$(\nabla_{E_i} E_i)(p) = (\bar\nabla_{E_i} E_i + \sigma_V(E_i, E_i))(p),$$

where $\bar\nabla_{E_i} E_i$ means the component of $\nabla_{E_i} E_i$ tangent to V, and σ_V denotes the second fundamental form of the imbedded V. We have a similar formula for $(\nabla_{E_i} E_i)(q)$ with the second fundamental form σ_W of W. Substituting these into the right-hand side of $L_i''(0)$, taking into consideration that $\bar\nabla_{E_i} E_i$ as tangential component is perpendicular to $\dot\gamma$, and summing up on i, we obtain

$$l\sum_{i=1}^{n-1} L_i''(0) = \left\langle \sum_i \sigma_V(E_i, E_i)(p), \dot\gamma(0)\right\rangle + \\ - \left\langle \sum_i \sigma_W(E_i, E_i)(q), \dot\gamma(l)\right\rangle - \int_0^l \sum_{i=1}^{n-1} K(\dot\gamma \wedge E_i)dt.$$

Now, since V and W are minimal hypersurfaces, $\sum_{i=1}^{n-1} \sigma_V(E_i, E_i)(p) = \sum_{i=1}^{n-1} \sigma_W(E_i, E_i)(q) = 0$. Further $\sum_{i=1}^{n-1} K(\dot\gamma(t) \wedge E_i(t)) = \mathrm{Ric}(\gamma(t)) \geq 0$ and at the endpoints of γ $\mathrm{Ric}(p)$ or $\mathrm{Ric}(q)$ is positive according to the assumption of our Theorem. So we obtain $l\sum_{i=1}^{n-1} L_i''(0) < 0$. However $L_i''(0) \geq 0$, for γ is a shortest curve between p and q. This contradiction shows that $V \cap W = \emptyset$ cannot hold. □

3 – Construction of Sasakian manifolds with k-positive bisectional curvature

Let $M^{2n+1} = (M, \varphi, \xi, \eta, G)$ be a $2n+1$ dimensional Sasakian manifold, where φ is a vector-valued 1-form, ξ is the structure field, η is a 1-form and G is a Riemannian metric. They satisfy the relations (see e.g. [1])

$$
(4)\qquad
\begin{aligned}
&\text{(a)}\quad (\nabla_X\varphi)Y = \langle X, Y\rangle\xi - \eta(Y)X\\
&\text{(b)}\quad \eta\circ\varphi = 0\\
&\text{(c)}\quad \varphi^2 = -\operatorname{id} + \eta\otimes\xi\\
&\text{(d)}\quad \varphi(\xi) = 0\\
&\text{(e)}\quad \langle\varphi X, \varphi Y\rangle = \langle X, Y\rangle - \eta(X)\eta(Y)\\
&\text{(f)}\quad \nabla_Y\xi = \varphi Y\\
&\text{(g)}\quad \langle\varphi X, Y\rangle + \langle X, \varphi Y\rangle = 0.
\end{aligned}
$$

The bisectional curvature of a Kähler manifold $M^n = (P, J, g)$ of complex dimension n with the complex structure J is defined on a plane spanned by X and Y as (see [5], [7])

$$H(X, Y) := \frac{R(X, JX, Y, JY)}{\langle X, X\rangle\langle Y, Y\rangle} = \frac{R(X, Y, X, Y) + R(X, JY, X, JY)}{\langle X, X\rangle\langle Y, Y\rangle},$$

where

$$R(X, Y, Z, V) = \left\langle \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, V\right\rangle .$$

Hence

$$H(X, Y) = K(X\wedge Y) + K(X\wedge JY),$$

where X, JX, Y, JY is an orthonormal system. M^n is said to be of k-positive (respectively, k-nonnegative) bisectional curvature ($1\le k\le n$) (see [7], p. 133) if for any orthonormal system $X, e_1, Je_1, \dots, e_k, Je_k \in T_pP$

$$\sum_{i=1}^{k} H(X, e_i) > 0 \qquad (\text{respectively} \ge 0),$$

at every $p\in P$.

Similarly we call a Sasakian manifold M^{2n+1} of *k-positive* (k-nonnegative) *bisectional curvature* ($k\le n$) if

$$\sum_{i=1}^{k} H(X\wedge E_i) := \sum_{i=1}^{k}(K(X\wedge E_i) + K(X\wedge\varphi E_i)) > 0 \qquad (\text{respectively} \ge 0),$$

for any orthonormal system $X, E_1, \varphi E_1, \dots, E_k, \varphi E_k \in T_pM$.

First we want to give examples for Sasakian manifolds with k-positive bisectional curvature. These examples show at the same time the existence of manifolds of this kind. To this end let us consider the Boothby-Wang fibration (see [2]) of a compact regular Sasakian manifold $(M, \varphi, \xi, \eta, G)$ on the Kähler manifold $(P = M/\xi, J, g)$ with fiber S^1 identified with an orbit of ξ (cf. Blair's book [1], for instance). It is known that the construction is reversible (see Y. Hatakeyama [6]). In other words if one starts with a compact Kähler manifold whose fundamental 2-form Ω defines an integral cocycle (this is called a Hodge manifold) then the Sasakian manifold can be constructed. (K. Yano and M. Kon [14] p. 291).

The above Boothby-Wang fibration $\pi : M \to P$ is, moreover, a Riemannian submersion with totally geodesic fiber S^1. Denoting with A and T the O'Neill tensors of the submersion (see O'Neill [10]) we have $T = 0$, because the fibers are totally geodesic. Let $\mathcal{H}, V$ be the horizontal, respectively vertical distributions of the submersion, and h, v the corresponding projectors. Obviously, V is locally spanned by ξ. We recall that $A_X Y = v\overset{M}{\nabla}_X Y$, $A_X U = h\overset{M}{\nabla}_X U$, $A_U X = 0$, $\quad X, Y, U \in \mathfrak{X}(M)$ for any horizontal X, Y and vertical U.

Let X^* be the horizontal lift on M of a vector field X on P. Then one can easily prove the following relations (see [14] page 456):

$$(JX)^* = \varphi X^*, \qquad G(X^*, Y^*) = g(X, Y),$$
$$(\overset{P}{\nabla}_X Y)^* = -\overset{M}{\nabla}_X Y + G(Y^*, \varphi X^*)\xi,$$

(the superscripts M, respectively P, denote notions in the Sasakian manifold M, respectively in the Kähler manifold P). Thus one deduces

$$\begin{aligned}(R^P(X, Y, Z))^* = R^M(X^*, Y^*)Z^* + G(Z^*, \varphi Y^*)\varphi X^* +\\ - 2G(Y^*, \varphi X^*)\varphi Z^* - G(Z^*, \varphi X^*)\varphi Y^*,\end{aligned}$$

and consequently, for the sectional curvature (X, Y orthonormal) one obtains:

$$K^P(X \wedge Y) = K^M(X^* \wedge Y^*) + 3G(X^*, \varphi Y^*)^2.$$

Note that $G(X^*, \varphi Y^*)^2 = \|A_X Y\|^2$. As for an other type of a 2-plane in M, using the O'Neill formulae for the curvature, we have:

$$K^M(X^* \wedge \xi) = \|A_X \xi\|^2 .$$

Consider now a local orthonormal system of vector fields $E_1, \varphi E_1, \dots, E_k, \varphi E_k$ on M as in the definition of the k-positive bisectional curvature. Three possibilities may arise:

(1) All E_i and X are horizontal. In this case they can be considered the "stars" of some vector fields $e_1, \dots, e_k, x$ on P. We have:

$$K^M(X \wedge E_i) + K^M(X \wedge \varphi E_i) = K^P(x \wedge e_i) + K^P(x \wedge Je_i) + \\ - 3(\|A_X E_i\|^2 + \|A_X \varphi E_i\|^2).$$

Since E_i and X are horizontal (normal to ξ) and mutually orthogonal, we obtain $(\overset{M}{\nabla}_X \varphi)E_i = 0$. Thus $\overset{M}{\nabla}_X \varphi E_i = \varphi \overset{M}{\nabla}_X E_i$. But φ preserves the horizontal distribution and $\varphi\xi = 0$, consequently $\overset{M}{\nabla}_X \varphi E_i$ is horizontal. This means that $A_X \varphi E_i = 0$. Moreover

$$A_X E_i = v\overset{M}{\nabla}_X E_i = \eta(\overset{M}{\nabla}_X E_i) - G(\overset{M}{\nabla}_X E_i, \xi) = \\ = G(E_i, \varphi X) = -G(\varphi E_i, X) = 0,$$

because X is orthogonal to all E_i and φE_i. Hence

$$\sum_{i=1}^{k} \{K^M(X \wedge E_i) + K^M(X, \varphi E_i)\} = \sum_{i=1}^{k} \{K^P(x \wedge e_i) + K^P(x, Je_i)\}.$$

(2) All E_i are horizontal, and $X = \xi$ is vertical. It is known that on Sasakian manifolds the two-planes containing ξ have sectional curvature 1. Hence the sum in the definition equals $2k$.

(3) X is horizontal, and one of the E_i-s is vertical. We may suppose $\xi = E_k$. Since $\varphi\xi = 0$ and, as above, $K^M(X \wedge \xi) = 1$, using the computation of case 1), we obtain

$$\sum_{i=1}^{k} \{K^M(X \wedge E_i) + K^M(X \wedge \varphi E_i)\} = \sum_{i=1}^{k-1} \{K^P(x \wedge e_i) + K^P(x \wedge Je_i)\} + 1.$$

In conclusion, in order to obtain examples of Sasakian manifolds with k-positive bisectional curvature it is enough to choose as base space of the Boothby-Wang fibration a Kähler and Hodge manifold with $(k-1)$-nonnegative bisectional curvature. Kähler manifolds with non-negative k-bisectional curvature are considered and investigated in Section 3 of Kenmotsu and Xia's paper [7]. The above statements are summarized in

Theorem 2. *The above construction yields* (*shows the existence of*) *Sasakian manifolds with k-positive bisectional curvature.*

4 – Orthonormal and parallel vector fields in a Sasakian manifold

Let $E_1(t), JE_1(t), \ldots, E_k(t), JE_k(t)$ be vector fields along a curve $\gamma(t)$ of a Kähler manifold (P, J, g). If they are orthonormal at $\gamma(0)$ and $E_1(t), \ldots, E_k(t)$ are parallel along $\gamma(t)$, then the whole system is orthonormal at every point of $\gamma(t)$. This is not true for $E_1(t), \varphi E_1(t), \ldots, E_k(t)$, $\varphi E_k(t)$ in a Sasakian manifold, because of $\nabla\varphi \neq 0$. However we want to prove the following

Proposition. *Let $\gamma(t)$ be a normal geodesic in a Sasakian manifold M^{2n+1} and N a linear subspace of $T_{\gamma(0)}M$ of dimension $2 \leq \dim N = m \leq n$, such that N is perpendicular to $\dot\gamma(0) \equiv T(0)$. Then there exist orthonormal vector systems $e_1, \ldots, e_k \in N$, $k = [\frac{m}{2}] - 1$, such that the parallel translated $E_i(t)$ $i, j = 1, 2, \ldots, k$ of $e_i = E_i(0)$ along $\gamma(t)$, completed with $\varphi E_i(t)$ (i.e. $E_1(t), \varphi E_1(t), \ldots, E_k(t), \varphi E_k(t)$) form an orthonormal system at any point of $\gamma(t)$.*

Proof. Let ν and ρ be the projections of $\varphi T(0)$ respectively $\xi(\gamma(0)) \equiv \xi_0$ on N. Thus $\varphi T(0) = (\varphi T(0))^n + \nu$ and $\xi_0 = \xi_0^n + \rho$, where $(\varphi T(0))^n$ and ξ_0^n are the normal components with respect to N. Let now $e_1, \ldots, e_k$ $k = [\frac{m}{2}] - 1$ be unit vectors in N with the following properties:

$$e_1 \in (\{\nu, \rho\}^\perp \cap N) \equiv \hat N_1\,;$$
$$e_2 \in (\{\nu, \rho, e_1, \varphi e_1\}^\perp \cap N) \equiv \hat N_2\,;$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$e_k \in (\{\nu, \rho, e_1, \varphi e_1, \ldots, e_{k-1}, \varphi e_{k-1}\}^\perp \cap N) \equiv \hat N_k\,.$$

These $e_1, \dots e_k$ exist, since for $i \leq k$

$$\dim \hat{N}_i = \dim(\{\nu, \rho, e_1, \varphi e_1, \dots, e_{i-1}, \varphi e_{i-1}\}^\perp \cap N) \geq$$
$$\geq m - 2i \geq m - 2\left(\left[\frac{m}{2}\right] - 1\right) \geq m - (m-2) = 2,$$

and $e_1, \varphi e_1, \dots, e_k, \varphi e_k$ are orthonormal.

Let us translate e_i $i = 1, 2, \dots, k$ parallel along $\gamma(t)$. We get $E_i(t)$, $E_i(0) = e_i$. We state that $E_1(t), \varphi E_1(t), \dots, E_k(t), \varphi E_k(t)$ is an orthonormal system at any point of $\gamma(t)$. Indeed:

(i) $\langle E_i, E_j \rangle = \langle e_i, e_j \rangle = \delta_{ij}$, (ii) For $\langle E_i, \varphi E_j \rangle$ we show that its covariant derivative along γ in the direction of $\dot{\gamma}(t) \equiv T(t)$ vanishes, and therefore it is constant. Indeed

$$\nabla_T \langle E_i, \varphi E_j \rangle = \langle E_i, (\nabla_T \varphi) E_j \rangle = \langle E_i, \langle T, E_j \rangle \xi - \eta(E_j) T \rangle = 0,$$

by (4) and the parallelism and orthogonality of E_i and T. Therefore $\langle E_i, \varphi E_j \rangle \mid_t = \langle e_i, \varphi e_j \rangle = 0$. (iii) Using again (4) and the parallelism and orthogonality of E_i and T, we obtain $\langle \varphi E_i, \varphi E_j \rangle = \langle E_i, E_j \rangle - \eta(E_i)\eta(E_j) = \delta_{ij} - \eta(E_i)\eta(E_j)$. Hence the φE_i are orthonormal if $\eta(E_i)$ vanish. We know that $\eta(E_i(t)) = \langle E_i(t), \xi \rangle$, and we denote this by $f_i(t)$. We show that $f_i(t)$ satisfies the differential equation $f_i'' = -f_i$ and $f_i(0) = f_i'(0) = 0$. Indeed $f_i'(t) = \nabla_T \langle E_i, \xi \rangle = -\langle E_i, \varphi T \rangle$ and $f_i''(t) = \nabla_T^2 \langle E_i, \xi \rangle = -\langle E_i, (\nabla_T \varphi) T \rangle = -\langle E_i, \langle T, T \rangle \xi - \eta(T) T \rangle = -\langle E_i, \xi \rangle = -f_i(t)$. Furthermore $f_i(0) = \langle E_i, \xi \rangle(0) = \langle e_i, \xi_0 \rangle = \langle e_i, \xi_0^n + \rho \rangle = 0$, and also $-f_i'(0) = \langle E_i, \varphi T \rangle(0) = \langle e_i, \varphi T(0) \rangle = \langle e_i, (\varphi T(0))^n + \nu \rangle = 0$. However, the solution of our differential equation with the given initial conditions is unique. Now $f_i(t) = 0$ satisfies this differential equation and the initial conditions. Therefore $f_i(t) = \eta(E_i) = 0$. ∎

Remarks. (1) If $\nu = 0$ or $\rho = 0$, then $\dim \hat{V}_k > 0$ is satisfied even for $k = [\frac{m}{2}]$. The other parts of the proof of the Proposition remain unaltered. Thus, in this case the statement of the Proposition is true also for $k = [\frac{m}{2}]$.

(2) If m is odd and $\nu = \rho = 0$, then $\dim \hat{V}_k > 0$ and the Proposition is true even for $k = [\frac{m}{2}] + 1$.

5 – Intersection of submanifolds in a Sasakian space

A submanifold V of a Sasakian space M is called invariant if it is tangent to the structure vector field ξ and, moreover, for any $X \in \mathfrak{X}(V)$ also $\varphi X \in \mathfrak{X}(V)$. We note that an invariant submanifold of M is odd dimensional and minimal (see for example [14] page 313).

THEOREM 3. *Let V and W be two complete invariant submanifolds of dimension $2r+1$, respectively $2s+1$, tangent to the structure vector field ξ of a complete connected Sasakian manifold M^{2n+1} and one of them compact. If M^{2n+1} has k-nonnegative bisectional curvature, k-positive bisectional curvature on V or W and $r+s \geq n+k-1$, then $V \cap W \neq \emptyset$.*

This theorem was proved for Kähler spaces P by KENMOTSU and XIA ([7], Theorem 3.2) and for compact regular Sasakian spaces M in case of $k=1$ by TANNO and BAIK ([13], Theorem 4.1). Now we want to prove this without the assumption of the regularity and the compactness of M, and for $k>1$.

PROOF. Assume that $V \cap W = \emptyset$. Now we can repeat the first part of the consideration of the Section 2. There exists a pair of points $p \in V$ and $q \in W$ which represent the distance of V and W, and there exists a normal geodesic $\gamma : [0,l] \to M$ such that $L(\gamma) = d(p,q) = l$ and $T(0) = \dot\gamma(0) \perp V$ and $T(l) = \dot\gamma(l) \perp W$. Then we can translate T_pV parallel along γ to q. Denoting the translated vector space of T_pV by $\widehat{TV}_q$, we obtain

$$\begin{aligned}\dim(\widehat{TV}_q \cap T_qW) &= 2r+1+2s+1-\dim(\widehat{TV}_q + T_qW) \geq \\ &\geq 2r+1+2s+1-(2n+1-1) = \\ &= 2(r+s-n)+2 \geq 2k,\end{aligned}$$

for $\dot\gamma(l) \perp \widehat{TV}_q \cap T_qW$. But also $\varphi\dot\gamma(l) \perp \widehat{TV}_q \cap T_qW$ is true. Let namely v_a $a = 1,\dots,2k$ a vector system which spans $\widehat{TV}_q \cap T_qW$. So $v_a \in T_qW$ and also $\varphi v_a \in T_qW$, for W is an invariant submanifold. Then $\langle \varphi\dot\gamma(l), v_a\rangle = -\langle \dot\gamma(l), \varphi v_a\rangle = 0$ showing that $\varphi\dot\gamma(l) \perp \widehat{TV}_q \cap T_qW$.

Now we want to apply our Proposition on $N = \widehat{TV}_q \cap T_qW$. Then $\dot\gamma(l), \varphi\dot\gamma(l) \perp N$ and so $\nu = 0$. Thus, according to the Proposition and Remark 1, there exists an orthonormal vector system $e_1,\dots,e_k \in N$,

$k = \left[\frac{\dim N}{2}\right]$, such that the parallel translated $E_i(t)$ of $e_i = E_i(l)$ along $\gamma(t)$, completed with $\varphi E_i(t)$ form an orthonormal system at any point of $\gamma(t)$.

Then we again can construct the variations α_i of γ with variational fields $E_i(t)$, and with curves $\alpha_i(t,s)$ in the variations. These curves run from V to W, for $E_i(0) \in T_pV$ and $E_i(l) \in T_qW$. We obtain $L'_{E_i}(0) = 0$, and formula (3) for $X(t) = E_i(t)$ yields again

$$lL''_{E_i}(0) = \langle \nabla_{E_i} E_i, \dot\gamma \rangle \mid_p^q - \int_0^l K(\dot\gamma \wedge E_i)dt.$$

Since V and W are invariant submanifolds, from $E_i(0) \in T_pV$ and $E_i(l) \in T_qW$ follow also $\varphi E_i(0) \in T_pV$ and $\varphi E_i(l) \in T_qW$. Thus we can construct variations α_i of γ also with variational fields $\varphi E_i(t)$, and with curves $\alpha_i(t,s)$ in the variations running from V to W. We obtain $L'_{\varphi E_i}(0) = 0$ and from the formula (2) for $X(t) = \varphi E_i(t)$:

$$\begin{aligned} lL''_{\varphi E_i}(0) = \langle \nabla_{\varphi E_i} \varphi E_i, \dot\gamma \rangle \mid_p^q + \int_0^l \| \nabla_{\dot\gamma} (\varphi E_i)^\perp \|^2 + \\ - \langle R((\varphi E_i)^\perp, \dot\gamma)\dot\gamma, (\varphi E_i)^\perp \rangle dt. \end{aligned} \tag{5}$$

We denote the expression on the right-hand side of (5) by A.

Now we calculate the first term of A:

$$\nabla_{\varphi E_i} \varphi E_i = (\nabla_{\varphi E_i} \varphi) E_i + \varphi(\nabla_{\varphi E_i} E_i).$$

Using (4,a) and the fact that ∇ is torsionfree, we get

$$\nabla_{\varphi E_i} \varphi E_i = \langle \varphi E_i, E_i \rangle \xi - \eta(E_i) \varphi E_i + \varphi(\nabla_{E_i} \varphi E_i + [\varphi E_i, E_i]).$$

An easy calculation yields $\langle \varphi E_i, E_i \rangle = 0$. Namely replacing in (4,e) X by φX and Y by X there results $\langle \varphi^2 X, \varphi X \rangle = \langle \varphi X, X \rangle$ with respect to (4,b). Then, applying (4,c) and (4,b) we get $-\langle X, \varphi X \rangle = \langle X, \varphi X \rangle$. Thus the first term drops out, and we obtain

$$= -\eta(E_i) \varphi E_i + \varphi\{(\nabla_{E_i} \varphi) E_i + \varphi \nabla_{E_i} E_i + [\varphi E_i, E_i]\}.$$

Finally, use of (4,a) and (4,c) yields

$$= -\nabla_{E_i} E_i + \eta(\nabla_{E_i} E_i)\xi + \varphi\{\langle E_i, E_i \rangle \xi - 2\eta(E_i) E_i + [\varphi E_i, E_i]\}.$$

Since $\varphi(\xi) = 0$ the third term drops out, and we obtain

$$\langle \nabla_{\varphi E_i} \varphi E_i, \dot\gamma \rangle \mid_p^q = -\langle \nabla_{E_i} E_i, \dot\gamma \rangle \mid_p^q + \eta(\nabla_{E_i} E_i)\langle \xi, \dot\gamma \rangle \mid_p^q + \\ - 2\eta(E_i)\langle \varphi E_i, \dot\gamma \rangle \mid_p^q + \langle \varphi[\varphi E_i, E_i], \dot\gamma \rangle \mid_p^q .$$

However, at p and q $E_i, \varphi E_i$, and hence $[\varphi E_i, E_i]$ and $\varphi[\varphi E_i, E_i]$ are perpendicular to $\dot\gamma$. Moreover, according to our assumption, ξ is tangent to V and W, i.e. $\xi \perp \dot\gamma$ at p and q. So the first term only remains alive on the right-hand side:

$$\langle \nabla_{\varphi E_i} \varphi E_i, \dot\gamma \rangle \mid_p^q = -\langle \nabla_{E_i} E_i, \dot\gamma \rangle \mid_p^q . \tag{6}$$

This is the firts term of A.

We show that the second term of A vanishes:

$$\nabla_{\dot\gamma}(\varphi E_i)^\perp = 0. \tag{7}$$

By the definition of $X^\perp$ (see Section 2)

$$\nabla_{\dot\gamma}(\varphi E_i)^\perp = \nabla_{\dot\gamma}(\varphi E_i - \langle \varphi E_i, \dot\gamma \rangle \dot\gamma) = \nabla_{\dot\gamma}(\varphi E_i) - \langle \nabla_{\dot\gamma}(\varphi E_i), \dot\gamma \rangle \dot\gamma,$$

for $\nabla_{\dot\gamma}\dot\gamma = 0$. However,

$$\nabla_{\dot\gamma} \varphi E_i = (\nabla_{\dot\gamma}\varphi) E_i + \varphi \nabla_{\dot\gamma} E_i.$$

Here $\nabla_{\dot\gamma} E_i = 0$, and because of (4,a)

$$= \langle \dot\gamma, E_i \rangle \xi - \eta(E_i)\dot\gamma = -\eta(E_i)\dot\gamma,$$

for $\dot\gamma \perp E_i$. Hence

$$\nabla_{\dot\gamma}(\varphi E_i)^\perp = -\eta(E_i)\dot\gamma - \langle -\eta(E_i)\dot\gamma, \dot\gamma \rangle \dot\gamma = \\ = -\eta(E_i)\dot\gamma + \eta(E_i)\dot\gamma = 0.$$

Finally, using again the relation $(\varphi E_i)^\perp = \varphi E_i - \langle \varphi E_i, \dot\gamma \rangle \dot\gamma$, we obtain for the integrand of the last term of A

$$\langle R((\varphi E_i)^\perp, \dot\gamma)\dot\gamma, (\varphi E_i)^\perp \rangle = \langle R(\varphi E_i, \dot\gamma)\dot\gamma, \varphi E_i \rangle. \tag{8}$$

Taking into consideration (6), (7), (8) we obtain

$$lL''_{\varphi E_i}(0) = -\langle \nabla_{E_i} E_i, \dot\gamma\rangle \,|_p^q - \int_0^l K(\dot\gamma \wedge \varphi E_i)dt,$$

and

$$l\sum_{i=1}^{k}(L''_{E_i}(0) + L''_{\varphi E_i}(0)) = -\int_0^l \sum_{i=1}^{k}(K(\dot\gamma \wedge E_i) + K(\dot\gamma \wedge \varphi E_i))dt = \\ = -\int_0^l \sum_{i=1}^{k} H(\dot\gamma \wedge E_i)dt.$$

However, according to the assumption of our theorem, the k-bisectional curvature $\sum_{i=1}^{k} H(\dot\gamma \wedge E_i) \geq 0$ on M and it is positive at p and q. So the right-hand side is negative, while the left is nonnegative, for $L''_{E_i}(0)$, $L''_{\varphi E_i}(0) \geq 0$, because γ is a shortest curve between p and q. This contradiction shows that $V \cap W = \emptyset$ cannot hold. □

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